Study of the general theories of gravitation: Lovelock-Cartan-Horndeski Gravity

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We explore generalized gravitational theories that have at most second-order equations of motion in the metric. In this way comes Lovelock's theory, which is the generalization of General Relativity and describes a gravitational theory in arbitrary dimensions. The next step is to include more degrees of freedom by admitting torsion in the theory and thus we arrive at the generalization of Lovelock's theory called the Cartan-Lovelock theory. Another generalization that can be obtained is to stay in 4 dimensions and include a scalar field so that second-order equations of motion are obtained in the metric and the scalar field, this theory is known as Horndeski's theory. In this thesis, we explore a step further from this generalized theory by including torsion and obtaining the Cartan-Lovelock-Horndeski theory. First-order formalism and differential forms are used for the construction of all these theories.

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Dedication

This is dedicated to my mon Roxana and my dad Lucio.

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Chapter 1

Introduction

The most successful theory in the description of gravity is undoubtedly General Relativity. With the principle of least action and the Einstein-Hilbert action, we can obtain the elegant Einstein field equations, which describe the dynamics of space-time. Einstein's equations can be arranged by placing the entire geometric part on one side, the curvature and the cosmological constant, and on the other side of the equation matter and energy. The covariant conservation of the energy-momentum tensor, matter and energy, requires that their divergence be zero, which is already satisfied by the geometric part thanks to the Bianchi identities. Here curiosity arises, Could we find a more general form for the geometric side of Einstein's equation, that remains at most the second order in derivatives of the metric, that still fulfills the null divergence? And so Lovelock's theory arises, which is more general than general relativity and helps us build gravitational theories in D-dimensions[16]. It is possible to build an endless number of gravitational theories but General Relativity contains symmetries that position it in a privileged place, now we must follow Lovelock's footsteps and include the symmetries of General Relativity to build more general theories in arbitrary dimensions and add new fields.

Although General Relativity is so successful, it is not free of problems. In the high-energy regime, the theory ceases to be useful, one finds a non-renormalizable theory and its predictive power disappears. On the other hand, the low-energy side gives us several surprises. New observations of the universe indicate that General Relativity must be modified [4]-[6], the existence of dark matter and dark energy. In principle, the cosmological acceleration is well explained by the cosmological constant and General Relativity. However, theoretical calculations on the value of this constant that includes the quantum fluctuations of matter fields are several orders of magnitude larger than what is observed [9]. Although one way to modify the Relavity is to respect the symmetries, there are alternatives that in principle we could use and ignore some symmetries, abandon the locality, or add new degrees of freedom or extra dimensions. In this thesis, we will explore the option of exploring more dimensions and including new degrees of freedom.

In the first-order formalism, the basic ingredients for the construction of gravitational theories are the Vielbein, the spin connection, curvature and torsion. To begin with, we need to generalize General Relativity, a 4 dimensional theory that uses only the Vielbein, and the curvature for the construction of its action. In 4 dimensions the torsion is identically zero. The first step was taken by Lovelock by building a general action of gravitation in D-dimensions using only the vielbein and curvature[16]. From this theory, it is learned that despite not including torsion, it appears in the field equations, that for $D < 5$ the torsion is zero, but for high dimensions, it is not necessarily so. One must impose the condition of zero torsion to preserve the diffeomorphism symmetry of the theory. The next step was the inclusion of torsion in the action of gravitation and this generalization was developed by Zanelli and Mardones[17]. Now the theory not only included the desired symmetries but also had a new field that would modify its behavior. For now, no observation has yielded traces of torsion in our universe, but they may be very weak and we have not detected it yet, so it is convenient to have a theory that includes it. Another alternative to adding degrees of freedom is the inclusion of a scalar field. Hordeski in his work [10] generalizes General Relativity in 4 dimensions by including a series of terms that include a scalar field in action that gives second-order gravitation equations in the derivative of the metric and the scalar field.

We organize this thesis as follows: In Chapter 2, we begin with a review of differential geometry. We start by developing concepts to understand differentiable manifolds, then we introduce differentiable forms and then do a quick review of Riemannian geometry and the generalized Kronecker delta. In Chapter 3, with the basic tools learned in Chapter 2, we can move on to the description of gravitation in the first-order formalism, where we define the Vielbein, the Lorentz connection, the Lorentz curvature, and torsion. In this chapter, we can construct gravitational theories making use of these four basic ingredients. In chapter 4, we are dedicated to presenting modified gravitational theories that generalize to General Relativity. In Chapter 5, we present the most general theory of gravitation that includes torsion and scalar fields in 4 dimensions and we end with the conclusions and make a summary of everything learned.

Chapter 2

An Overview of Differential Geometry

For the study of gravitational theories, it is essential to know geometry, in this chapter we present the most general aspects of geometry so that the reader has the necessary tools to understand all the work. My main objective is that anyone curious about gravitation reads this thesis and has a good base to consult other more specialized books.

When studying geometry it is important to understand the basic concepts, and these are *metricity* and *affinity*. With the notion of metricity, we can understand and define lengths, areas and volumes of objects that are defined locally in space-time. On the other hand, affinity refers to properties that remain invariant under translational transformations or more generally affine transformations.

We begin the chapter by quickly describing differentiable manifolds, then we introduce differentiable forms, and then a brief review of Riemannian geometry.

2.1 Differentiable manifolds

2.1.1 Manifold

A differentiable manifold $\mathcal{M}$, in D dimentions, is some smooth surface, a topological space, which locally looks Euclidean but not necessarily in its global extent. Via a homeomorphism to $\mathbb{R}^D$ we attach to each point of $\mathcal{M}$ a set of coordinates and require the traansition from one set to another to be smooth.

2.1.2 Chart

Let $\mathcal{M}$ be a topological space. A *chart* $(V_\alpha, \varphi_\alpha)$ is a homeomorphism φ_α from an open set $V_\alpha \subset \mathcal{M}$ into a open set $R_\alpha \subset \mathbb{R}^D$

$$V_\alpha \xrightarrow{\varphi_\alpha} R_\alpha. \tag{2.1}$$

Two charts are compatible if the overlap maps are continuously differentiable diffeomorphisms

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} \in C^\infty, \\ \varphi_2 \circ \varphi_1^{-1} \in C^\infty, \end{aligned} \tag{2.2}$$

or if $V_1 \cap V_2 = \emptyset$ (see Fig. 2.1).

2.1.3 Atlas

The set of compatible charts $\{(V_\alpha, \varphi_\alpha)\}$ covering $\mathcal{M}$ is named, for obvious reasons, an atlas. Two atlases are compatible if all their charts are compatible.

2.1.4 Differentiable manifold

A differentiable manifold satisfies the following conditions:

1. $\mathcal{M}$ is a topological space.
2. $\mathcal{M}$ is equipped with a family of pairs $\{(V_\alpha, \varphi_\alpha)\}$.
3. $\{V_\alpha\}$ is a family of open sets which cover $\mathcal{M} : \bigcup_\alpha V_\alpha = \mathcal{M}$ and φ_α is a homeomorphism from V_α onto an open subset $R_\alpha \subset \mathbb{R}^D : V_\alpha \xrightarrow{\varphi_\alpha} R_\alpha$.
4. Given two open sets V_α, V_β with $V_\alpha \cap V_\beta \neq \emptyset$ the overlap maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ from subset $\varphi_\alpha(V_\alpha \cap V_\beta)$ to $\varphi_\beta(V_\alpha \cap V_\beta)$ or $\varphi_\alpha \circ \varphi_\beta^{-1}$ from subset $\varphi_\beta(V_\alpha \cap V_\beta)$ to $\varphi_\alpha(V_\alpha \cap V_\beta)$ are infinitely differentiable (C^∞ funtions, diffeomorphisms). (See Fig. 2.1)

Hence a differentiable manifold is given by a topological space $\mathcal{M}$ and the equivalence class of atlases. The features 2) and 3) imply that the topological space is locally Euclidean. $\mathcal{M}$ is covered with patches V_α and via the homeomorphisms φ_α we attach coordinates in $\mathbb{R}^D$ to these patches. So within one patch the manifold looks like the Euclidean $\mathbb{R}^D$ but not necessarily globally.

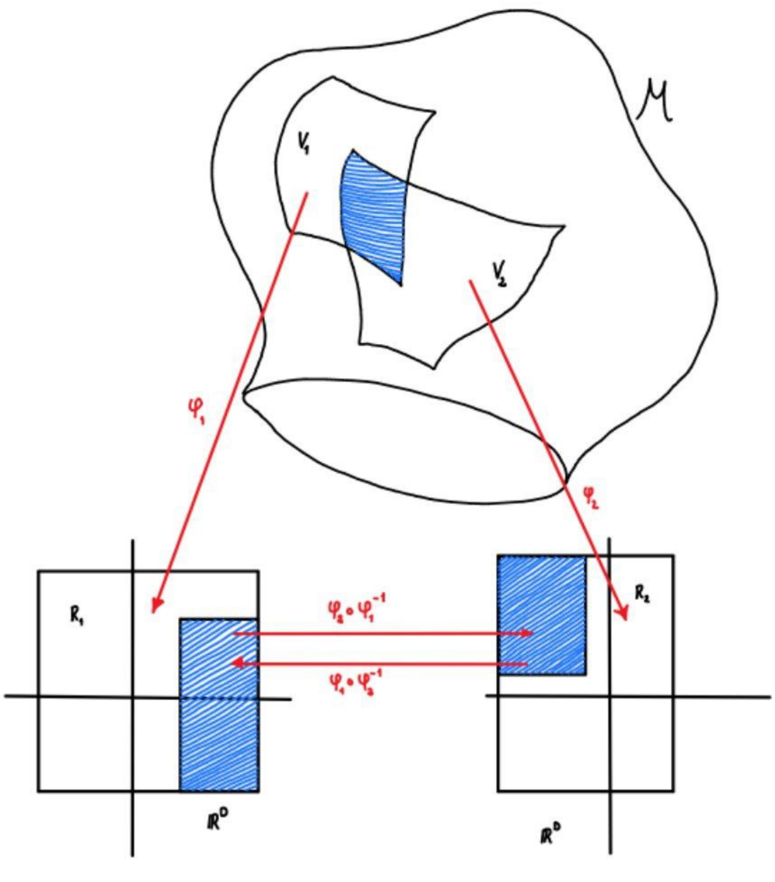

Figure 2.1: A differentiable manifold is shown.

Feature 4) means that in the overlap region of two patches V_α, V_β we establish two coordinates in $\mathbb{R}^D$, $\varphi_\alpha(V_\alpha \cap V_\beta)$ and $\varphi_\beta(V_\alpha \cap V_\beta)$ and we can change these coordinates smoothly via $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$.

The dimention of the manifold is given by the dimention of the Euclidean vector space

$$dim\mathcal{M} \equiv dim\mathbb{R}^D = D. \tag{2.3}$$

2.1.5 Tangent space to a manifold

Vectors on manifold $\mathcal{M}$ always describe tangent vectors to a curve in $\mathcal{M}$. Let $p(u)$, $u \in I$ be some curve passing through a chart. The coordinates of this curve are $x^i(p(u))$, $i = 1, ..., D$ and the tangent vector to the curve is given by

$$\frac{d}{du} x^i(p(u)). \tag{2.4}$$

Next we consider a funtion $f(p) : \mathcal{M} \to \mathbb{R}^D$ defined on $\mathcal{M}$. The change of the funtion along the curve is

$$\frac{d}{du} f(p(u)) \tag{2.5}$$

and in terms of local coordinates

$$\frac{\partial f}{\partial x^i} \frac{d}{du} x^i(p(u)). \tag{2.6}$$

Defining the operator

$$X = X^i \frac{\partial}{\partial x^i} \tag{2.7}$$

with $X^i = \frac{d}{du} x^i(p(u))$ we obtain

$$\frac{d}{du} f(p(u)) = Xf. \tag{2.8}$$

We consider the differential operator X as the *tangent vector to the manifold* $\mathcal{M}$ at the point $p = p(u_0)$, with u_0 fixed, in the direction of the curve $p(u)$.

Applying, the operator X to the position coordinates we obtain the velocity

$$X[x^i] = \frac{dx^j}{du} \frac{\partial x^i}{\partial x^j} = \frac{dx^i}{du}. \tag{2.9}$$

For every differentiable curve through a point $p \in \mathcal{M}$ of the manifold $\mathcal{M}$ there exists a tangent vector. So we can define the *tangent space* $T_p(\mathcal{M})$ to the manifold $\mathcal{M}$ at p to be the space of all posible tangents at the point p. $T_p(\mathcal{M})$ describes a vector space and in terms of local coordinates the set of differential operators

$$\left\{\frac{\partial}{\partial x^i}\right\}, \quad i = 1, ..., D \tag{2.10}$$

forms a *basis* in $T_p(\mathcal{M})$. Clearly the dimention is

$$dim T_p(\mathcal{M}) = dim \mathcal{M} = D. \tag{2.11}$$

This notation of tangent vector–they represent *contravariant vectors*–is a coordinate independent description.

2.1.6 Cotangent space to a manifold

To the contravariant vector, which we have considered up to now, there also exist their duals–the covariant vectors. The dual space to $T_p(\mathcal{M})$ is the *cotangent space* $T_p^*(\mathcal{M})$ where duality is defined via the inner product

$$\left(dx^i, \frac{\partial}{\partial x^i}\right) = \delta^i_j. \tag{2.12}$$

So the set of differentials

$$\{dx^i\}, \quad i = 1, ..., D, \tag{2.13}$$

which are dual to $\left\{\frac{\partial}{\partial x^i}\right\}$ forms a basis in $T_p^*(\mathcal{M})$.

Generally, an element of $T_p^*(\mathcal{M})$ is given by the so-called 1-form

$$\omega = \omega_i dx^i \tag{2.14}$$

where ω_i represents some *covariant vector*. The action of 1-form on a vector is deternined by

$$(\omega, X) = \left(\omega_i dx^i, X^j \frac{\partial}{\partial x^j}\right) = \omega_i X^j \delta^i_j = \omega_i X^i. \tag{2.15}$$

The notion of a 1-form ω is also a coordinate independent description.

