Caio Vaz Pereira de Brito

Nonequilibrium dynamics of ultrarelativistic gases from the method of moments

Niterói

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Universidade Federal Fluminense

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Supervisor: Prof. Dr. Gabriel Silveira Denicol

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Caio Vaz Pereira de Brito

NONEQUILIBRIUM DYNAMICS OF ULTRARELATIVISTIC GASES FROM THE METHOD OF MOMENTS

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BANCA AVALIADORA

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Prof. Dr. Gabriel Silveira Denicol
(Orientador)
IF/UFF

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Profa. Dra. Raissa Fernandes Pessoa Mendes
IF/UFF

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“I could never get very excited about other people’s ambitions for my life.”
(Jesse on ‘Before Sunrise’)

Abstract

In this thesis, we investigate solutions of the Boltzmann equation from the method of moments. This approach, originally proposed by Grad, consists in expanding the single-particle distribution function in terms of its irreducible moments using a complete and orthogonal basis of irreducible momenta. The problem of solving an integro-differential equation for the single-particle distribution function is then converted into solving a set of coupled differential equations for its moments. Traditional fluid-dynamical theories can be obtained by systematically truncating the so-called moment expansion including solely the 14 independent degrees of freedom that appear in the continuity equations that describe the conservation of particle, energy and momentum. We first extend this truncation by also including nonhydrodynamic currents and derive a third-order formulation of fluid dynamics. One expects that the inclusion of more terms in the moment expansion leads to more accurate solutions for the Boltzmann equation, which typically requires the inclusion of moments that lie far beyond the hydrodynamic limit. We then systematically derive a general equation of motion for irreducible moments of arbitrary rank and show that, as more moments are included in the hierarchy of equations, the solutions for the moments themselves converge. We then reconstruct the single-particle distribution from the solutions for its moments, thus actually solving the Boltzmann equation. We demonstrate that the moment expansion for an ultrarelativistic gas in Bjorken flow is actually divergent and physically meaningful results can only be obtained with the implementation of resummation schemes. In particular, we observe that the exact energy-momentum tensor is well described by fluid dynamics, whereas the single-particle distribution function itself is not. Thus, the nonequilibrium dynamics of a massless gas is not quantitatively well captured by a fluid-dynamical description and is primarily governed by nonhydrodynamic contributions. This implies that, for a classical massless gas undergoing a Bjorken expansion, fluid dynamics works even though the system is not in equilibrium.

Keywords: Kinetic theory, Boltzmann equation, method of moments, relativistic dissipative fluid dynamics.

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Introduction

Quantum chromodynamics (QCD) describes the strong interaction in terms of fundamental particles, namely quarks and gluons [3, 4]. Quarks possess two intrinsic quantum numbers: flavor – six types have been observed so far (up, down, charm, strange, top, bottom), each with specific mass and electric charge; color charge, the analogue of electric charge in quantum electrodynamics (QED), but with three types (red, green, blue) that combine into color-neutral states. The interactions between color-charged particles are mediated by gluons, which themselves carry color, allowing not only quark–quark but also gluon–gluon interactions. This self-interaction, which is absent in QED since the photon has zero electric charge, renders first-principles calculations extremely challenging. Furthermore, QCD exhibits two unique properties: asymptotic freedom [5–7], where the interaction between quarks weakens as the energy is increased, and color confinement [8], which forbids the existence of free color-charged particles. In particular, at extremely high temperatures, asymptotic freedom dictates that quarks and gluons become weakly interacting and thus cannot form any bound states. Thus, such property predicts a deconfined phase of nuclear matter, the so-called quark-gluon plasma (QGP), a state in which hadrons cannot exist and the system is composed solely of weakly-interacting quarks and gluons.

Over the past decades, several experiments were launched with the common goal of producing this novel state of nuclear matter and understanding the thermodynamic and transport properties of QCD. These experiments consist in colliding heavy ions at ultra-relativistic velocities with the purpose of creating systems with the highest energies and densities ever achieved in a controlled environment. The first efforts of this experimental endeavor started over five decades ago, with heavy-ions being collided at ~ 1 GeV per nucleon at the Bevatron, Berkeley. Later on, heavy-ion experiments were launched at the Alternating Gradient Synchrotron (AGS) [9], Relativistic Heavy-Ion Collider (RHIC), both located in Brookhaven National Lab, in Upton, USA, as well as at the Super Proton Synchrotron (SPS) [10] and the Large Hadron Collider (LHC), at CERN, at Geneva. The current ongoing experiments both at RHIC and LHC achieve energies in the order of TeV per nucleon, thus providing a rather powerful tool to investigate the QCD phase diagram.