2.1.7 Tensors

We can construct *tensor type* (a, b) by mapping a elements of $T_p^*(\mathcal{M})$ and b elements of $T_p(\mathcal{M})$ into $\mathbb{R}$. So the space of these tensors is defined by

$$T^a{}_b = \underbrace{T_p(\mathcal{M}) \otimes ... \otimes T_p(\mathcal{M})}_{\text{a factors}} \otimes \underbrace{T_p^*(\mathcal{M}) \otimes ... \otimes T_p^*(\mathcal{M})}_{\text{b factors}} \tag{2.16}$$

An element of $T^a{}_b$, a mixed tensor with contravariant rank a and covariant rank b, is given in terms of local coordinates

$$T(x) = T^{i_1 \dots i_a}{}_{j_1 \dots j_b}(x) \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_a}} dx^{j_1} \dots dx^{j_b}. \tag{2.17}$$

The action of T on 1-forms $\omega_1, \dots, \omega_a$ and vectors $X_1, \dots, X_b$ gives the number

$$T(\omega_1, \dots, \omega_a, X^1, \dots, X^b) = T^{i_1 \dots i_a}{}_{j_1 \dots j_b} \omega_{a_1 1} \dots \omega_{i_a a} X_1^{j_1} \dots X_b^{j_b}. \tag{2.18}$$

Allowing the point p to vary smoothly over the whole manifold $\mathcal{M}$ the vectors and tensors also vary smoothly over $\mathcal{M}$ and we achieve so-called vector fields and tensor fields on $\mathcal{M}$.

2.2 Differential forms

Differential forms are practical objects. Their notation is independent of the choise of a coordinate system and the mathematical formalism becames simple.

2.2.1 Wedge product

Let us begin with the antisymmetric tensor product

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \tag{2.19}$$

which is the *wedge product.*

Then we have

$$\begin{aligned} dx^\mu \wedge dx^\nu &= -dx^\nu \wedge dx^\mu \\ dx^\mu \wedge dx^\mu &= 0 \end{aligned} \tag{2.20}$$

by definition.

2.2.2 Differential form

We can define the *differential forms*

$$\begin{aligned} 0-form \quad & \omega = \omega(x) \\ 1-form \quad & \omega = \omega_\mu(x) dx^\mu \\ 2-form \quad & \omega = \frac{1}{2!} \omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu \\ \dots \quad & \dots \\ p-form \quad & \omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \end{aligned} \tag{2.21}$$

where $\omega_{\mu_1 \ldots \mu_p}(x)$ is a totally antisymmetric covariant tensor field of rank $p \leq D$. ω vanishes for $p > D$.

We denote the set of all p-forms by Λ^p. This vector space of dimention

$$dim\Lambda^p = \frac{D!}{p!(D-p)!}. \tag{2.22}$$

Conmuting the forms $\alpha_p \in \Lambda^p$ and $\beta_q \in \Lambda^q$ we obtain

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p. \tag{2.23}$$

So odd forms always anticommute.

2.2.3 Exterior derivative

Let us introduce the *exterior derivative*

$$d = \frac{\partial}{\partial x^\mu} dx^\mu \tag{2.24}$$

acting on a p-form in the following way

$$d\omega = \frac{1}{p!} \frac{\partial}{\partial x^\mu} \omega_{\nu_1 \ldots \nu_p} dx^\mu \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p}. \tag{2.25}$$

The exterior derivative is a map $d : \Lambda^p \to \Lambda^{p+1}$ which transforms p-forms into (p+1)-forms. It satisfies the importan propety

$$d^2 = 0. \tag{2.26}$$

Furthermore d obeys the *antiderivation rule*

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \tag{2.27}$$

2.2.4 Interior product

We construct a contracted multiplication of a p-form $\omega \in \Lambda^p$ with a vector field $X \in T(\mathcal{M})$. It is defined

$$\begin{aligned} &1) \quad I_X \quad \textit{antiderivative} \\ &2) \quad I_X f = 0 \\ &3) \quad I_X dx^\mu = X^\mu, \end{aligned} \tag{2.28}$$

where f denotes some funtion. Viewed as a mapping the operation $I_X : \Lambda^p \to \Lambda^{p-1}$ maps opposite to the exterior derivative d. Also we have:

$$I_X^2 = 0. \tag{2.29}$$

For a p-form we have

$$I_X\omega = \frac{1}{(p-1)!} X^\nu \omega_{\nu\mu_1 \dots \mu_p} dx^{\mu_2} \wedge .... \wedge dx^{\mu_p} \tag{2.30}$$

2.2.5 Hodge star operation

Let us recall the space of all p-forms Λ^p and the space Λ^{D-p}. Both vector spaces have the same dimention $dim\Lambda^{D-p} = dim\Lambda^p$. There is a duality between these 2 spaces, an isomorphism given by the *Hodge * operation*: $\Lambda^p \xrightarrow{*} \Lambda^{D-p}$. The star operation transforms p-forms into (D-p)-forms and its action is defined by

$$*dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} = \frac{1}{(D-p)!} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_D} dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_D} \tag{2.31}$$

where

$$\epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_D} = g^{\mu_1\nu_1} ... g^{\mu_p\nu_p} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_D}. \tag{2.32}$$

$g_{\mu\nu}$ is the tensor and $e = detg_{\mu\nu}$. In a curved space endowed with metric $g_{\mu\nu}$ we have $\epsilon_{1 \dots D} = \sqrt{|e|}$.

We notice that raise and lower the indices by

$$\epsilon^{\mu_1 \dots \mu_p} = g^{\mu_1\nu_1} ... g^{\mu_p\nu_p} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_D} = e^{-1} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_D}. \tag{2.33}$$

Given a p-form

$$\omega_p = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \tag{2.34}$$

then the *dual p-form* denotes

$$*\omega_p = \frac{1}{p!(D-p)!} \omega_{\mu_1 \dots \mu_p} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D}. \tag{2.35}$$

2.3 An Overview of Riemannian geometry

Now briefly we discuss what the geometry of spacetime means.

2.3.1 Metric

Einstein's General Relativity requires the definition of pseudo-Riemannian geometry on a manifold that locally approximates a Minkowski geometry.

Let $\mathcal{M}$ be a differentiable manifold, then the inner product of two vectors $X, Y \in T_p(\mathcal{M})$ is defined in the tangent space $T_p(\mathcal{M})$. We introduce the following metrics:

Riemannian metric

In Riemannian geometry the most relevant objects that are defined in the manifold $\mathcal{M}$ as distance, angles, areas, parallel transports, curvature, torsion, etc., can be built from the metric. The metric provides us with the notion of distance between two infinitesimally close points. A Riemannian metric g is a tensor field of type $(0, 2)$ on $\mathcal{M}$ with subsequent properties ateach point $p \in \mathcal{M}$

1. $g_p(X, Y) = g_p(Y, X)$
2. $g_p(X, X) \geq 0$, equality only for $X = 0$.

So $g_p \equiv g|_p$, which means the tensor g evaluated al point p is a symmetric positive definite bilinear form.

The metric g_p is called a *pseudo-Riemannian metric* if we have

3. $g_p(X, Y) = 0$, $\forall X \in T_p(\mathcal{M}) \to Y = 0$

insted of property 2.

Let us recall our discussion in Section 2.2, the inner product of a vector $X \in T_p(\mathcal{M})$ with a dual vector, a 1-form $\omega \in T_p^*(\mathcal{M})$. It represents the map

$$(,) : T_p^*(\mathcal{M}) \otimes T_p(\mathcal{M}) \to \mathbb{R} \tag{2.36}$$

where

$$(\omega, X) = (\omega_\mu dx^\mu, X^\nu \frac{\partial}{\partial x^\nu}) = \omega_\mu X^\mu \in \mathbb{R}. \tag{2.37}$$

When we have a metric we can define the *inner product* between two vectors $X, Y \in T_p(\mathcal{M})$ by the map

$$g_p(X, Y) : T_p(\mathcal{M}) \otimes T_p(\mathcal{M}) \to \mathbb{R}. \tag{2.38}$$

Then we may define a linear map

$$g_p(X,) : T_p(\mathcal{M}) \to \mathbb{R} \tag{2.39}$$

by

$$Y \to g_p(X, Y) \tag{2.40}$$

and we identify this map with a 1-form

$$g_p(X,) \iff \omega \in T_p^*(\mathcal{M}). \tag{2.41}$$

On the other hand, the 1-form $\omega \in T_p^*(\mathcal{M})$ induces a vector $X \in T_p(\mathcal{M})$ by

$$(\omega, Y) = g_p(X, Y), \tag{2.42}$$

where X is unique due to property 3. So the metric g_p establishes an isomorphism between

$$T_p(\mathcal{M}) \simeq T_p^*(\mathcal{M}). \tag{2.43}$$

On a given chart (V, φ) of the manifold $\mathcal{M}$ with coordinates x^μ we can express the *metric tensor* $g \in T_2^0(\mathcal{M})$ as

$$g_p = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{2.44}$$

where

$$g_{\mu\nu}(x) = g_p \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right). \tag{2.45}$$

We regard $g_{\mu\nu}$ as a matrix and $g^{\mu\nu}$ as its inverse according to

$$g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma. \tag{2.46}$$

We denote the *determinant of the metric* by

$$g \equiv det(g_{\mu\nu}). \tag{2.47}$$

and we have

$$det(g^{\mu\nu}) = g^{-1}. \tag{2.48}$$

The isomorphism (2.43) between vestors and covestors is supplied by the metric

$$\begin{aligned} \omega_\mu &= g_{\mu\nu} X^\nu, \\ X^\mu &= g^{\mu\nu} \omega_\nu \end{aligned} \tag{2.49}$$

and

$$g_p(X, Y) = (\omega, Y) = \omega_\mu Y^\mu = g_{\mu\nu} X^\nu Y^\mu. \tag{2.50}$$

So we raise and lower the indices with the help of the metric tensor $g^{\mu\nu}$, $g_{\mu\nu}$.

The *norm* $|X|$, of a vector $X \in T_p(\mathcal{M})$ is defined by

$$|X|^2 = g_p(X, X) = g_{\mu\nu} X^\mu X^\nu. \tag{2.51}$$

The *line element* can be found by inserting the infinitesimal displacement $dx^\mu \frac{\partial}{\partial x^\mu} \in T_p(\mathcal{M})$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.52}$$

The quadratic form $g_{\mu\nu}$ can be diagonalized. If $g_{\mu\nu}$ is Riemannian all diagonal terms are positive; in the case of a pseudo-Riemannian metric may have positive and negative diagonal terms $diag(-1, ..., 1)$ with -1 occurring t times and 1 occurring s times. The pair (t, s) is called the *signature of the metric*. If $t = 1$ we have a *Lorentz metric*. The pair $(\mathcal{M}, g)$ is called a Riemannian, a pseudo-Riemannian or a Lorentz manifold.

2.3.2 Christoffel connection and curvature

Christoffel connection

For a Riemannian manifold $(\mathcal{M}, g)$ there exists a unique *connection 1-form* called Christoffel connection

$$\Gamma^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\mu} dx^\mu, \tag{2.53}$$

where $\Gamma^\alpha{}_{\beta\mu}$ is the familiar Christoffel simbol. It is determined by the metric tensor via formula

$$\Gamma^\alpha{}_{\beta\mu} = \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\beta\lambda} + \partial_\beta g_{\mu\lambda} - \partial_\lambda g_{\mu\beta}) \tag{2.54}$$

and it is symmetric in the lower indices

$$\Gamma^\alpha{}_{\beta\mu} = \Gamma^\alpha{}_{\mu\beta}.$$

From now on we use the notation

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

Covariant derivative

We define the *covariant derivative* as

$$\nabla = d + [\Gamma,\ ] \tag{2.55}$$

with $\nabla = \nabla_\mu dx^\mu$, $d = \partial_\mu dx^\mu$, $\Gamma_\mu dx^\mu$ and Γ is the matrix notation of $\Gamma^\alpha{}_\beta$. If we apply to a tensor-valued p-form we get

$$\begin{aligned} \nabla T^\mu{}_\nu &= dT^\mu{}_\nu + [\Gamma, T]^\mu{}_\nu \\ &= dT^\mu{}_\nu + \Gamma^\mu{}_\alpha T^\alpha{}_\nu - \Gamma^\alpha{}_\nu T^\mu{}_\alpha. \end{aligned} \tag{2.56}$$

A connection is called *a Riemannian connection* if

- the metric is covariantly constant: $\nabla_\alpha g_{\mu\nu} = 0$
- the connection has cero torsion: $T^\alpha = \nabla dx^\alpha = 0$.

Sometimes a Riemannian connection is called the *Levi-Civita connection*.

Torsion

The torsion is defined as

$$T^\alpha = \frac{1}{2} T^\alpha{}_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2.57}$$

2.4 Generalized Kronecker delta

We define the *generalized delta* of order n as the determinant formed by Kronecker deltas $T^{n,n}_p \mathcal{M} \to \{-1, 0, 1\}$

$$\delta^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_2} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \cdots & \delta^{\mu_1}_{\nu_n} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \cdots & \delta^{\mu_2}_{\nu_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\mu_n}_{\nu_1} & \delta^{\mu_n}_{\nu_2} & \cdots & \delta^{\mu_n}_{\nu_n} \end{vmatrix} \tag{2.58}$$

Generalized delta properties:

1. The definition (2.58) can be written as

$$\delta^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_2} = \sum_{p=1}^{n} (-1)^{p+1} \delta^{\mu_1}_{\nu_p} \delta^{\mu_2 \dots\dots\dots \mu_n}_{\nu_1 \dots \hat{\nu}_p \dots \nu_n}$$

   where $\hat{\nu}_p$ means skip index p.

2. Is completely antisymmetric under the change of indices

$$\delta^{\mu_2 \dots \mu_i \dots \mu_j \dots \mu_n}_{\nu_1 \dots\dots\dots\dots\dots\dots \nu_n} = -\delta^{\mu_2 \dots \mu_j \dots \mu_i \dots \mu_n}_{\nu_1 \dots\dots\dots\dots\dots\dots \nu_n}$$

$$\delta^{\mu_1 \dots\dots\dots\dots\dots \mu_n}_{\nu_2 \dots \nu_i \dots \nu_j \dots \nu_n} = -\delta^{\mu_1 \dots\dots\dots\dots\dots \mu_n}_{\nu_2 \dots \nu_j \dots \nu_i \dots \nu_n}$$

3. It is an antisymmetrization operator, which allows us to construct the exterior product from the tensor product as

$$\frac{1}{p!} \delta^{\mu_1 \dots \mu_n}_{\nu_2 \dots \nu_n} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_n} = dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}.$$

If we have two tensors $\alpha^{\mu_1...\mu_n}$ and $\beta_{\nu_1...\nu_n}$ we get that

$$\alpha^{[\mu_1...\mu_n]} = \delta^{\mu_1...\mu_n}_{\nu_1...\nu_n}\alpha^{\nu_1...\nu_n}$$

$$\beta_{[\nu_1...\nu_n]} = \delta^{\mu_1...\mu_n}_{\nu_1...\nu_n}\beta_{\mu_1...\mu_n}.$$

For completely antisymmetric tensors it holds that

$$n!\alpha^{\mu_1...\mu_n} = \delta^{\mu_1...\mu_n}_{\nu_1...\nu_n}\alpha^{\nu_1...\nu_n}$$

$$n!\beta_{\nu_1...\nu_n} = \delta^{\mu_1...\mu_n}_{\nu_1...\nu_n}\beta_{\mu_1...\mu_n}.$$

The same is true for the differential forms.

4. It satisfies the following identity

$$\delta^{\mu_1...\mu_r\mu_{r+1}...\mu_n}_{\nu_1...\nu_r\mu_{r+1}...\mu_n} = \frac{(D-r)!}{(D-n)!}\delta^{\mu_1...\mu_r}_{\nu_1...\nu_r}$$

where D is the dimension of the space.

We define the Levi-Civita symbol as

$$\begin{aligned}\epsilon^{\mu_1...\mu_n} &= \delta^{\mu_1...\mu_n}_{1......n}, \\ \epsilon_{\nu_1...\nu_n} &= \delta^{1......n}_{\nu_1...\nu_n}.\end{aligned} \tag{2.59}$$

With the Levi civita symbol we can recover the generalized Kronecker delta

$$\epsilon^{\mu_1...\mu_n}\epsilon_{\nu_1...\nu_n} = \delta^{\mu_1...\mu_n}_{\nu_1...\nu_n}. \tag{2.60}$$

It also allows us to relate a matrix of components M^μ_ν in D-dimensions with its determinant

$$\epsilon_{\mu_1...\mu_n}M^{\mu_1}_{\nu_1}...M^{\mu_n}_{\nu_n} = det(M)\epsilon_{\nu_1...\nu_n}. \tag{2.61}$$

Chapter 3

Gravity in the first-order formalism

In this chapter, the fundamental blocks for the construction of a theory of gravitation in first-order formalism are introduced. First, they are introduced to the fundamental fields which are the vielbein and the spin connection. Then, the Lorentz curvature and torsion are constructed. With these materials, we can build an action for gravitation that is invariant under Lorentz transformations and diffeomorphisms. References consulted for this chapter are [2], [5], [18].

3.1 Fundamental fields

Equivalence principle

The equivalence principle can be understood as follows: In a laboratory in free fall, we cannot feel gravity and any object would float around us as if we were in space in a spaceship. Everything said above can be achieved if and only if we do it locally. The laboratory must be small enough and the time in which they do the experiments must be short enough. Under these conditions, any experiment should be indistinguishable from events in the absence of gravity and from events with gravity, and the laws of physics reflected in the experiment must be the same as the laws of physics valid in a Minkowski space, that is say that locally you cannot perceive the curvature of spacetime as illustrated in Fig. 3.1. In short, in a local neighborhood of space-time, for the Lorentz transformations, there is an invariance. Einstein argued that in the absence of gravity, the gravitational field can be emulated by applying acceleration in the laboratory. This idea is known as the equivalence principle, where gravitation and acceleration are equivalent in a small region of spacetime.

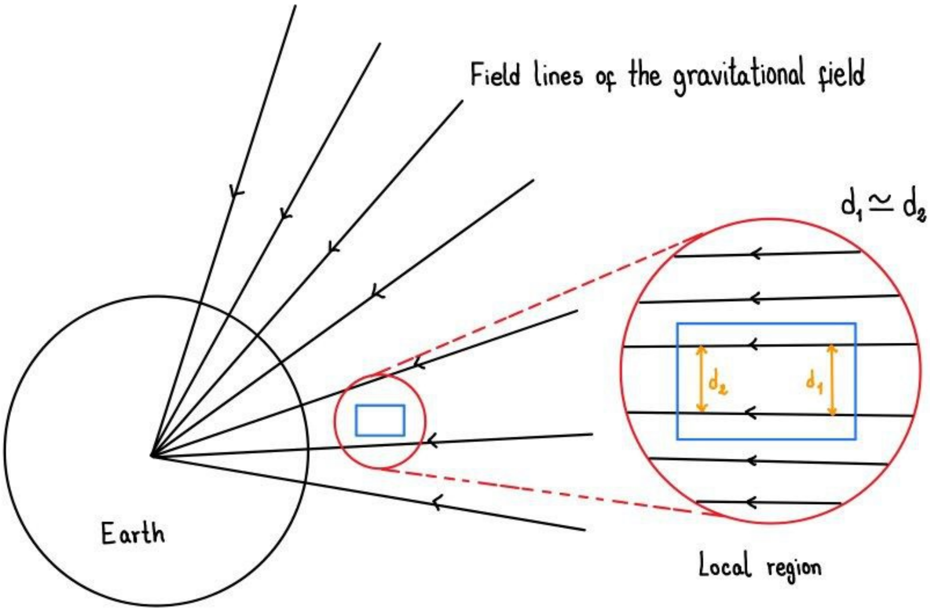

Figure 3.1: A schematic of the gravitational field lines produced by the Earth is shown. It is observed that globally there is a curvature in these lines, but locally they are perceived as parallel, the distances d_1 and d_2 are approximately equal $d_1 \simeq d_2$.

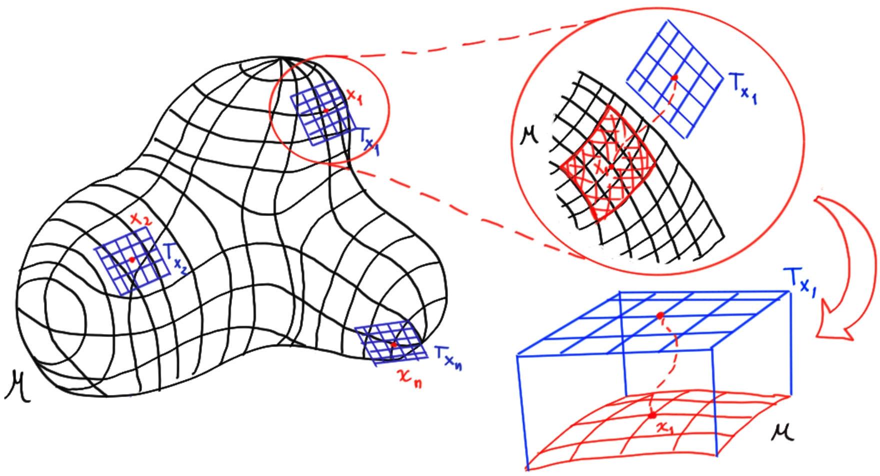

Figure 3.2: A smooth manifold $\mathcal{M}$ is shown and a tangent space T_x is inserted at each point of it. The neighborhood around x_1 approximates a flat space, that is, the tangent space to x_1.

3.1.1 The vielbein

From the principle of equivalence we learned that there will always exist a sufficiently small region where spacetime can be approximated as flat spacetime. In the language of differential geometry, this flat space-time corresponds to a tangent space that can be defined at each point of a smooth manifold.

Let spacetime be a smooth (d+1)-dimensional manifold $\mathcal{M}$. At each point $x \in \mathcal{M}$ there exists a (d+1)-dimensional tangent space T_x with Lorentzian signature (-,+,...,+), see Fig.3.2. There is an isomorphism between $\mathcal{M}$ and T_x called **vielbein**. The vielbein helps us to map tensors in $\mathcal{M}$ to T_x and vice versa.

The isomorphism between $\mathcal{M}$ and the set of tangential spaces $\{T_x\}$ can be constructed as a coordinate transformation between a local coordinate system $\{x^\mu\}$ in a neighborhood of $\mathcal{M}$ and a Minkowskian reference system in the tangent space T_x of coordinates $\{x^a\}$.

The vielbein is:

$$e^a{}_\mu(x) = \frac{\partial x^a}{\partial x^\mu}, \tag{3.1}$$

and has $\frac{n(n-1)}{2}$ independent components, where $n = d+1$. The vielbein is sufficient to define a relation between tensors $T^{\mu\nu\rho\cdots} \in \mathcal{M}$ and tensors $T^{abc\cdots} \in T_x$. For example,

$$T^{abc} = e^a{}_\mu e^b{}_\nu e^c{}_\rho T^{\mu\nu\rho}.$$

This isomorphism induces a metric in $\mathcal{M}$:

$$g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu(x) e^b{}_\nu(x), \tag{3.2}$$

where,

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3.3}$$

From (3.2) we note that the vielbein is the square root of the metric. Since $e^a{}_\mu$ determines the metric, all the properties of spacetime are conteined in it. However, given a metric $g_{\mu\nu}$, there are infinitely many vielbeins that reproduce the same metric. This arbitrariness corresponds to the freedom to choose any local orthogonal system in T_x.

Under a Lorentz transformation the vielbein transforms as:

$$e^{\bar{a}}{}_\mu(x) = \Lambda^{\bar{a}}{}_a e^a{}_\mu(x), \tag{3.4}$$

where the matrix $\Lambda \in SO(d,1)$. This transformation leaves the metric $\eta_{ab} \in T_x$ invariant,

$$\eta_{ab} = \Lambda^{\bar{a}}{}_a \Lambda^{\bar{b}}{}_b \eta_{\bar{a}\bar{b}}. \tag{3.5}$$

Consequently $g_{\mu\nu}$ is also invariant under these local Lorentz transformations.

3.1.2 The Lorentz connection

The Lorentz invariance in the tangent space T_x at each point of the manifold $\mathcal{M}$, endows it with gauge symmetry. Given this local Lorentz symmetry we can define a covariant derivative, for this we need to introduce a **spin or Lorentz connection** denoted by $\omega^a{}_{b\mu}(x)$.

Let $\Phi^a(x)$ be a field that transforms as a vector under the Lorentz group, and its covariant derivative is:

$$D_\mu \Phi^a(x) = \partial_\mu \Phi^a(x) + \omega^a{}_{b\mu}(x)\Phi^b(x), \tag{3.6}$$

and the spin connection transforms as:

$$\omega^{\bar{a}}{}_{\bar{b}\mu} = \Lambda^{\bar{a}}{}_a \Lambda^b{}_{\bar{b}} \omega^a{}_{b\mu} + \Lambda^{\bar{a}}{}_a \partial_\mu \Lambda^b{}_{\bar{b}}, \tag{3.7}$$

and defines the parallel transport of the Lorentzian tensors in the tangent spaces T_x and T_{x+dx}.

Types of covariant derivatives

Spacetime is a pseudo-Riemannian manifold $\mathcal{M}$ endowed with a metric $g_{\mu\nu}$. The covariant derivative is defined by the Levi-Civita connection:

$$(\overset{\circ}{\nabla}_\mu)^\alpha_\nu \equiv \delta^\alpha_\nu \partial_\mu + \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} \tag{3.8}$$

where $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$, is the Christoffel's symbol.

$\overset{\circ}{\nabla}_\mu$ transforms a tensor, under diffeomorphism, into another tensor with an additional index. The Lorentz covariant operator D_μ performs a similar task, transforms a vector in the tangent space T_x by translating it to T_{x+dx}, and with the vielbein we map it to $\mathcal{M}$.

We will compare the actions of $\overset{\circ}{\nabla}_\mu$ and D_μ on an arbitrary vector field V.

$$\overset{\circ}{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \overset{\circ}{\Gamma}{}^\nu{}_{\mu\alpha} V^\alpha,$$

mapping it to the tangent space, $V^a = e^a{}_\mu V^\mu$, we have,

$$D_\mu V^a = \partial_\mu V^a + \omega^a{}_{b\mu} V^b.$$

D and $\overset{\circ}{\nabla}$ are well defined for tensors in their respective spaces, and for tensors with mixed indices, such as the vielbein, it is necessary to use a total covariant derivative, which we will denote by $\mathcal{D}$, and applied to a mixed tensor we are left with:

$$\mathcal{D}_\mu S^{a\nu} \equiv \partial_\mu S^{a\nu} + \omega^a{}_{b\mu} S^{b\nu} + \overset{\circ}{\Gamma}{}^\nu{}_{\mu\alpha} S^{a\alpha},$$

the total covariant derivative follows the Leibniz rule:

$$\mathcal{D}_\mu V^a = \mathcal{D}_\mu (e^a{}_\nu V^\nu) = (\mathcal{D}_\mu e^a{}_\nu) V^\nu + e^a{}_\nu (\mathcal{D}_\mu V^\nu) \tag{3.9}$$

where the total covariant derivative for the vielbein is

$$\mathcal{D}_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu - \mathring{\Gamma}^\alpha{}_{\mu\nu} e^a{}_\alpha . \tag{3.10}$$

The equation (3.10) is easily deducible from the equation (3.9), it is only necessary to replace $\mathcal{D}_\mu V^a \to D_\mu V^a$ and $\mathcal{D}_\mu V^\nu \to \nabla_\mu V^\nu$ and we get

$$(\mathcal{D}_\mu e^a{}_\nu - \partial_\mu e^a{}_\nu - \omega^a{}_{b\mu} e^b{}_\nu + \mathring{\Gamma}^\alpha{}_{\mu\nu} e^a{}_\alpha) V^\nu = 0.$$

We can see that Christofel's symbol is related to e and ω.