The far-from-equilibrium system formed shortly after the collision evolves into the quark-gluon plasma, whose dynamics is governed by relativistic fluid-dynamical equations. In particular, phenomenological computations estimate that such system takes $\tau \sim 1$ fm/c to reach a hydrodynamic behavior [11]. However, if and how the system thermalizes so fast still remains an open question in the field [12]. In particular, it is not trivial to understand how the competition between the system's expansion and the interactions between the particles drive it to a hydrodynamic regime.

Numerous experimental evidence corroborate the fluid behavior of the QGP. In particular, the QGP is expected to be formed in the region where the two incoming nuclei overlap. In general, the overlap region is not perfectly isotropic in the transverse plane, but instead has an almond-like shape, that is, the bulk matter is more squeezed in one direction in the transverse plane of reaction. Hydrodynamics thus dictates that larger pressure gradients are generated in this direction, implying that particles are expected to be emitted with larger momenta in this particular direction. Therefore, spatial anisotropies in the system are converted into pressure gradients in the presence of collective behavior. This is imprinted in the so-called flow harmonics – denoted by v_n –, coefficients of a Fourier expansion of the azimuthal distribution of the outgoing particles in the transverse plane which quantify the momentum anisotropies of the emitted particles. The first three coefficients are known as the *directed* flow (v_1), the *elliptic* flow (v_2), and the *triangular* flow (v_3). Measurements of the flow harmonics determine the flow configuration of the QGP, which in particular reflects the system's initial geometry, thus serving as striking evidence of the fluid character of the quark-gluon plasma [13–17].[1]

Another evidence for the formation of the QGP comes from jet quenching [20–23]. The energy of the hot and dense system produced in ultrarelativistic heavy-ion collisions is sufficiently large to allow for the production of parton–anti-parton pairs. Naturally, conservation momentum impose that these sprays of particles, known as jets, are emitted in opposite directions. One of the tails of the jet travels inward the hot and dense medium, losing energy from the strong interactions with the particles of the system, thus being significantly stopped from reaching the azimuthally opposite side of the detectors. This leads to an asymmetry in the detected final-state particles, as the other end of the jet, traveling outside the QGP region, does not experience the same effects and reaches the detector rather intact. These observations have been reported by various experiments and provide reliable probes of the QGP [24–29]. This phenomenon is known as *jet quenching* and further supports the hypothesis of the formation of a strongly coupled fluid of quarks and gluons.

The QGP is the fluid with the smallest specific shear viscosity, η/s, ever observed [30–35]. Nevertheless, dissipative effects should not be disregarded in order to consistently describe the evolution of this system. Therefore, one must invoke a formalism of relativistic dissipative fluid dynamics. The most intuitive approach is to resort to a relativistic extension of Navier-Stokes theory, the first description of relativistic fluids, derived independently by Eckart [36] and Landau [37]. The derivation of the Navier-Stokes theory relies on a nonequilibrium entropy 4-current that contains only terms up to first order in the dissipative currents. Then, from the second law of thermodynamics, one obtains the so-called Navier-Stokes equations, a set of constitutive relations in which the dissipative currents are expressed as gradients of velocity, temperature

[1] Nevertheless, we remark that, for ultracentral collisions, hydrodynamic models fail to quantitatively predict the elliptic and triangular flows – in particular, the first is too large or the latter is too small, or even both [18, 19]. Even though this problem has been pursued by several authors in the recent years, the so-called *ultracentral puzzle* remains an open problem in the field.

and chemical potential. However, this is an ill-defined formalism, since perturbations around a system initially in equilibrium propagate with infinite speed and increase with time [38, 39]. Such unphysical pathologies, which can be traced back to the parabolic and acausal nature of Navier-Stokes equations, forbid its application to describe realistic systems.

The derivation of causal and stable theories of relativistic hydrodynamics has a long history. It started with Israel and Stewart in the 1970s [40]. Israel and Stewart generalized the formalism proposed by Grad in the nonrelativistic regime [41] and showed that the unphysical behavior of Navier-Stokes theory can be circumvented by introducing a transient theory of fluid dynamics [40, 42]. In these formulations, the dissipative currents are treated as independent fields that relax dynamically to the usual constitutive relations employed in Navier-Stokes theory, thus satisfying equations of motion rather than constitutive relations. Israel and Stewart have performed this task in two different ways, namely from phenomenological [40] and microscopic [42] calculations. In the first, the nonequilibrium entropy 4-current considered in Navier-Stokes theory was extended to also incorporate terms of second order in the dissipative currents. In this case, the second law of thermodynamics yields the Israel-Stewart equations, a set of hyperbolic equations of motion for the dissipative currents. The hyperbolic character of Israel-Stewart-like equations is a crucial feature for causality and stability of such formulations [43]. Nevertheless, the transport coefficients cannot assume arbitrary values and must be appropriately constrained from a linear stability and causality analysis, a problem that has been addressed by several authors in the past [44–57].