Metricity Postulate: The metric must be invariant under parallel transport transformations

$$\mathring{\nabla}_\lambda g_{\mu\nu} \equiv 0. \tag{3.11}$$

Writing this postulate in the first-order formalism we get:

$$2\eta_{ab} (\mathcal{D}_\lambda e^a{}_\mu) e^b{}_\nu = 0,$$

and assuming that the vielbein has an inverse, we get **the vielbein postulate**:

$$\mathcal{D}_\lambda e^a{}_\mu = 0. \tag{3.12}$$

SO(d,1) invariant tensors

The group $SO(d,1)$ has two invariant tensors, the Minkowski metric η_{ab}, and the fully antisymmetric Levi-Civita tensor $\epsilon_{a_1 a_2 \ldots a_{d+1}}$. These tensors are defined by the algebraic structure of the Lorentz group, therefore they are the same for all tangent spaces, consequently, they are also constant in the variety $\mathcal{M}$, that means $\partial_\mu \eta_{ab} = 0$ and $\partial_\mu \epsilon_{a_1 a_2 \ldots a_{d+1}} = 0$. These tensors must also be covariantly constant:

$$\partial_\mu \eta_{ab} = D_\mu \eta_{ab} = 0,$$

$$\partial_\mu \epsilon_{a_1 a_2 \ldots a_{d+1}} = D_\mu \epsilon_{a_1 a_2 \ldots a_{d+1}} = 0.$$

- As a requirement of metric invariance, the connection must be antisymmetric:

$$D_\mu \eta_{ab} = \cancelto{0}{\partial_\mu \eta_{ab}} - \omega^c{}_{b\mu} \eta_{ca} - \omega^c{}_{a\mu} \eta_{cb} = 0,$$

$$\omega_{ab\mu} + \omega_{ba\mu} = 0 \Rightarrow \omega_{ab\mu} = -\omega_{ba\mu},$$

- On the other hand, for the invariance of the Levi-Civita tensor we have the Bianchi identity:

$$D_\mu \epsilon_{a_1 a_2 \ldots a_{d+1}} = \cancelto{0}{\partial_\mu \epsilon_{a_1 a_2 \ldots a_{d+1}}} - \omega^c{}_{a_1 \mu} \epsilon_{c a_2 \ldots a_{d+1}} - \ldots - \omega^c{}_{a_{d+1} \mu} \epsilon_{a_1 a_2 \ldots c} = 0,$$

$$\omega^c{}_{a_1 \mu} \epsilon_{c a_2 \ldots a_{d+1}} + \ldots + \omega^c{}_{a_{d+1} \mu} \epsilon_{a_1 a_2 \ldots c} = 0.$$

  An example for $d + 1 = 3$:

$$\omega^d{}_{a\mu} \epsilon_{dbc} + \omega^d{}_{b\mu} \epsilon_{adc} + \omega^d{}_{c\mu} \epsilon_{abd} = 0.$$

The connection $\omega^a{}_{b\mu}$ has $n^2(n-1)/2$ independent components, and has fewer components than the Christoffel Symbol, $n^2(n+1)/2$. The Christoffel Symbol is determined by the vielbein and the Lorentz connection, $\mathring{\Gamma} = \mathring{\Gamma}(e, \omega)$. If we join the independent components of the vielbein and the Lorentz connection, we obtain precisely the number of components of the Christoffel symbol.

3.2 Curvature and Torsion

We can write the vielbein and the connection in the form:

$$\begin{aligned} e^a &= e^a{}_\mu dx^\mu, \\ \omega^a{}_b &= \omega^a{}_{b\mu} dx^\mu, \end{aligned} \tag{3.13}$$

are 1-local forms. This implies that all the geometric properties of the manifold $\mathcal{M}$ can be expressed with these 1-forms, e^a, $\omega^a{}_b$, their products and exterior derivatives. These 1-forms do not have Greek spacetime indices, and that means they are invariant to general coordinate transformations (diffeomorphisms) on $\mathcal{M}$.

The exterior derivative operator $d = dx^\mu \partial_\mu$, acts on a p-form, α_p, and transforms it into a (p+1)-form, $d\alpha_p$. One of the fundamental properties of the exterior calculus is that the second exterior derivative of a differential form becomes zero.

$$d(d\alpha_p) = d^2 \alpha_p = \partial_\mu \partial_\nu \alpha_p dx^\mu \wedge dx^\nu = 0. \tag{3.14}$$

The above property is easy to derive, as there is a contraction between commuting indices $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ and anticommuting indices $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$.

3.2.1 Lorentz curvature

A consequence of $d^2\alpha_p = 0$ is that D^2 is not a differential operator but an algebraic operator, here the external covariant derivative is $D = dx^\mu D_\mu$. Consider the second covariant derivative of a vector field,

$$\begin{aligned} D^2V^a &= D(dV^a + \omega^a{}_b \wedge V^b), \\ &= (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge V^b, \\ &= \frac{1}{2}(\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}) dx^\mu \wedge dx^\nu \wedge V^b, \end{aligned} \tag{3.15}$$

this can be easily deduced using the following property: $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta$. The expression in parentheses is an antisymmetric second-order tensor called the Lorentz curvature,

$$\begin{aligned} R^a{}_b &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \\ &= \frac{1}{2}(\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}) dx^\mu \wedge dx^\nu, \\ &= \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu. \end{aligned} \tag{3.16}$$

The connection $\omega^a{}_b(x)$ and the gauge potential in Yang-Mills theory, $A(x) = A_\mu dx^\mu$, are 1-forms with similar properties. Both are spin connections of a gauge group; their transformation laws have the same forms, and the curvature $R^a{}_b$ is completely analogous to the curvature in Yang-Mills theory

$$F = dA + A \wedge A. \tag{3.17}$$

The fields ω and e play different roles in the theory and this is reflected in their different transformation rules for local Lorentz transformations: the vielbein transforms as a vector and not as a connection. In gauge theories vector fields play the role of matter, and connection fields are responsible for interaction.

3.2.2 Torsion

The covariant derivative of the vielbein is a 2-form called torsion,

$$\begin{aligned} T^a = De^a &= de^a + \omega^a{}_b \wedge e^b, \\ &= \frac{1}{2}(D_\mu e^a{}_\nu - D_\nu e^a{}_\mu) dx^\mu \wedge dx^\nu, \\ &= \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{3.18}$$

the torsion depends on e and ω. In contrast, the Lorentz curvature $R^a{}_b$ is not the covariant derivative of some field.

We can notice that the symmetric part of the torsion is arbitrary, therefore we can write it as,

$$T^a = (\frac{1}{2}T^a{}_{\mu\nu} + \frac{1}{2}S^a{}_{\mu\nu})dx^\mu \wedge dx^\nu, \tag{3.19}$$

where $S^a{}_{\mu\nu}$ is an arbitrary three-indexed object and is symmetric in μ and ν. Mapping this object to the manifold $\mathcal{M}$ we have,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}T^\lambda{}_{\mu\nu} + \frac{1}{2}S^\lambda{}_{\mu\nu} = (\frac{1}{2}T^a{}_{\mu\nu} + \frac{1}{2}S^a{}_{\mu\nu})e_a{}^\lambda. \tag{3.20}$$

$\Gamma^\lambda{}_{\mu\nu}$ is a connection with torsion, we have the torsion tensor $T^\lambda{}_{\mu\nu}$, and the tensor $S^\lambda{}_{\mu\nu}$ is arbitrary and does not depend on torsion. To fix a $S^\lambda{}_{\mu\nu}$, we will use the metricity postulate

$$\begin{aligned} \mathring{\nabla}_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \mathring{\Gamma}^\alpha{}_{\mu\nu} g_{\alpha\rho} - \mathring{\Gamma}^\alpha{}_{\mu\rho} g_{\alpha\nu} = 0, \\ \mathring{\Gamma}^\alpha{}_{\mu\nu} &= \frac{1}{2}g^{\alpha\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = \frac{1}{2}S^\alpha{}_{\mu\nu}. \end{aligned} \tag{3.21}$$

3.2.3 Riemann curvature and Lorentz curvature

We are now interested in finding the relationship between the Riemann curvature and the Lorentz curvature. We will start by projecting the Riemann curvature, $\widetilde{R}^\alpha{}_{\beta\mu\nu}$, to the tangent space

$$\widetilde{R}^a{}_b = \frac{1}{2}e_\alpha{}^a e^\beta{}_b \widetilde{R}^\alpha{}_{\beta\mu\nu} dx^\mu \wedge dx^\nu. \tag{3.22}$$

The expression above is similar to the Lorentz curvature (3.16). We will prove that these objects will coincide only when an algebraic property is satisfied, zero torsion.

Let us consider the torsion

$$T^a = de^a + \omega^a{}_b \wedge e^b.$$

The equation to nullify the torsion is given by

$$de^a + \widetilde{\omega}^a{}_b \wedge e^b = 0, \tag{3.23}$$

where $\widetilde{\omega}^a{}_b$ is a solution of this equation. The torsion-free connection $\widetilde{\omega}^a{}_b(e, \partial e)$ is completely determined by the metric structure of the manifold $\mathcal{M}$ and is expressed as:

$$\mathcal{D}e^a = 0 \rightarrow \widetilde{\omega}^a{}_{b\mu} = -e_a{}^\nu(\partial_\mu e^a{}_\nu - \mathring{\Gamma}^\lambda{}_{\mu\nu} e^a{}_\lambda).$$

The difference between the Lorentz connection and the torsion-free connection is a 1-form known as the contorsion tensor.

$$\kappa^a{}_b = \omega^a{}_b - \widetilde{\omega}^a{}_b \tag{3.24}$$

and is related to the torsion as

$$T^a = \kappa^a{}_b \wedge e^b. \tag{3.25}$$

Then, the Lorentz curvature is

$$\begin{aligned} R^a{}_b &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \\ &= d(\widetilde{\omega}^a{}_b + \kappa^a{}_b) + (\widetilde{\omega}^a{}_c + \kappa^a{}_c) \wedge (\widetilde{\omega}^c{}_b + \kappa^c{}_b), \\ &= (d\widetilde{\omega}^a{}_b + \widetilde{\omega}^a{}_c \wedge \widetilde{\omega}^c{}_b) + (d\kappa^a{}_b + \widetilde{\omega}^a{}_c \wedge \kappa^c{}_b - \widetilde{\omega}^c{}_b \wedge \kappa^a{}_c) + \kappa^a{}_c \wedge \kappa^c{}_b, \\ R^a{}_b &= \widetilde{R}^a{}_b + \widetilde{D}\kappa^a{}_b + \kappa^a{}_c \wedge \kappa^c{}_b, \end{aligned} \tag{3.26}$$

we can conclude by saying that the Lorentz curvature and the Riemann curvature coincide when the torsion is zero. We should note something else, if $R^a{}_b$ becomes zero it does not imply that $\widetilde{R}^a{}_b$ is zero and vice versa.

3.3 Building blocks for a gravitational theory

Let's start with the identity of Bianchi,

$$DR^a{}_b = dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0, \tag{3.27}$$

$$DT^a = R^a{}_b \wedge e^b. \tag{3.28}$$

We must remember that this is an identity and not a set of equations; will be satisfied by any well-defined connection, and in no way constrains the form of the field $\omega^a{}_{b\mu}$. As a consequence of this identity, taking successive covariant derivatives of e, ω, R, T, no new independent tensors appear.

Then, the basic building blocks of an action for gravitation in the first-order formalism are

$$e^a,\ \omega^a{}_b,\ R^a{}_b,\ T^a,\ \eta_{ab},\ \epsilon_{a_1,\ldots,a_{d+1}}$$

If we want to describe a (d+1)-dimensional spacetime, the Lagrangean must be a (d+1)-form. With these objects and their exterior products we can construct a limited number of actions, in each dimension, that are locally Lorentz invariant.

3.3.1 Some Lorentz invariant actions

Under a local infinitesimal Lorentz transformation with parameter $\lambda^a{}_b$ the dynamic fields are transformed as ,

$$\begin{aligned} e'^a &= e^a + \lambda^a{}_b e^b \\ \omega'^a{}_b &= \omega^a{}_b - D\lambda^a{}_b \end{aligned} \tag{3.29}$$

We are going to show some examples of actions invariant to (3.29):

In d+1=3: The Einstein-Hilbert action with cosmological constant

$$S[e,\omega] = \int \epsilon_{abc}(aR^{ab} \wedge e^c + be^a \wedge e^b \wedge e^c)$$

In d+1=4: The Einstein-Hilbert action with cosmological constant

$$S[e,\omega] = \int \epsilon_{abcd}(aR^{ab} \wedge e^c \wedge e^d + be^a \wedge e^b \wedge e^c \wedge e^d)$$

In d+1=5: The Gauss–Bonnett term

$$S[e,\omega] = \int \epsilon_{abcde}(aR^{ab} \wedge R^{cd} \wedge e^e)$$

In 4 and 3 dimensions the terms $be^a \wedge e^b$ and $be^a \wedge e^b \wedge e^c \wedge e^d$ respectively are related to the cosmological constant. The Gauss-Bonnett is a topological term.

Chapter 4

Generalized Lagrangeans for a (d+1)-dimensional gravitation

General gravitational theories are explored in this chapter. We start with Lovelock's theory that generalizes the theory of General Relativity. By including torsion we have the Lovelock-Cartan theory that generalizes Lovelock's theory. On the other hand, we have Horndeski's theory that generalizes Lovelock's theory including a scalar field. Finally, By including torsion in the Horndeski theory we can obtain the most general theory of gravitation called the Lovelock-Cartan-Horndeski theory, see (Fig. 4.1). For this chapter we consulted [16], [18], [14], [10] and [1].

4.1 Is Torsion necessary?

At a macroscopic scale, space-time appears to be (3+1)-dimensional and without torsion. There are good theoretical reasons to think that there are more than four dimensions. However, there appear to be no experimental or theoretical reasons to believe that space-time possesses dynamic degrees of freedom for torsion. All the experimental proofs made to General Relativity seem to require no more than a metric structure.

In General Relativity the matter and energy of space-time are sources of curvature. In the same way, the intrinsic angular momentum of matter is a source of torsion. In some extended gravitational theories, torsion sometimes appears as a combination of other fields and not necessarily as an independent geometric feature of the manifold. Curvature, on the other hand, cannot be expressed as a local function of matter fields and does not require the presence of anything other than the metric. However, curvature and torsion are characteristics of the same status, and there seems to be no principle to rule out their presence.

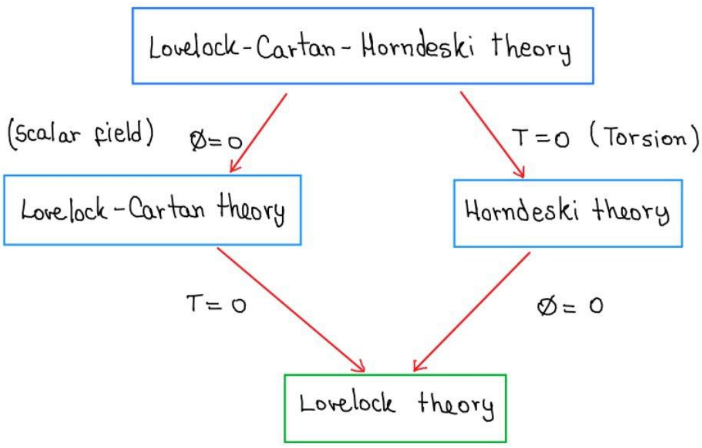

Figure 4.1: The figure shows a map of general gravitation theories. The most general theory of gravitation is the Lovelock-Cartan-Horndeski theory which includes torsion and scalar fields. By making the torsion zero in the Lovelock-Cartan-Horndeski theory, the Horndeski theory is obtained, on the other hand, if we make zero the scalar field we go to the Lovelock-Cartan theory which is a gravitational theory that includes torsion. From the Lovelock-Cartan theory, by making the torque zero, the Lovelock theory is obtained. If in Horndeski's theory, we nullify the scalar field we obtain Lovelock's theory.

4.1.1 Curvature and torsion again

The 2-forms, torsion and curvature respectively are

$$T^a = De^a,$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b,$$

and satisfy the Bianchi identities

$$DR^a{}_b = 0,$$

$$DT^a = R^a{}_b \wedge e^b.$$

The analogy between R and T can be manifested if the vielbein and the Lorentz connection are combined in a 1-form connection for a (d+2)-dimensional Lorentz group. So (d+1)-dimensional curvature and torsion are just two pieces of a (d+2)-dimensional curvature.

The curvature appears as a necessary characteristic, whereas the torsion may or may not be present. So, from the point of view of the local symmetric structure of space-time, it is arbitrary to set the torsion equal to zero and not the curvature. If torsion is admitted as a legitimate attribute of space-time, it is necessary to know how to include this term in the action. In this chapter, we will show the construction of a general gravitational theory that includes torsion.

4.2 Lovelock theory

The most general action for gravitation, which is Lorentz invariant, which does not contain torsion and which gives us second-order equations of motion for the metric, has the form

$$S_L = \int \sum_{p=0}^{D/2} a_p L_L^{(D,p)}, \tag{4.1}$$

and was proposed by Lovelock [16]. Where $D = d + 1$, a_p are arbitrary constants, for an odd D we use $\sum_{p=0}^{d/2}$.

The Lovelock Lagrangean, $L_L^{(D,p)}$, is given by

$$L_L^{(D,p)} = \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \ldots \wedge e^{a_D} \tag{4.2}$$

The Lovelock action is a function of the vielbein and the Lorentz connection, since we assume that the torsion is zero, we have $\omega^a{}_b = \omega^a{}_b(e)$. If we vary the action of Lovelock with respect to the vielbein we obtain the Einstein-Lovelock equations,

$$\frac{\delta S_L}{\delta e^{a_D}} = \sum_{p=0}^{D/2} a_p(D-2p)\epsilon_{a_1...a_D} R^{a_1a_2} \wedge ... \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge ... \wedge e^{a_d} = 0, \tag{4.3}$$

and by varying the action with respect to the connection we obtain a term proportional to the torsion. Since we assume that the torsion is zero this term does not contribute to the equations of motion

$$\frac{\delta S_L}{\delta \omega^{a_1a_2}} = \sum_{p=0}^{D/2} a_p(D-2p)p\epsilon_{a_1...a_D} R^{a_3a_4} \wedge ... \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge ... \wedge e^{a_d} \wedge T^{a_D} = 0. \tag{4.4}$$

It is somewhat awkward to impose $T^a = 0$, as it seems like an ace in the hole and it would be nice if the equation $T^a = 0$ was derived from the action, and indeed for dimensions $D \leq 4$ the torsion is canceled identically without the need to impose anything. In these cases, the action can vary with respect to e or ω independently or not, and the result will be the same.

In high dimensions $D > 4$ the equations of motion resulting from the variation of the action with respect to ω do not imply that the torsion is zero, in these cases, it must be assumed that the torsion is zero, this will bring additional restrictions in the connection. It is not yet clear how to impose restrictions naturally. Furthermore, for high dimensions $D > 4$ the equations of motion resulting from the variation of ω cannot be solved algebraically for ω.

Some examples of Lovelock actions and their field equations in different dimensions will be shown:

In D=2: The action reduces to a linear combination of the Euler characteristic, χ_2 and the volume of space-time

$$\begin{aligned} S_L^2 &= \int \sum_{p=0}^{1} a_p L_L^{(2,p)} \\ &= \int (a_0 L_L^{(2,0)} + a_1 L_L^{(2,1)}) \\ &= \int (a_0 \epsilon_{ab} e^a \wedge e^b + a_1 \epsilon_{ab} R^{ab}) \end{aligned} \tag{4.5}$$

rewriting this equation to the metric formalism we have

$$\begin{aligned} S_L^2 &= a_1 \int \sqrt{|g|} R d^2x + 2a_0 \int \sqrt{|g|} d^2x \\ &= a_1 \chi_2 + 2a_0 V_2 \end{aligned} \tag{4.6}$$

We vary the action and obtain

$$\delta S_L^2 = \int \epsilon_{ab}(a_1 \delta R^{ab} + 2a_0 \delta e^a \wedge e^b)$$
$$= \int [a_1 D(\epsilon_{ab}\delta\omega^{ab}) + 2a_0 \epsilon_{ab}\delta e^a \wedge e^b] = 0. \tag{4.7}$$

The first term of (4.7) is an integral over the manifold boundary, leaving us with no equations for ω. This reflects the fact that the Euler characteristic, χ_2, is a topological invariant and does not change under continuous deformations such as those done with variation. On the other hand, demanding that the action be stationary under variations of e we obtain that the vielbein is zero.

In D=3: The action reduces to the Einstein-Hilbert action with cosmological constant

$$\begin{aligned} S_L^3 &= \int \sum_{p=0}^{1} a_p L_L^{(3,p)} \\ &= \int (a_0 L_L^{(3,0)} + a_1 L_L^{(3,1)}) \\ &= \int (a_0 \epsilon_{abc} e^a \wedge e^b \wedge e^c + a_1 \epsilon_{abc} R^{ab} \wedge e^c), \end{aligned} \tag{4.8}$$

rewriting it to the metric formalism we have

$$S_L^3 = a_1 \int \sqrt{|g|} R d^3x + 6a_0 \int \sqrt{|g|} d^3x. \tag{4.9}$$

Varying the action we obtain the field equations:

$$\begin{aligned} \delta S_L^3 &= \int (3a_0 \epsilon_{abc} \delta e^a \wedge e^b \wedge e^c + a_1 \epsilon_{abc} \delta(R^{ab} \wedge e^c)), \\ &= \int \{a_1 \epsilon_{abc}[D(\delta\omega^{ab}) - \delta\omega^{ab} \wedge T^c] + \epsilon_{abc}(a_1 R^{ab} + 3a_0 e^a \wedge e^b) \wedge \delta e^c\} = 0, \end{aligned} \tag{4.10}$$

for the variation of ω we have

$$\epsilon_{abc} T^c = 0,$$

and for the variation of e we have

$$\epsilon_{abc}(a_1 R^{ab} + 3a_0 e^a \wedge e^b) = 0.$$

These field equations describe a torsion-free geometry with a negative curvature constant, a (2+1)-dimensional space AdS. We should note that despite not including torsion in the action, it appears in the field equations and is identically canceled without us asking for it. Here the affine structure and the metric are considered dynamically independent and vary independently.

In D=4: The Lovelock action contains the Einstein-Hilbert term, the cosmological constant, and the Euler form χ_4.

$$\begin{aligned} S_L^4 &= \int \sum_{p=0}^{2} a_p L_L^{(4,p)} \\ &= \int (a_0 L_L^{(4,0)} + a_1 L_L^{(4,1)} + a_2 L_L^{(4,2)}) \\ &= \int (a_0 \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + a_1 \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d + a_2 \epsilon_{abcd} R^{ab} \wedge R^{cd}), \end{aligned} \tag{4.11}$$

rewriting the action to the metric formalism we have

$$\begin{aligned} S_L^4 &= a_2 \int \sqrt{|g|}(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 4R^{\alpha\beta} R_{\alpha\beta} + R^2) d^4x + 2a_1 \int \sqrt{|g|} R d^4x + 24a_0 \int \sqrt{|g|} d^4x, \\ &= a_2 \chi_4 + 2a_1 \int \sqrt{|g|} R d^4x + 24 a_0 V_4. \end{aligned} \tag{4.12}$$

Varying the action we get

$$\begin{aligned} \delta S_L^4 &= \int [4a_0 \epsilon_{abcd} \delta e^a \wedge e^b \wedge e^c \wedge e^d + a_1 \epsilon_{abcd} \delta(R^{ab} \wedge e^c \wedge e^d) + a_2 \epsilon_{abcd} \delta(R^{ab} \wedge R^{cd})], \\ &= \int \epsilon_{abcd} [2a_2 D(R^{ab} \wedge \delta\omega^{cd}) + a_1 D(\delta\omega^{ab} \wedge e^c \wedge e^d) - 2a_1 \delta\omega^{ab} \wedge T^c \wedge e^d + \\ &+ 2(a_1 R^{ab} \wedge e^c + 2a_0 e^a \wedge e^b \wedge e^c) \wedge \delta e^d] = 0, \end{aligned} \tag{4.13}$$

the field equations are:

$$\epsilon_{abcd} T^c \wedge e^d = 0,$$
$$\epsilon_{abcd}(a_1 R^{ab} \wedge e^c + 2a_0 e^a \wedge e^b \wedge e^c) = 0.$$

The first of the equations tells us that the theory is free of torsion; the second equation is the Einstein-Hilbert equation with cosmological constant. Since the torsion is zero, we have that the Lorentz curvature is equal to the Riemann curvature $\widetilde{R}^{ab} = R^{ab}$.

4.3 Lovelock-Cartan theory

Lovelock's Lagrangian was found thanks to the search for a generalization of Einstein's theory that gives us second-order field equations for the metric as a result. To obtain the Lovelock Lagrangean, L_L^D, we must consider two conditions:

- a) The Lagrangian L_L^D must be invariant under local Lorentz transformations, and constructed with e^a, $\omega^a{}_b$, their exterior derivatives and products between them, and without using the Hodge operator $\star$.
- b) The connection is, by definition, torsion-free (Spacetime is a Riemannian manifold).

a) implies that the field equations must be tensorial and second order in the metric: The use of exterior derivatives restricts the equations of motion to be first-order differential equations for e and ω (since $d^2 = 0$), Lorentz invariance guarantees that the field equations are tensor and also only contain metric, curvature and torsion. Excluding the Hodge operator $\star$ we obtain the following restrictions: Using $d \wedge \star d \neq 0$, this operator makes it possible for higher order derivatives of the fields to exist, and in that case, the Lagrangian does not only depend on the metric and its first derivative, but on higher order derivatives. If we want to consider a more general theory than Lovelock's, we must ignore condition b) and include torsion.

A general construction was worked on [17]. The main difference with the free torsion case is that the Levi-Civita tensor $\epsilon_{a_1 \ldots a_D}$ is not the only term used for index contraction since we will now use η_{ab}.

Using only the tensors e^a, $\omega^a{}_b$, $R^a{}_b$, T^a and products between them through $\epsilon_{a_1 \ldots a_D}$ and η_{ab} we can form the following Lorentz invariants:

$$\begin{aligned} L_L^{(D,p)} &= \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{2p+1} \wedge \ldots \wedge e^D, \\ L_R^{2n} &= R^{a_1}{}_{a_2} \wedge \ldots \wedge R^{a_{2n}}{}_{a_1}, \\ L_E^{2n} &= e_{a_1} \wedge R^{a_1}{}_{a_2} \wedge \ldots \wedge R^{a_{2n}}{}_b \wedge e^b, \\ L_B^{2m+2} &= T_{a_1} \wedge R^{a_1}{}_{a_2} \wedge \ldots \wedge R^{a_{2n}}{}_b \wedge T^b, \\ L_K^{2l+1} &= T_{a_1} \wedge R^{a_1}{}_{a_2} \wedge \ldots \wedge R^{a_{2n}}{}_b \wedge e^b. \end{aligned} \tag{4.14}$$

where $n \geq 2$ are even, $m \geq 1$ are odd, and $l \geq 1$ is any number. Any other invariant can be written as a linear combination of the products of these basic invariants. The Lovelock-Cartan Lagrangian must be

$$L_{LC}^D = L_L^D + L_T^D \tag{4.15}$$

where L_L^D is the Lovelock Lagrangian:

$$L_L^D = \sum_{p=0}^{D/2} a_p L_L^{(D,p)}, \tag{4.16}$$

and L_T^D is the Lagrangian containing terms with torsion:

$$L_T^D = \sum b_p L_p^{(D,p)}, \tag{4.17}$$

$N(D)$	0	0	1	3	1	0	4	10	4	1	13	27	13
D	1	2	3	4	5	6	7	8	9	10	11	12	13
$N(D)$	5	36	69	36	18	91	161	92	53	213	361	140	217
D	14	15	16	17	18	19	20	21	22	23	24	25	26

Table 4.1: This table shows the number of terms with torsion $N(D)$ for each dimension D.

$$L_p^{(D,p)} = L_R^{2n_1} \wedge ... \wedge L_R^{2n_r} \wedge L_E^{2n_1} \wedge ... \wedge L_E^{2n_s} \wedge ... \wedge L_B^{2m_1+2} \wedge ... \wedge L_B^{2m_v+2} \wedge L_K^{2l_1+1} \wedge ... \wedge L_K^{2l_t+1}, \tag{4.18}$$

the indices $L_p^{(D,p)}$ are given such that that Lagrangean is a D-form and r, s , l and t are any positive number including zero and $p = (2n_1, ..., 2n_r, 2n_1, ..., 2n_s, 2m_1 + 2, ..., 2m_v + 2, 2l_1 + 1, ..., 2l_t + 1)$, and b_p are arbitrary constants.