So far, we have only discussed phenomenological derivations of fluid-dynamical theories. Such approaches rely on the construction of an entropy 4-current from the dissipative currents and then imposing the non-negativity of the entropy production to obtain equations for the dissipative currents. Nevertheless, this is by no means the unique form to derive a fluid-dynamical theory [58]. One possible approach is to derive such equations from a microscopic perspective, in particular using kinetic theory. For this purpose, the Boltzmann equation is traditionally used as a starting point, since it allows for the most complete derivation of a hydrodynamic theory, including all the second-order terms as well as nonlinear terms. The Boltzmann equation describes the dynamics of the momentum distribution in a *dilute* gas, encoded in the so-called distribution function, defined as the density of particles per momentum [59, 60]. However, in practice, obtaining general solutions for the Boltzmann equation is a rather nontrivial task, and it is often necessary to resort to approximate methods.

The Chapman-Enskog method is the most widespread form to obtain solutions in the hydrodynamic limit of the Boltzmann equation [61]. It consists in expanding the single-particle distribution function in powers of gradients and solving the Boltzmann equation order by order perturbatively. The single-particle distribution is then reconstructed from the solution at each order. In particular, truncating the Chapman-Enskog expansion at zeroth order yields the Euler equations of ideal fluid dynamics. A truncation at first order leads to the relativistic Navier-Stokes

theory. As was previously discussed, this formulation suffers from mathematical pathologies that lead to unphysical instabilities that forbid its application to the description of realistic systems. Furthermore, truncations at second order and beyond lead to Burnett and super-Burnett theories, which are unstable even in the non-relativistic regime [62]. Thus, we conclude that the Chapman-Enskog method yields theories that are unsuitable for practical purposes.

An alternative to obtain fluid-dynamical theories from the Boltzmann equation that does not display non-physical features of the Chapman-Enskog method is the method of moments. It was originally developed by Grad for non-relativistic gases [63], introducing an expansion for the single-particle distribution function in terms of a complete and orthogonal basis of generalized Hermite polynomials [41]. Three decades later, Israel and Stewart were the pioneers in extending Grad's work to relativistic systems [42]. In the absence of a convenient orthogonal basis, they simply expanded the single-particle distribution function using a basis of 4-momenta, $1, k_\mu, k_\mu k_\nu, \cdots$. This expansion is then truncated at second order, so that the distribution function is described solely in terms of degrees of freedom that can be matched to the independent fields appearing in the conserved currents. The truncated distribution function is finally inserted into the first three moments of the Boltzmann equation, leading to the so-called Israel-Stewart transient theory of relativistic fluid dynamics. The derived equations are shown to be linearly causal and stable as long as the transport coefficients satisfy certain constraints, cf. Refs. [45, 47, 48, 53, 64]. We remark that a major difference between the method of moments and the Chapman-Enskog method is the fact that the first relies on a truncation in degrees of freedom, while the latter corresponds to a truncation in a small parameter, characterized by gradients of the fluid-dynamical variables.

In Ref. [2], the relativistic version of the method of moments was generalized and the single-particle distribution function was expanded in terms of a basis of *irreducible*[2] momenta, $1, k_{\langle\mu\rangle}, k_{\langle\mu}k_{\nu\rangle}, \cdots$. In contrast to the basis chosen by Israel and Stewart, the irreducible momenta form a complete and *orthogonal* basis [58, 60]. Hence, the expansion coefficients can be obtained in exact form without relying on a matching procedure, and their equations of motion stem from the Boltzmann equation. In practice, one replaces the Boltzmann equation by an, in principle, infinite set of coupled partial differential equations for the moments of the single-particle distribution function, and fluid-dynamical theories are then obtained from a systematic truncation of these equations. As a matter of fact, fluid dynamics requires a coarse-graining of this myriad of variables in order to provide a macroscopic description in terms of a finite number of degrees of freedom. In their original work, Israel and Stewart proposed a truncation for the moment expansion such that the single-particle distribution function is completely expressed in terms of the traditional fluid-dynamical variables, namely net-charge and energy densities, bulk viscous pressure, fluid 4-velocity, net-charge diffusion and shear-stress tensor, comprising a total of 14 degrees of freedom.[3] This truncation is commonly referred to as the 14-moment approximation.

[2] with respect to Lorentz transformations that leave the fluid 4-velocity unchanged, i.e., $\Lambda^\mu{}_\nu u^\nu = u^\mu$ [60].
[3] The energy diffusion is set to zero using Landau matching conditions [37].

In this case, the moment equations are truncated including terms up to second order in gradients. Overall, the generalization of the method of moments proposed in Ref. [2], has been consistently employed to obtain and study causal fluid-dynamical theories from the Boltzmann equation, cf. Refs. [2, 65–82].