We can note that $L_p^{(D,p)}$ is zero if the torsion is zero, except in the particular case of $s = v = t = 0$, which corresponds to the Pontryagin density,

$$P_p = L_R^{2n_1} \wedge ... \wedge L_R^{2n_r}, \tag{4.19}$$

where $\sum_{i=1}^{r} 2n_i = \frac{D}{2}$. For a general case it must be satisfied that

$$2[\sum_{i=1}^{r} 2n_i + \sum_{i=1}^{s} 2n_i + \sum_{i=1}^{t}(2l_i + 1)] + 4s + 2v + 3t = D,$$

when we are in an odd dimension we use d instead of D. Any solution of this equation, subject to the above conditions, should contribute in the Lagrangian. The restrictions on the coefficients make it extremely difficult to even count the number of torsion terms, $N(D)$. Using a table 4.1, extracted from [17], we can visualize the number of torsion terms in the Lagrangian.

We show explicitly the Lovelock-Cartan Lagrangians in different dimensions.

In D=2: In this dimension we only have the Lagrangian of Lovelock since the terms that contain torsion are D-forms with D>2,

$$L_{LC}^2 = L_L^2.$$

In D=3: In this dimension, in addition to Lovelock's Lagrangian, we have 1 term with torsion to add, we use l=1:

$$L_T^3 = b_0 L_K^3 = b_0 T_a \wedge e^a,$$

the Lovelock-Cartan Lagrangian is

$$L_{LC}^3 = L_L^3 + L_T^3 \tag{4.20}$$

varying the Lagrangian (and remembering that $\delta T^a = D(\delta e^a) + \delta\omega^{ab} \wedge e_b$):

$$\begin{aligned}\delta L^3_{LC} &= \delta L^3_L + b_0 \delta L^3_K,\\ &= \delta L^3_L + b_0(\delta T^a \wedge e_a + T^a \wedge \delta e_a),\\ &= a_1\epsilon_{abc}[D(\delta\omega^{ab}) - \delta\omega^{ab} \wedge T^c] + \epsilon_{abc}(a_1 R^{ab} + 3a_0 e^a \wedge e^b) \wedge \delta e^c\\ &+ b_0[D(\delta e_a \wedge e^a) - \delta\omega^{ab} \wedge e_a \wedge e_b],\end{aligned} \tag{4.21}$$

we get its equations of motion:

$$a_1\epsilon_{abc}T^c + b_0 e_a \wedge e_b = 0,$$

$$\epsilon_{abc}(a_1 R^{ab} + 3a_0 e^a \wedge e^b) = 0.$$

In D=4: In this dimension we have the Lovelock Lagrangian and two torsion terms and a topological invariant:

$$L^4_R = R^a{}_b \wedge R^b{}_a,$$

$$L^4_E = e_a \wedge R^a{}_b \wedge e^b = e_a \wedge DT^a,$$

$$L^4_B = T_a \wedge T^a,$$

with $n = 2$ and $m = 1$. The Lagrangian with torsion is

$$L^4_T = b_0 L^4_B + b_1 L^4_E + b_2 L^4_R,$$

and Lovelock-Cartan Lagrangian is

$$L^4_{LC} = L^4_L + L^4_T. \tag{4.22}$$

We vary the Lagrangian and obtain:

$$\begin{aligned}\delta L^4_{LC} &= \delta L^4_L + \delta L^4_T,\\ &= \delta L^4_L + b_2\delta(R^a{}_b \wedge R^b{}_a) + b_1\delta(e_a \wedge R^a{}_b \wedge e^b) + b_0\delta(T_a \wedge T^a),\\ &= \delta L^4_L + 2b_2 D(\delta\omega^a{}_b \wedge R^b{}_a) + b_1[2\delta e_a \wedge R^a{}_b \wedge e^b + 2\delta\omega^{ab} \wedge T_a \wedge e_b\\ &+ D(\delta\omega^{ab} \wedge e_a \wedge e_b)] + 2b_0[D(\delta e^a \wedge T_a) + \delta e^a \wedge DT_a + \delta\omega^{ab} \wedge e_b \wedge T_a],\\ &= \epsilon_{abcd}[2a_2 D(R^{ab} \wedge \delta\omega^{cd}) + a_1 D(\delta\omega^{ab} \wedge e^c \wedge e^d) - 2a_1\delta\omega^{ab} \wedge T^c \wedge e^d\\ &+ 2(a_1 R^{ab} \wedge e^c + 2a_0 e^a \wedge e^b \wedge e^c) \wedge \delta e^d] + 2b_2 D(\delta\omega^a{}_b \wedge R^b{}_a)\\ &+ b_1[2\delta e_a \wedge R^a{}_b \wedge e^b + 2\delta\omega^{ab} \wedge T_a \wedge e_b + D(\delta\omega^{ab} \wedge e_a \wedge e_b)]\\ &+ 2b_0[D(\delta e^a \wedge T_a) + \delta e^a \wedge DT_a + \delta\omega^{ab} \wedge e_b \wedge T_a].\end{aligned} \tag{4.23}$$

The field equations are:

$$[a_1\epsilon_{abcd} + (b_0 + b_1)\eta_{ca}\eta_{db}]T^c \wedge e^d = 0,$$

$$\epsilon_{abcd}(a_1 R^{ab} \wedge e^c + a_0 e^a \wedge e^b \wedge e^c) - (b_0 + b_1) D T_d = 0.$$

We can form an invariant by placing $b_0 = 1$ and $b_1 = -1$, known as the Nieh-Yan form:

$$N_4 = T^a \wedge T_a - e_a \wedge R^a{}_b \wedge e^b,$$

note that by including this term we recover the Lovelock-Einstein equations of motion. From the field equations, we see that this theory is free of torsion.

For higher dimensions only their Lagrangians are shown. For the construction of the Lovelock action, we use a useful recipe that simplifies our work.

The recipe: A recipe to obtain Lovelock Lagrangians for $D = 2n$ $(n = 1, 2, ...)$ is the following:

- 1) We start with the D-dimensional Euler density:

$$L_E^D = \epsilon_{a_1...a_D} R^{a_1 a_2} \wedge ... \wedge R^{a_{D-1} a_D}$$

- 2) We replace each curvature with a concircular curvature:

$$R^{ab} \longrightarrow R^{ab} - \beta e^a \wedge e^b$$

  we use a different constant β on each substitution. And this will bring us to Lovelock's Lagrangean:

$$L_L^D = \beta_0 \epsilon_{a_1...a_D} (R^{a_1 a_2} - \beta_1 e^{a_1} \wedge e^{a_2}) \wedge ... \wedge (R^{a_{D-1} a_D} - \beta_n e^{a_{D-1}} \wedge e^{a_D}) \tag{4.24}$$

- 3) If it is required to generate a Lagrangian in odd dimension, we only have to modify the Euler density by increasing one dimension:

$$L_E^D = \epsilon_{a_1...a_{D+1}} R^{a_1 a_2} \wedge ... \wedge R^{a_{D-1} a_D} \longrightarrow L_E^{D+1} = \epsilon_{a_1...a_D} R^{a_1 a_2} \wedge ... \wedge R^{a_{D-1} a_D} \wedge e^{a_{D+1}}$$

  and then we just continue with step 1) and 2). The coefficients β_i in (4.24) are roots of (4.16).

In D=5: In this dimension we have the Lovelock Lagrangian and a torsion term. We use the Recipe:

We start from the Euler density for 4 dimensions and increase one dimension

$$L_E^4 = \epsilon_{abcd} R^{ab} \wedge R^{cd} \longrightarrow L_E^5 = \epsilon_{abcde} R^{ab} \wedge R^{cd} \wedge e^e.$$

Substitute $R^{ab} \longrightarrow R^{ab} - \beta e^a \wedge e^b$ and obtain the 5-dimensional Lovelock Lagrangian

$$L_L^5 = \beta_0 \epsilon_{abcde} (R^{ab} - \beta_1 e^a \wedge e^b) \wedge (R^{cd} - \beta_2 e^c \wedge e^d) \wedge e^e. \tag{4.25}$$

On the other hand, the Lagrangian containing torsion is

$$L_T^5 = b_0 L_K^5 = b_0 T_a \wedge R^a{}_b \wedge e^b, \tag{4.26}$$

and finally we can build the general Lagrangian for 5 dimensions

$$L_{LC}^5 = L_L^5 + L_T^5.$$

In D=6: In this dimension we only have Lovelock's Lagrangian, there are no terms with torsion.
We use the recipe:

We start from the Euler density for 6 dimensions

$$L_E^6 = \epsilon_{abcdef} R^{ab} \wedge R^{cd} \wedge R^{ef},$$

substitute in $R^{ab} \longrightarrow R^{ab} - \beta e^a \wedge e^b$ and get the 6-dimensional Lovelock Lagrangian

$$L_L^6 = \beta_0 \epsilon_{abcdef} (R^{ab} - \beta_1 e^a \wedge e^b) \wedge (R^{cd} - \beta_2 e^c \wedge e^d) \wedge (R^{ef} - \beta_3 e^e \wedge e^f). \tag{4.27}$$

In D=7: In this dimension we have the Lovelock Lagrangian and four terms with torsion.
We use the Recipe:

We start from the Euler density for 6 dimensions and increase one dimension

$$L_E^6 = \epsilon_{abcdef} R^{ab} \wedge R^{cd} \wedge R^{ef} \longrightarrow L_E^7 = \epsilon_{abcdefg} R^{ab} \wedge R^{cd} \wedge R^{ef} \wedge e^g.$$

Substitute $R^{ab} \longrightarrow R^{ab} - \beta e^a \wedge e^b$ and obtain the 5-dimensional Lovelock Lagrangian

$$L_L^7 = \beta_0 \epsilon_{abcdefg} (R^{ab} - \beta_1 e^a \wedge e^b) \wedge (R^{cd} - \beta_2 e^c \wedge e^d) \wedge (R^{ef} - \beta_3 e^e \wedge e^f) \wedge e^g. \tag{4.28}$$

On the other hand, the terms with torsion are

$$\begin{aligned} L_K^7 &= T_a \wedge R^a{}_b \wedge R^b{}_c \wedge e^c, \\ L_K^3 \wedge L_E^4 &= T_a \wedge e^a \wedge e_b \wedge R^b{}_c \wedge e^c, \\ L_K^3 \wedge L_B^4 &= T_a \wedge e^a \wedge T_b \wedge T^b, \\ L_K^3 \wedge L_R^4 &= T_a \wedge e^a \wedge R^b{}_c \wedge R^c{}_b, \end{aligned} \tag{4.29}$$

and the Lagrangian with torsion will be

$$L_T^7 = b_0 L_K^7 + b_1 L_K^3 \wedge L_E^4 + b_2 L_K^3 \wedge L_B^4 + b_3 L_K^3 \wedge L_R^4. \tag{4.30}$$

Finally, we can construct the general Lagrangian for 7 dimensions

$$L^7_{LC} = L^7_L + L^7_T.$$

In D=8: In this dimension we have the Lovelock Lagrangian and ten torsion terms. We use the recipe:

We start from the Euler density for 8 dimensions

$$L^8_E = \epsilon_{abcdefgh} R^{ab} \wedge R^{cd} \wedge R^{ef} \wedge R^{gh},$$

substitute in $R^{ab} \longrightarrow R^{ab} - \beta e^a \wedge e^b$ and get the 8-dimensional Lovelock Lagrangian

$$L^6_L = \beta_0 \epsilon_{abcdef} (R^{ab} - \beta_1 e^a \wedge e^b) \wedge (R^{cd} - \beta_2 e^c \wedge e^d) \wedge (R^{ef} - \beta_3 e^e \wedge e^f) \wedge (R^{gh} - \beta_4 e^g \wedge e^h). \quad (4.31)$$

the terms with torsion are

$$\begin{aligned} L^8_E &= e_a \wedge R^a{}_b \wedge R^b{}_c \wedge R^c{}_d \wedge e^d, \\ L^8_B &= T_a \wedge R^a{}_b \wedge R^b{}_c \wedge T^c, \\ L^8_R &= R^a{}_b \wedge R^b{}_c \wedge R^c{}_d \wedge R^d{}_a, \\ L^4_E \wedge L^4_E &= e_a \wedge R^a{}_b \wedge e^b \wedge e_c \wedge R^c{}_d \wedge e^d, \\ L^4_E \wedge L^4_B &= e_a \wedge R^a{}_b \wedge e^b \wedge T_c \wedge T^c, \\ L^4_B \wedge L^4_B &= T_a \wedge T^a \wedge T_b \wedge T^b, \\ L^3_K \wedge L^5_K &= T_a \wedge e^a \wedge T_b \wedge R^b{}_c \wedge e^c, \\ L^4_R \wedge L^4_E &= R^a{}_b \wedge R^b{}_a \wedge e_c \wedge R^c{}_d \wedge e^d, \\ L^4_R \wedge L^4_B &= R^a{}_b \wedge R^b{}_a \wedge T_c \wedge T^c, \\ L^4_R \wedge L^4_R &= R^a{}_b \wedge R^b{}_a \wedge R^c{}_d \wedge R^d{}_c, \end{aligned} \quad (4.32)$$

and the Lagrangian with torsion is

$$\begin{aligned} L^8_T = &b_0 L^8_E + b_1 L^8_B + b_2 L^8_R + b_3 L^4_E \wedge L^4_E + b_4 L^4_E \wedge L^4_B + b_5 L^4_B \wedge L^4_B \\ &+ b_6 L^3_K \wedge L^5_K + b_7 L^4_R \wedge L^4_E + b_8 L^4_R \wedge L^4_B + b_9 L^4_R \wedge L^4_R. \end{aligned} \quad (4.33)$$

Finally, we can construct the general Lagrangian for 8 dimensions

$$L^8_{LC} = L^8_L + L^8_T.$$

Chapter 5

Lovelock-Cartan-Horndeski theory

The theory of General Relativity with cosmological constant is the most general metric theory in four dimensions. For this reason, modified 4-dimensional gravity necessarily involves some additional field, which means an increase in the degrees of freedom of the theory. The simplest case that illustrates this modification is the introduction of a scalar field.

In the 1920s, one of the first works to introduce a scalar field to gravitational theory is that of T. Kaluza [11] and O. Klein [13] in an attempt to unify gravity with electromagnetism, they built a theory with an extra dimension and a scalar field. Motivated by the appearance of the scalar field and its possible role as a generalized gravitational constant, P. Jordan in 1949 was the first to formally introduce stress-scalar theories, replacing Newton's constant with a time-dependent scalar field. Then, in 1961, Jordan's work was continued by Robert Dicke and Carl Brans[4]. His work aimed to find a gravitational theory where the metric properties of space-time are completely determined by the masses of the bodies, and this corresponds to a theory that respects Mach's principle. The Jordan-Brans-Dicke theory became the alternative theory to Einstein's General Relativity for about forty years [9].

In 1974, Gregory Horndeski [10] developed the most general theory in four dimensions with a non-minimally coupled scalar field, having second-order field equations in the derivatives of the fundamental fields. Besides, in 2009 the theory of Galileones [19] was developed, which corresponds to the most general theory in four dimensions, for a scalar field in flat space, which has field equations that are polynomial in the derivatives of second-order field and does not contain derivatives of lesser or greater order than two. The same year this theory is generalized to a curved space [7, 8], calling generalized Covariant Galileons. In 2012, it was shown that the theory of generalized Galileons in four dimensions is equivalent to the Horndeski theory [15], this work promoted the revival of the study of Horndeski's theory.

5.0.1 Horndeski theory

G. Horndeski develops a uniqueness theorem similar to Lovelock's theorem. The most general second-order field equations can be obtained from the Lagrangian which has the form of

$$L_H = L(g_{\mu\nu}, \partial_{i_1} g_{\mu\nu}, ..., \partial_{i_p}...\partial_{i_1} g_{\mu\nu}, \phi, \partial_{i_1}\phi, ..., \partial_{i_q}...\partial_{i_1}\phi),$$

where $p, q \geq 2$ in a space 4-dimensional. The Lagrangian depends on the metric tensor, its derivatives, a scalar field and its derivatives, and is built on a Lorentzian manifold endowed with a Levi Civita connection, given by the Christoffel symbols. The equations of motion associated with L_H are $E^{\mu\nu}$ and E. In general $E^{\mu\nu}$ is of $2p$th order in the derivatives of $g_{\mu\nu}$ and $(p+q)$th order in the derivatives of ϕ; whereas E is of $2q$th order in the derivatives of ϕ and $(p+q)$th order in derivatives of $g_{\mu\nu}$.

The Horndeski Lagrangean is

$$\begin{aligned}
L_H =& \delta^{\alpha\beta\gamma}_{\mu\nu\sigma} \Bigg[K_1(\phi, X)\nabla^\mu\nabla_\alpha\phi \widetilde{R}_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\frac{\partial K_1}{\partial X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi + \\
& K_3(\phi, X)\nabla_\alpha\phi\nabla^\mu\phi\widetilde{R}_{\beta\gamma}{}^{\nu\sigma} + 2\frac{\partial K_3}{\partial X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\Bigg] \\
& + \delta^{\alpha\beta}_{\mu\nu}\left[(F(\phi, X) + 2W(\phi))\widetilde{R}_{\alpha\beta}{}^{\mu\nu} + 2\frac{\partial F}{\partial X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi + 2K_8(\phi, X)\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\right] \\
& - 6\left[\frac{\partial}{\partial\phi}(F(\phi, X) + 2W(\phi)) - XK_8(\phi, X)\right]\nabla_\mu\nabla^\mu\phi + K_9(\phi, X),
\end{aligned} \tag{5.1}$$

where $X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. L_H is the only Lagrangean from which second order field equations are obtained.

The equations of motion are obtained by varying the action:

$$S_H = \int d^4x\sqrt{-g}L_H,$$

and they are

$$\begin{aligned}
E_{\mu\nu} &= -\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = 0, \\
E &= -\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta\phi} = 0,
\end{aligned} \tag{5.2}$$

and as a consequence of the invariance of the action before transformations of diffeomorphisms, we have the property

$$\nabla_\mu E^{\mu\nu} = E\nabla^\nu\phi. \tag{5.3}$$

This theory has been well studied and the reader interested in explicit results can consult [14, 15].

5.0.2 Horndeski theory in first order formalism

The behavior of the Horndeski Lagrangian is analyzed without imposing the condition of zero torsion on it. Through the language of differential forms, it is possible to rewrite the Horndeski Lagrangian. When making the language change, we encounter the dual Hodge several times.

It is very useful to define an operator $\Sigma_{a_1 \dots a_q}$ that maps p-forms into (p-q)-forms, and is defined by its action on a p-form α as

$$\Sigma^{a_1 \dots a_q} \alpha = -(-1)^{p(p-q)} \star (e^{a_1} \wedge \dots \wedge e^{a_q} \wedge \star \alpha). \tag{5.4}$$

When $q = 1$ we got

$$\Sigma^a \alpha = -(e^a \wedge \star \alpha). \tag{5.5}$$

This case is particularly interesting, since Σ_a behaves like an exterior derivative, and satisfies Leibniz's rule,

$$\Sigma_a(\alpha \wedge \beta) = \Sigma_a \alpha \wedge \beta + (-1)^p \alpha \wedge \Sigma_a \beta, \tag{5.6}$$

and is nilpotent,

$$\Sigma_a \Sigma^a = 0. \tag{5.7}$$

An interesting difference between Σ_a and d is the fact that while d increases the degree of the differential form by one, Σ_a decreases the order of the form by one.

In order to describe the field ϕ and its derivatives in the first-order formalism, we will define the 0-form

$$\begin{aligned} Z^a &= e^a{}_\mu \partial^\mu \phi \\ &= - \star (e^a \wedge \star d\phi) \\ &= \Sigma^a d\phi, \end{aligned} \tag{5.8}$$

and the 1-forms

$$\begin{aligned} \pi^a &= e^a{}_\mu \nabla^\mu \nabla_\nu \phi dx^\nu \\ &= DZ^a, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \theta^a &= e^a{}_\nu \nabla^\nu \phi \nabla_\mu \phi dx^\mu \\ &= Z^a d\phi. \end{aligned} \tag{5.10}$$

We also have the following properties:

- $e^a Z_a = d\phi$,

- $\theta^a \wedge \theta^b = 0$, $e^a \wedge \theta_a = 0$,
- $D\theta^a = \pi^a \wedge d\phi$,
- $\Sigma^a \theta^b = Z^a Z^b$,
- $X = -\frac{1}{2}\Sigma^a \theta_a$,
- $\Sigma_a dX = -\frac{1}{2}\Sigma_a d\Sigma_b \theta^b$,
- $D\Sigma_a + \Sigma_a D = e_a{}^\mu D_\mu$,
- $\Sigma_b e^a = \delta_b^a$.

Using the operator Σ^a and the properties above, it is easy to map the Horndeski Lagrangean to the first-order formalism. Written in this formalism, the Lagrangian remains as

$$\begin{aligned} L_H^4 =& \epsilon_{abcd} \Big\{ 2K_1 R^{ab} \wedge e^c \wedge \pi^d + \frac{2}{3}\frac{\partial K_1}{\partial X}\pi^a \wedge \pi^b \wedge \pi^c \wedge e^d + 2K_3 R^{ab} \wedge e^c \wedge \theta^d \\ & + 2\frac{\partial K_3}{\partial X}\theta^a \wedge \pi^b \wedge \pi^c \wedge e^d + (F+2W)R^{ab} \wedge e^c \wedge e^d + \frac{\partial F}{\partial X}\pi^a \wedge \pi^b \wedge e^c \wedge e^d \\ & + K_8 \theta^a \wedge \pi^b \wedge e^c \wedge e^d - \left[\frac{\partial}{\partial \phi}(F+2W) - XK_8\right]\pi^a \wedge e^b \wedge e^c \wedge e^d \\ & + \frac{K_9}{4!}e^a \wedge e^b \wedge e^c \wedge e^d \Big\}, \end{aligned} \tag{5.11}$$

where $F = F(\phi, X)$, $K_i = K_i(\phi, X)$ with $(i = 1, 3, 8, 9)$ and $W = W(\phi)$. And satisfy the relationship

$$\mathcal{C}(\phi, X) = \frac{\partial F}{\partial X} - 2\left(2K_3 + 2X\frac{\partial K_3}{\partial X} - \frac{\partial K_1}{\partial \phi}\right) = 0, \tag{5.12}$$

with $X = -\frac{1}{2}Z_a Z^a$.

We can notice how the Hodge dual operator $\star$ appears in the Horndeski Lagrangian through the operator Σ^a. This operator allows us to cast the Horndeski Lagrangian into an effective Lovelock type of Lagrangian, with the 1-forms π^a and θ^a playing a role similar to that of the vielbein.

The equation (5.11) gives us the Horndeski Lagrangian in the first-order Cartan formalism. Horndeski's Theorem [10] establishes that, when the torsion is zero, the most general tensor-scalar Lagrangian is precisely that of Horndeski (5.1) and gives us second-order field equations for the metric and the scalar field. When torsion is allowed, Horndeski's theorem is no longer valid. It is almost simple to incorporate into the Lagrangian new terms that contain the torsion explicitly, and that does not spoil the second-order nature of the field equations. In the next subsection, we will include the simpler terms that contain torsion explicitly. For now, we will work only with (5.11).

Before moving on to the Horndeski Lagrangian movement equations (5.11), the most general Lagrangians is shown in d=2 and d=3 that contain a scalar field and its field equations are Degree two in derivatives for the metric and for the scalar field:

$$L_H^2 = M_1(\phi, X)\nabla_\mu\nabla^\mu\phi + \left(\frac{\partial M_1}{\partial\phi} - 2\frac{\partial M_2}{\partial X}\right)X + 2M_2(\phi, X), \tag{5.13}$$

$$\begin{aligned} L_H^3 =& \delta^{\mu\nu}_{\alpha\beta}\left(M_3(\phi, X)\widetilde{R}^{\alpha\beta}{}_{\mu\nu} - 4\frac{\partial M_3}{\partial\phi}\nabla_\mu\nabla^\alpha\phi\nabla_\nu\nabla^\beta\phi + M_4(\phi, X)\nabla_\mu\phi\nabla^\alpha\phi\nabla_\nu\nabla^\beta\phi\right) \\ &- 2\left(XM_4 + 4\frac{\partial M_3}{\partial\phi}\right)\nabla_\mu\nabla^\mu\phi - \left(X\frac{\partial M_4}{\partial\phi} + 4\frac{\partial^2 M_4}{\partial\phi^2} + 2\frac{\partial M_5}{\partial X}\right)X + 3M_5(\phi, X), \end{aligned} \tag{5.14}$$

and expressed in first-order language we have:

$$L_H^2 = \epsilon_{ab}\left[a_1 R^{ab} + M_1\pi^a \wedge e^b + \left(\frac{\partial M_1}{\partial\phi} - 2\frac{\partial M_2}{\partial X}\right)\theta^a \wedge e^b + M_2 e^a \wedge e^b\right], \tag{5.15}$$

$$\begin{aligned} L_H^3 =& \epsilon_{abc}\left[M_3 R^{ab} \wedge e^c - 4\frac{\partial M_3}{\partial\phi}\pi^a \wedge \pi^b \wedge e^c + M_4\theta^a \wedge \pi^b \wedge e^c\right. \\ &- \left(XM_4 + 4\frac{\partial M_3}{\partial\phi}\right)\pi^a \wedge e^b \wedge e^c - \left(\frac{X}{2}\frac{\partial M_4}{\partial\phi} + 2\frac{\partial^2 M_4}{\partial\phi^2} + \frac{\partial M_5}{\partial X}\right)\theta^a \wedge e^b \wedge e^c \\ &\left.+\frac{1}{2}M_5 e^a \wedge e^b \wedge e^c\right]. \end{aligned} \tag{5.16}$$

In the equation (5.15) the term $a_1\epsilon_{ab}R^{ab}$ was introduced, with a_1 an arbitrary constant, which does not appear in (5.13) and the reason is that this term is a boundary term and does not contribute to the field equations, but to have something general it is included.

Field equations

To obtain the field equations in the first-order formalism, the fields ω^{ab}, e^a and ϕ are treated as independent degrees of freedom.