The relativistic method of moments constructed in Ref. [2] is, nevertheless, incomplete. This happens because the equations of motion for the moments of the single-particle distribution function were not fully derived – only the equations of motion for the moments that contributed to the hydrodynamic equations were obtained. While this is not a problem for deriving the traditional transient second-order fluid-dynamical theories, it is insufficient for solving the Boltzmann equation itself using the method of moments or deriving higher-order hydrodynamic formulations. In this thesis, we bridge this gap and derive for the first time equations of motion for arbitrary moments of the single particle distribution function [83]. We then use these equations to derive higher-order hydrodynamic theories [72] and to understand how the Boltzmann equation can be solved using the method of moments [84]. In this thesis we demonstrate for the first time that, while the solutions for the moment equations display a convergent behavior, the moment expansion itself diverges. This divergence is completely unexpected and makes it rather complicated to solve the Boltzmann equation using its moments.

This thesis is organized as follows. In Chapter 1, we discuss the phenomenological derivation of relativistic hydrodynamics. We start discussing the equations of motion for ideal fluids and then demonstrate how dissipation can be included into the description. Two fundamental theories are approached: the relativistic Navier-Stokes theory [36, 37] and Israel-Stewart theory [40]. We detail how these theories are constructed from the second law of thermodynamics and discuss their corresponding linear stability of the global equilibrium state. We clearly demonstrate that Navier-Stokes theory is linearly acausal and unstable [38], as already stated above, and obtain constraints for the transport coefficients that ensure the causality and stability of Israel-Stewart theory [45–48, 53].

In Chapter 2, we derive the relativistic Boltzmann equation and explain how the traditional Navier-Stokes equations are derived from it. This is accomplished by the well known Chapman-Enskog expansion [61], a perturbative expansion of the Boltzmann equation in powers of gradients or, equivalently, the Knudsen number. In this Chapter, the constitutive equations employed in Navier-Stokes theory are derived, as well as microscopic expressions for the transport coefficients that appear in such relations. In Chapter 3, the derivation of fluid dynamics from the Boltzmann equation is further discussed, now via the method of moments – the main formalism used and developed in this thesis. In this case, we explain the traditional 14-moment approximation and derive the famous Israel-Stewart equations [42]. The remainder of this Chapter is dedicated to the first new results presented in this thesis: a derivation of third-order fluid-dynamical equations [72], that arise as a correction to Israel-Stewart theory to describe rapidly expanding plasmas.

In Chapter 4, we develop one of the main results of this thesis and establish a complete relativistic version of the method of moments [83]. As already mentioned, the equations of motion for the moments originally derived in Ref. [2] were not complete and only contained the moments that appeared or were required to derive the fluid-dynamical equations of motion. In this Chapter, a complete set of moment equations are derived and the convergence of their solutions is analyzed and confirmed. These results were published in Ref. [83].

Finally, in Chapter 5, we analyze for the first time the convergence of the moment expansion itself, another new result from this thesis. Considering an ultrarelativistic plasma undergoing Bjorken flow, we demonstrate that the moment expansion does not converge for this toy model of a heavy ion collision [84]. We then show how the moment expansion can be resummed via a Borel-Padé scheme and analyze if an ultrarelativistic plasma in these conditions can be close to local equilibrium. We then end this thesis with our Conclusions and final remarks. The results in this thesis are also complemented by several Appendices, which contain some of the more tedious steps of our derivations.

Throughout this work we employ natural units, $c = \hbar = k_B = 1$, and adopt the mostly-minus convention for the Minkowski metric tensor, $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

1 Relativistic fluid dynamics

Fluid dynamics is a macroscopic effective theory that describes the evolution of a many-body system in terms of a reduced number of degrees of freedom, called the hydrodynamic variables – namely particle and energy densities and fluid velocity. The application of fluid dynamics requires a coarse-graining of the system, which is then discretized into a collection of fluid cells. Each fluid cell must be large enough to contain a sufficiently large number of particles, so that it can be described as a thermodynamic system in *local* equilibrium. On the other hand, fluid cells must also be much smaller than the typical length scale of the system in order to enable a consistent description of fluid cells as point-like particles, that is, disregarding its dimensions. In this sense, the application of fluid dynamics requires a separation of scales in the system, which is typically quantified by the Knudsen number, a ratio between a microscopic and a macroscopic scale of the system, $\mathrm{Kn} \sim \ell_{\mathrm{micro}}/L_{\mathrm{macro}}$. In principle, one can consistently evoke fluid dynamics when $\mathrm{Kn} \ll 1$. The goal of this chapter is to discuss the fundamentals aspects of relativistic fluid dynamics.

1.1 Thermodynamic identities

A fluid-dynamical description is expected to be valid for systems *close* to an equilibrium state. In this context, we first develop the thermodynamic relations that describe a macroscopic description of a system in equilibrium.

The first law of thermodynamics describes the conservation of the energy of an arbitrary system, E,

$$dE = TdS - PdV + \mu dN, \tag{1.1}$$

where T is the temperature, S is the entropy, P is the thermodynamic pressure, V is the volume, μ is the chemical potential and N is the number of particles. It then follows that

$$\left(\frac{\partial E}{\partial S}\right)_{V,N} = T, \quad \left(\frac{\partial E}{\partial V}\right)_{S,N} = -P, \quad \left(\frac{\partial E}{\partial N}\right)_{S,V} = \mu. \tag{1.2}$$

We remark that the energy of a system is an *extensive quantity*, i.e., it is directly proportional to its macroscopic variables [85]. Therefore, if the system's entropy, volume and number of particles are changed by a given factor, its energy also changes proportionally,

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N), \tag{1.3}$$

with λ being an arbitrary real number. Hence,

$$\lambda E = S\left[\frac{\partial E}{\partial(\lambda S)}\right]_{\lambda V,\lambda N} + V\left[\frac{\partial E}{\partial(\lambda V)}\right]_{\lambda S,\lambda N} + N\left[\frac{\partial E}{\partial(\lambda N)}\right]_{\lambda S,\lambda V}. \tag{1.4}$$

Taking $\lambda = 1$ and employing Eqs. (1.2), we obtain the Euler equation

$$E = TS - PV + \mu N. \tag{1.5}$$

From the first law of thermodynamics and the Euler equation, Eqs. (1.1) and (1.5), one obtains the Gibbs-Duhem equation

$$SdT - VdP + Nd\mu = 0. \tag{1.6}$$

In the applications that follow in this thesis, we are not particularly interested in the total volume of the system. In this context, it is convenient to express all these relations in terms of densities of the thermodynamic variables with respect to the volume. The Euler and Gibbs-Duhem equations then become

$$\varepsilon + P = Ts + \mu n, \tag{1.7}$$

$$sdT - dP + nd\mu = 0, \tag{1.8}$$

where we defined the energy density, $\varepsilon \equiv E/V$, the entropy density, $s \equiv S/V$, and the particle density, $n \equiv N/V$. Combining the two equations above, we obtain the first law of thermodynamics in terms of densities,

$$d\varepsilon = Tds + \mu dn. \tag{1.9}$$

An additional useful thermodynamic relation can be obtained by combining the Euler and Gibbs-Duhem equations, Eqs. (1.7) and (1.8), respectively, leading to

$$d\beta = \frac{n}{\varepsilon + P} d\alpha - \frac{\beta}{\varepsilon + P} dP, \tag{1.10}$$

where we defined the inverse temperature $\beta = 1/T$ and the thermal potential $\alpha = \mu/T$.

The thermodynamic relations obtained in this section will be essential in the derivation of the fluid-dynamical equations of motion.

1.2 Ideal Relativistic Fluid Dynamics

We first study ideal fluids, in which the system is assumed to be always in a state of local thermodynamic equilibrium. That is, each fluid element is a thermodynamic system in equilibrium itself. In this case, we shall demonstrate that the dynamics of such systems can be completely described in terms of conservation laws and an equation of state. The conserved quantities considered in this chapter are the net-charge, energy and momentum, which satisfy the following conservation laws

$$\partial_\mu N^\mu = 0, \tag{1.11a}$$

$$\partial_\mu T^{\mu\nu} = 0, \tag{1.11b}$$

where N^μ is the net-charge 4-current and $T^{\mu\nu}$ is the energy-momentum tensor.

Under the assumption of local thermodynamic equilibrium, it is possible to determine the exact form of the conserved currents N^μ and $T^{\mu\nu}$. In practice, this is done by performing a Lorentz-boost to their *local rest frame* (LRF), in which the fluid elements are at rest. In this case, there is no energy flux, and thus $T^{i0}_{\rm LRF} = 0$. Furthermore, the momentum density is zero, and therefore the components $T^{0i}_{\rm LRF}$ also vanish. Finally, due to the assumption of thermodynamic equilibrium, the force per surface element between adjacent fluid elements is isotropic and equal to the thermodynamic pressure P_0, therefore $T^{ij}_{\rm LRF} = \delta^{ij} P_0$. Hence, the energy-momentum tensor has the following form in this frame

$$T^{\mu\nu}_{\rm LRF} = \begin{pmatrix} \varepsilon_0 & 0 & 0 & 0 \\ 0 & P_0 & 0 & 0 \\ 0 & 0 & P_0 & 0 \\ 0 & 0 & 0 & P_0 \end{pmatrix}. \tag{1.12}$$

We remark that the thermodynamic pressure is determined by the net-charge density and energy density via an equation of state, $P_0 = P_0(n_0, \varepsilon_0)$. The subscript '0' is employed to characterize fluid-dynamical quantities in an equilibrium state.

Furthermore, in the local rest frame there is no net-charge flux, and thus the spatial components $N^i_{\rm LRF}$ are identically zero. Therefore, the net-charge 4-current in the local rest frame can be written as

$$N^\mu_{\rm LRF} = \begin{pmatrix} n_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.13}$$

where here n is the net-charge density. Finally, the entropy 4-current in the local rest frame can be written analogously as

$$S^\mu_{\rm LRF} = \begin{pmatrix} s_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.14}$$

Since the fluid elements of an ideal fluid are in local thermodynamic equilibrium, the entropy flux, $S^i_{\rm LRF}$, is zero.