Varying the Lagrangian we obtain

$$\delta L_H^4 = W_{ab} \wedge \delta\omega^{ab} + E_a \wedge \delta e^a + \Phi\delta\phi \tag{5.17}$$

where

$$
\begin{aligned}
W_{ab} =&\epsilon_{abcd}e^c \wedge \left[dK_1 \wedge \pi^d + K_1 R^d{}_e Z^e + dK_3 \wedge \theta^d - K_3 d\phi \wedge \pi^d + \frac{1}{2}d(F+2W) \wedge e^d\right] \\
&- \epsilon_{abcd}T^c \wedge [K_1\pi^d + K_3\theta^d + (F+2W)e^d] \\
&- \frac{1}{2}(Z_a\epsilon_{bcde} - Z_b\epsilon_{acde})\left\{K_1 R^{cd} + \pi^c \wedge \left(\frac{\partial K_1}{\partial X}\pi^d + 2\frac{\partial K_3}{\partial X}\theta^d + \frac{\partial F}{\partial X}e^d\right)\right. \\
&\left.+\frac{1}{2}\left(K_8\theta^c - \left[\frac{\partial}{\partial\phi}(F+2W) - XK_8\right]e^c\right) \wedge e^d\right\} \wedge e^e,
\end{aligned}
\tag{5.18}
$$

$$
E_a = \mathcal{E}_a + \Sigma^b(\mathcal{S}_b + \mathcal{T}_b + \mathcal{U}_b)Z_a, \tag{5.19}
$$

$$
\Phi = \mathcal{E} + \mathcal{Z} - d\Sigma^b(\mathcal{S}_b + \mathcal{T}_b + \mathcal{U}_b), \tag{5.20}
$$

with

$$
\begin{aligned}
\mathcal{E}_d =&\epsilon_{abcd}\left\{2K_1 R^{ab} \wedge \pi^c + \frac{2}{3}\frac{\partial K_1}{\partial X}\pi^a \wedge \pi^b \wedge \pi^c + 2K_3 R^{ab} \wedge \theta^c + 2\frac{\partial K_3}{\partial X}\theta^a \wedge \pi^b \wedge \pi^c\right. \\
&+ 2(F+2W)R^{ab} \wedge e^c + 2\frac{\partial F}{\partial X}\pi^a \wedge \pi^b \wedge e^c + 2K_8\theta^a \wedge \pi^b \wedge e^c \\
&\left.+\frac{1}{3!}K_9 e^a \wedge e^b \wedge e^c - 3\left[\frac{\partial}{\partial\phi}(F+2W) - XK_8\right]\pi^a \wedge e^b \wedge e^c\right\},
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
\mathcal{E} =&\epsilon_{abcd}\left\{2\left(\frac{\partial K_1}{\partial\phi} - K_3\right)R^{ab} \wedge e^c \wedge \pi^d + 2\left(\frac{1}{3}\frac{\partial^2 K_1}{\partial\phi\partial X} - \frac{\partial K_3}{\partial X}\right)\pi^a \wedge \pi^b \wedge \pi^c \wedge e^d\right. \\
&+ 2\left(\frac{\partial K_3}{\partial\phi}R^{ab} \wedge e^c \wedge \theta^d + \frac{\partial^2 K_3}{\partial\phi\partial X}\theta^a \wedge \pi^b \wedge \pi^c \wedge e^d\right) + \frac{1}{4!}\frac{\partial K_9}{\partial\phi}e^a \wedge e^b \wedge e^c \wedge e^d \\
&\left.+ \left[\frac{\partial}{\partial\phi}(F+2W)R^{ab} + \left(\frac{\partial^2 F}{\partial\phi\partial X} - K_8\right)\pi^a \wedge \pi^b + \frac{\partial K_8}{\partial\phi}\theta^a \wedge \pi^b\right] \wedge e^c \wedge e^d\right\},
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
\mathcal{Z} =& \left[2dK_3 \wedge R^{ab} + 2d\frac{\partial K_3}{\partial X} \wedge \pi^a \wedge \pi^b + D\pi^a \wedge \left(4\frac{\partial K_3}{\partial X}\pi^b + K_8 e^b\right)\right] \wedge e^c Z^d \\
&+ 2\epsilon_{abcd}\left[K_3 R^{ab} + \frac{\partial K_3}{\partial X}\pi^a \wedge \pi^b + K_8\pi^a \wedge e^b\right] \wedge T^c Z^d,
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
\mathcal{S}_d =&2\epsilon_{abcd}\left\{D\pi^a \wedge e^b \wedge \left(2\frac{\partial K_1}{\partial X}\pi^c + 2\frac{\partial K_3}{\partial X}\theta^c + \frac{\partial F}{\partial X}e^c\right)\right. \\
&+ \pi^a \wedge e^b \wedge dX \wedge \left(\frac{\partial^2 K_1}{\partial X^2}\pi^c + 2\frac{\partial^2 K_3}{\partial X^2}\theta^c + \frac{\partial^2 F}{\partial X^2}e^c\right) \\
&\left.+\frac{1}{2}e^a \wedge e^b \wedge dX \wedge \left[\frac{\partial K_8}{\partial X}\theta^c - \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial\phi} - XK_8\right)e^c\right]\right\},
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
\mathcal{T}_d =& 2\epsilon_{abcd}\left[K_1 R^{ab} + \frac{\partial K_1}{\partial X}\pi^a\wedge\pi^b + 2\frac{\partial K_3}{\partial X}\pi^a\wedge\theta^b + 2\frac{\partial F}{\partial X}\pi^a\wedge e^b + \frac{1}{2}K_8 e^a\wedge\theta^b\right. \\
&\left. -\frac{2}{3}\left(\frac{\partial}{\partial\phi}(F+2W) - XK_8\right)e^a\wedge e^b\right]\wedge T^c,
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
\mathcal{U}_e =& \epsilon_{abcd}\left[-R^{ab}\wedge e^c\wedge\left(C^d{}_e + 2\frac{\partial K_1}{\partial X}\delta^{gd}_{ef}Z_g\pi^f\right) + \pi^a\wedge e^b\wedge e^c\wedge M^d{}_e\right. \\
&\left. -\pi^a\wedge\pi^b\wedge e^c\wedge\left(N^d{}_e + \frac{2}{3}\frac{\partial^2 K_1}{\partial X^2}\pi^d Z_e\right) + e^a\wedge e^b\wedge e^c\wedge K^d{}_e\right],
\end{aligned}
\tag{5.26}
$$

in the last equation above the terms $C^d{}_e$, $N^d{}_e$, $M^d{}_e$ y $K^d{}_e$ are

$$
C^a{}_b = 2d\phi\left[\frac{\partial K_3}{\partial X}Z^a Z_b - \left(K_3 - \frac{\partial K_1}{\partial\phi}\right)\delta^a_b\right] + e^a Z_b\frac{\partial F}{\partial X}, \tag{5.27}
$$

$$
N^a{}_b = 2d\phi\left[\frac{\partial^2 K_3}{\partial X^2}Z^a Z_b - \left(3\frac{\partial K_3}{\partial X} - \frac{\partial^2 K_1}{\partial\phi\partial X}\right)\delta^a_b\right] + e^a Z_b\frac{\partial^2 F}{\partial X^2}, \tag{5.28}
$$

$$
K^a{}_b = \left[\frac{\partial^2}{\partial\phi^2}(F+2W) - X\frac{\partial K_8}{\partial\phi}\right]d\phi\delta^a_b - \frac{1}{4!}e^a Z_b\frac{\partial K_9}{\partial X}, \tag{5.29}
$$

$$
M^a{}_b = \left[2\left(K_8 - \frac{\partial^2 F}{\partial\phi\partial X}\right)\delta^a_b - \frac{\partial K_8}{\partial X}Z^a Z_b\right]d\phi + e^a Z_b\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial\phi} - XK_8\right). \tag{5.30}
$$

and the 1-forms satisfy $C^d{}_e$ and $N^d{}_e$ properties

$$
\begin{aligned}
\Sigma^b C^a{}_b &= Z^a\mathcal{C}, \\
\Sigma^b N^a{}_b &= Z^a\frac{\partial\mathcal{C}}{\partial X},
\end{aligned}
\tag{5.31}
$$

where $\mathcal{C}(\phi, X) = 0$ is the Horndeski constraint (5.12). With all this above, we have the field equations:

$$
\begin{aligned}
W_{ab} &= 0, \\
E_a &= 0, \\
\Phi &= 0.
\end{aligned}
\tag{5.32}
$$

We should note that in $\mathcal{Z}$ and $\mathcal{T}_a$ the torsion appears explicitly as a result of non-minimal couplings. Torsion degrees of freedom are also present in Lorentz curvature through torsion.

5.0.3 How to build the most general theory

In the previous section, we were able to generalize Horndeski's theory by writing it in the first-order formalism, this is achieved by using an arbitrary connection. Now we will follow the steps of the work of Mardones and Zanelli and we will include torsion terms to the action. It is possible to include new terms that some of them contain explicit torsion and are

$$\begin{aligned}
&f_1(\phi,X)T^a\wedge T_a,\ f_2(\phi,X)T^a\wedge R_{ab}Z^b,\ f_3(\phi,X)T^a\wedge D\theta^a,\\
&f_4(\phi,X)T^a\wedge e_a\wedge d\phi,\ f_5(\phi,X)Z_aR^a{}_b\wedge R^b{}_cZ^c,\ f_6(\phi,X)D\theta_a\wedge R^a{}_bZ^b,\\
&f_7(\phi,X)T_aZ^a\wedge T_bZ^b,\ f_8(\phi,X)\epsilon_{abcd}Z^aT^b\wedge e^c\wedge e^d,\ f_9(\phi,X)T_aZ^a\wedge Z_bD\theta^b,\\
&f_{10}(\phi,X)R^a{}_b\wedge R^b{}_a,\ f_{11}(\phi,X)e_a\wedge R^{ab}\wedge e_b.
\end{aligned} \tag{5.33}$$

Where the funtions $f_i(\phi,X)$ with $i=1,...,11$ are arbitrary.

The modified Horndeski action that includes all the terms above would be as follows:

$$\begin{aligned}
L^4_{LCH}=&\epsilon_{abcd}\Big\{2K_1R^{ab}\wedge e^c\wedge\pi^d+\frac{2}{3}\frac{\partial K_1}{\partial X}\pi^a\wedge\pi^b\wedge\pi^c\wedge e^d+2K_3R^{ab}\wedge e^c\wedge\theta^d\\
&+2\frac{\partial K_3}{\partial X}\theta^a\wedge\pi^b\wedge\pi^c\wedge e^d+(F+2W)R^{ab}\wedge e^c\wedge e^d+\frac{\partial F}{\partial X}\pi^a\wedge\pi^b\wedge e^c\wedge e^d\\
&+K_8\theta^a\wedge\pi^b\wedge e^c\wedge e^d-\left[\frac{\partial}{\partial\phi}(F+2W)-XK_8\right]\pi^a\wedge e^b\wedge e^c\wedge e^d\\
&+\frac{K_9}{4!}e^a\wedge e^b\wedge e^c\wedge e^d\Big\}+f_1T^a\wedge T_a+f_2T^a\wedge R_{ab}Z^b+f_3T^a\wedge D\theta_a\\
&+f_4T^a\wedge e_a\wedge d\phi+f_5Z_aR^a{}_b\wedge R^b{}_cZ^c+f_6D\theta_a\wedge R^a{}_bZ^b+f_7T_aZ^a\wedge T_bZ^b\\
&+f_8\epsilon_{abcd}Z^aT^b\wedge e^c\wedge e^d+f_9T_aZ^a\wedge Z_bD\theta^b+f_{10}R^a{}_b\wedge R^b{}_a+f_{11}e_a\wedge R^{ab}\wedge e_b.
\end{aligned} \tag{5.34}$$

One could conclude by saying that this is the most general action of gravitation, but we hope that it gives us second-order equations in the metric and scalar fields. To achieve this we must somehow restrict $f_i(\phi,X)$ to functions that are arbitrary and could generate field equations of order higher than two. To find a constraint for the functions and thus obtain first-degree equations we are going to impose the symmetry of diffeomorphisms in the modified Horndeski action (5.34).

Chapter 6

Conclusions

The main objective of this work was the construction of general gravitational theories using first-order formalism. We mainly focus on the study of the Lovelock-Cartan-Horndeski theory, which is a modified theory of gravitation that includes torsion and a scalar field. It also shows the possibility of further generalizing this theory by including terms with torsion in the Lagrangian. It makes use of a very useful operator Σ^a, which allows us to work with the Horndeski Lagrangian more efficiently. Before starting with the description of this theory, we are going to summarize what we have learned in the other two theories that generalize to General Relativity.

Lovelock Theory[16]: The most general action for gravitation, which is Lorentz invariant, does not contain torsion and gives us second-order equations of motion for the metric is (4.1).

- It uses as base ingredients the vielbein, the spin connection and the curvature.
- It describes a gravitational theory of torsion free in D-dimensions and General Relativity is a special case when $D = 4$.
- The null torsion condition must be imposed so that diffeomorphism invariance is satisfied. This only happens for dimensions greater than 4, otherwise, it is satisfied without the need to impose anything.

Lovelock-Cartan Theory[17]: The most general action for gravitation, which is Lorentz invariant, contains torsion and gives us second-order equations of motion for the metric is (4.15).

- It uses as base ingredients the vielbein, the spin connection, the curvature and torsion.

Theory	Ingredients	Lagrangian	Dimention
General Relativity	$R^{ab},\ e^a$	$\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d$	4
Horndeski	$R^{ab},\ e^a,\phi$	L^4_H	4
Lovelock-Cartan-Horndeski	$R^{ab},T^a,\ e^a,\phi$	L^4_{LCH}	4

Table 6.1: This table shows the generalized theories of gravity where a scalar field is used. Where L^4_H is given by (5.11), and L^4_{LCH} is given by (5.34).

- It describes a gravitational theory with torsion in D-dimensions and General Relativity is a special case with $T^a = 0$ and $D = 4$.

Hornseski theory[10]: Firstly we present Horndeski's theory in the second-order formalism where we impose the zero torsion condition. This is a general theory in 4 dimensions that gives us second-order field equations in the derivative of the metric and the scalar field. To approach a generalization of this theory, the first step that was taken was to rewrite it in the primer formalism, this forces us to use an arbitrary connection that forces us not to impose zero torsion. The Horndeski Lagrangian in 4 dimensions with torsion is (5.11). With recent mathematical techniques based on the operator Σ^a we were able to rewrite and obtain the field equations of Horndeski theory with torsion (5.32). With this result, we conclude that the torsion that appears in the equations of motion is generated by:

- Second order derivatives of the scalar field in the Lagrangian (5.11)
- The non-minimal coupling of the scalar field to the geometric terms, that is, to the coupling with the Lorentz curvature.

The torsion T^a has a fairly close relationship with the scalar field and its dynamic terms.

Lovelock-Cartan-Horndeski theory: Horndeski's theory is the most general theory that was known, however, there are still possibilities to generalize it. An alternative that is still open is to generalize it to higher dimensions, and the other is the one that was touched on in the last section that tries to generalize it by including new terms in the Horndeski action (5.33), said terms can now contain explicit torsion.

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