So far, the energy-momentum tensor, the net-charge 4-current and the entropy 4-current of an ideal fluid have been obtained only in the local rest frame. In order to obtain the general form of the conserved currents, a Lorentz-boost must be performed with the 4-velocity $u^\mu = \gamma(1, \mathbf{V})$, with $\mathbf{V}$ being the fluid 3-velocity, and $\gamma \equiv 1/\sqrt{1 - V^2}$ being the Lorentz factor. We remark that the 4-velocity is a normalized 4-vector, i.e., $u_\mu u^\mu = 1$. Then, the following expressions are

obtained

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} = \varepsilon_0 u^\mu u^\nu - \Delta^{\mu\nu} P_0, \tag{1.15}$$

$$N^\mu = N^\mu_{\rm eq} = n_0 u^\mu, \tag{1.16}$$

$$S^\mu = S^\mu_{\rm eq} = s_0 u^\mu, \tag{1.17}$$

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is the projection operator onto the 3-space orthogonal to u^μ. We remark that these expressions are valid only for a fluid in thermodynamic equilibrium, denoted by the underscript "eq". The most general form of the conserved currents for nonequilibrium system will be the topic of the next section.

Since the thermodynamic pressure is given by an equation of state and the 4-velocity is normalized, the conserved currents N^μ and $T^{\mu\nu}$ contain a total of five independent degrees of freedom, namely n, ε, and $\mathbf{V}$. Therefore, the conservation of net-charge, energy and momentum, Eqs. (1.11), which constitute five equations, are sufficient to completely describe the dynamics of an ideal fluid. The next step is to explicitly obtain these equations of motion.

First, it is convenient to decompose the conservation of energy and momentum in terms of its parallel and orthogonal components with respect to the fluid 4-velocity. This is accomplished by projecting Eq. (1.11b) with u^μ and Δ^λ_μ, respectively. Thus, the conservation of net-charge number, energy and momentum in an ideal fluid can be expressed as

$$u_\nu \partial_\mu T^{\mu\nu}_{\rm eq} = 0, \tag{1.18}$$

$$\Delta^\lambda_\nu \partial_\mu T^{\mu\nu}_{\rm eq} = 0, \tag{1.19}$$

$$\partial_\mu N^\mu_{\rm eq} = 0, \tag{1.20}$$

which lead to the following equations of motion for the hydrodynamic variables

$$\dot{\varepsilon}_0 + (\varepsilon_0 + P_0)\theta = 0, \tag{1.21a}$$

$$(\varepsilon_0 + P_0)\dot{u}^\lambda - \nabla^\lambda P_0 = 0, \tag{1.21b}$$

$$\dot{n}_0 + n_0 \theta = 0. \tag{1.21c}$$

Here, the dot denotes the comoving derivative, $\dot{A} = u^\mu \partial_\mu A$, $\theta \equiv \partial_\mu u^\mu$ is the expansion rate and $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is the spatial projection of the covariant derivative. Equations (1.21) are the Euler equations of ideal fluid dynamics.

We now derive an equation of motion for the entropy density. The thermodynamic relation (1.9) implies that,

$$\dot{s}_0 = \frac{\dot{\varepsilon}_0}{T_0} - \alpha_0 \dot{n}_0. \tag{1.22}$$

Then, using Eqs. (1.21a) and (1.21c), we have

$$\dot{s}_0 = -\left(\frac{\varepsilon_0 + P_0}{T_0} - \alpha_0 n_0\right)\theta. \tag{1.23}$$

Finally, from the Euler relation, Eq. (1.7), it follows that

$$\dot{s}_0 + s_0\theta = 0. \tag{1.24}$$

In particular, this equation can be cast in the following form,

$$\partial_\mu(s_0 u^\mu) \overset{(1.17)}{=} \partial_\mu S^\mu_{\text{eq}} = 0. \tag{1.25}$$

The occurrence of a continuity equation implies that the entropy of an ideal fluid is conserved. However, this is no longer the case when dissipative effects are taken into account. Nevertheless, in order to actually conclude this, first it is essential to understand how the energy-momentum tensor and the net-charge 4-currents must be modified to consistently accommodate dissipative effects. This will be the topic of the next sections.

1.3 Dissipative Relativistic Fluid Dynamics

In the last section, we have derived expressions for the energy-momentum tensor, net-charge and entropy 4-current for an ideal fluid. In this case, we obtained a closed system of five fluid-dynamical equations that, supplemented by an equation of state, completely describe the evolution of an ideal fluid. The main goal of this section is to understand how to consistently include nonequilibrium effects in the fluid-dynamical equations.

In the case of dissipative fluids, one assumes that energy can be transferred between fluid elements also in the form of heat as well as friction. Thus, the system can no longer be assumed to be always in local thermodynamic equilibrium. This implies that the fluid cannot be described solely in terms of its equilibrium variables and the addition of novel nonequilibrium degrees of freedom is required. Therefore, the conservation laws must then be supplemented by additional relations for the dissipative currents for closure and the relations derived in the previous section must be revised. In what follows, we thoroughly investigate the derivation of two different prescriptions for dissipative fluid dynamics, namely Navier-Stokes and Israel-Stewart theories.

For the sake of simplicity, we assume that the fluid is *close* to a local equilibrium state. In fact, fluid dynamics is expected to be valid in this regime. Naturally, the presence of dissipative effects in the system imply that it is *never* in an equilibrium state, but the assumption of proximity to a local equilibrium state enables a consistent description of the system in terms of a reduced number of degrees of freedom. However, we remark that there are infinitely many different possible equilibrium states accessible by the system. The reference equilibrium state is then uniquely defined by the imposition of the so-called matching conditions, as will be discussed in Sec. 1.3.1.

In general, the energy-momentum tensor and net-charge 4-current can be expressed as

$$N^\mu = N^\mu_{\text{eq}} + \delta N^\mu, \tag{1.26}$$

$$T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \delta T^{\mu\nu}, \tag{1.27}$$

where N^μ_{eq} and $T^{\mu\nu}_{\text{eq}}$ are the equilibrium net-charge 4-current and energy-momentum tensor, given respectively by Eqs. (1.16) and (1.15), whereas δN^μ and $\delta T^{\mu\nu}$ are the corresponding dissipative corrections. In what follows, we obtain the exact form of these terms.

We now analyze the explicit form of the nonequilibrium terms δN^μ and $\delta T^{\mu\nu}$. We remark that the energy and net-charge densities in the local rest frame of the fluid remain being defined as, $\varepsilon = u_\mu u_\nu T^{\mu\nu}$ and $n = u_\mu N^\mu$, respectively. We consider the system to be *close* to a fictitious local equilibrium state, with temperature T and chemical potential μ, such that the net-charge and energy densities can be expressed as

$$n = n_0(T,\mu) + \delta n, \tag{1.28a}$$

$$\varepsilon = \varepsilon_0(T,\mu) + \delta\varepsilon, \tag{1.28b}$$

where n_0 and ε_0 denote the net-charge and energy densities of the aforementioned fictitious equilibrium state, with δn and $\delta\varepsilon$ being the corresponding nonequilibrium corrections. In particular, we remark that ε and n are the energy and net-charge densities of a fluid out of equilibrium, and therefore, in general do not satisfy the thermodynamic relations (1.7)–(1.10), since these are only valid for systems in equilibrium. Such relations are satisfied by the equilibrium densities n_0 and ε_0.

Without loss of generality, δN^μ and $\delta T^{\mu\nu}$ can be decomposed in terms of the fluid 4-velocity as

$$\delta N^\mu = \delta n\, u^\mu + n^\mu, \tag{1.29}$$

$$\delta T^{\mu\nu} = \delta\varepsilon\, u^\mu u^\nu + u^\mu h^\nu + u^\nu h^\mu + \Pi^{\mu\nu}, \tag{1.30}$$

where n^μ is the net-charge diffusion 4-current, h^μ is the energy 4-diffusion and $\Pi^{\mu\nu}$ is the component of $\delta T^{\mu\nu}$ that is doubly orthogonal to u^μ, which can be further separated in terms of a traceless part, defined as $\pi^{\mu\nu}$, and a component that carries the trace, defined as -3Π. In this case, one obtains

$$\delta T^{\mu\nu} = \delta\varepsilon\, u^\mu u^\nu + u^\mu h^\nu + u^\nu h^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi, \tag{1.31}$$

with $\Pi \equiv -\frac{1}{3}\Delta^{\alpha\beta}\Pi_{\alpha\beta}$ being the bulk viscous pressure and $\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta}\Pi^{\alpha\beta}$ being the shear-stress tensor. Here, we have defined the double traceless symmetric projection operator onto the 3-space orthogonal to the fluid 4-velocity u^μ.

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}. \tag{1.32}$$

Then, it is possible to recast the conserved currents in the following form

$$T^{\mu\nu} = (\varepsilon_0 + \delta\varepsilon)u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + u^\mu h^\nu + u^\nu h^\mu + \pi^{\mu\nu}, \tag{1.33a}$$

$$N^\mu = (n_0 + \delta n)u^\mu + n^\mu. \tag{1.33b}$$

Here, the bulk viscous pressure Π can immediately be understood as a viscous correction to the thermodynamic pressure P_0, defined by an equilibrium equation of state $P_0 = P_0(\mu, T)$.

Naturally, the conservation laws (1.11) remain valid also in the presence of nonequilibrium effects in the system. In this case, the inclusion of dissipative terms in Eqs. (1.33a) and (1.33b) lead to the occurrence of new terms in addition to the ones that already appeared in the equations of motion for an ideal fluid, Eqs. (1.21). The general equations of motion for a dissipative relativistic fluid then read

$$\dot{\varepsilon}_0 + \dot{\delta\varepsilon} + (\varepsilon + P_0 + \Pi)\theta + \partial_\mu h^\mu + u_\mu \dot{h}^\mu - u_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.34a}$$

$$(\varepsilon + P_0 + \Pi)\dot{u}^\lambda - \nabla^\lambda (P_0 + \Pi) + \Delta^{\lambda\nu}\dot{h}_\nu + h^\mu \theta + h^\mu \Delta^{\lambda\nu}\partial_\mu u_\nu + \Delta^\lambda_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.34b}$$

$$\dot{n}_0 + \dot{\delta n} + n\theta + \partial_\mu n^\mu = 0. \tag{1.34c}$$

One can straightforwardly recover the equations of motion for the ideal case, Eqs. (1.21), by simply setting to zero the dissipative currents Π, n^μ, h^μ and $\pi^{\mu\nu}$, as well as the nonequilibrium corrections to the net-charge and energy densities, δn and $\delta\varepsilon$, respectively.

In the case of dissipative fluids, the conservation laws (1.34) comprise five equations that feature nineteen independent degrees of freedom, namely n_0, ε_0, δn, $\delta\varepsilon$, Π, u^μ, h^μ, n^μ and $\pi^{\mu\nu}$. Nevertheless, the conserved currents N^μ and $T^{\mu\nu}$ contain a total of fourteen degrees of freedom. The five additional degrees of freedom can be removed by the imposition of the so-called matching conditions, which fix the values of T, μ and u^μ of the fictitious equilibrium state. This will be discussed in the following.

1.3.1 Matching conditions

Before proceeding to the actual derivation of the explicit form of the dissipative currents and their respective contributions to the nonequilibrium equations of motion, it is crucial to state the physical meaning of the fluid-dynamical variables that feature Eqs. (1.33). The most widespread approaches are due to Eckart [36] and Landau and Lifshitz [37], the so-called Eckart and Landau picture, respectively. In both cases, n_0 and ε_0 are fixed as the net-charge density and the energy density of the fluid in the local rest frame, respectively,

$$\varepsilon = u_\mu u_\nu T^{\mu\nu} \equiv \varepsilon_0(T, \mu), \tag{1.35}$$

$$n = u_\mu N^\mu \equiv n_0(T, \mu). \tag{1.36}$$

This implies that the corresponding nonequilibrium corrections to the energy and net-charge densities are identically zero,

$$\delta n = 0, \quad \delta\varepsilon = 0. \tag{1.37}$$

The next step is to provide a definition of the fluid 4-velocity. For dissipative fluids, there is no frame where the flow of energy and net-charge are simultaneously zero. In the following, we shall discuss the two main definitions widely used in relativistic fluid dynamics.

In the Eckart picture [36], the 4-velocity is defined by following the net-charge 4-current,

$$N^\mu \equiv n u^\mu, \tag{1.38}$$

which implies that the net-charge diffusion 4-current is zero, $n^\mu = 0$.

In the Laudau-Lifshitz picture [37], the 4-velocity is defined by following the energy 4-current, as an eigenvector of the energy-momentum tensor

$$u_\nu T^{\mu\nu} \equiv \varepsilon u^\mu, \tag{1.39}$$

which in turn implies that the energy diffusion 4-current is zero, $h^\mu = 0$. Throughout this thesis, we shall only employ the Landau-Lifshitz picture. Within this choice of matching conditions, the fluid-dynamical equations (1.34) reduce to

$$\dot{\varepsilon}_0 + (\varepsilon + P_0 + \Pi)\theta - u_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.40a}$$

$$(\varepsilon + P_0 + \Pi)\dot{u}^\lambda - \nabla^\lambda (P_0 + \Pi) + \Delta^\lambda_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{1.40b}$$

$$\dot{n}_0 + n\theta + \partial_\mu n^\mu = 0. \tag{1.40c}$$

In fact, we note that the above equations comprise five independent equations that feature fourteen independent fields, $n, \varepsilon, \Pi, u^\mu, n^\mu$ and $\pi^{\mu\nu}$. Hence, nine additional relations for the dissipative currents are required to close this set of equations. In the following sections, we revisit two widely explored approaches to this problem, namely the relativistic Navier-Stokes theory and the Israel-Stewart theory.

1.4 Navier-Stokes theory

It was shown in Sec. 1.2 that the entropy 4-current of an ideal fluid satisfies a continuity equation, which implies that for an ideal fluid, the entropy itself is always conserved, as displayed in Eq. (1.25). This occurs due to the fact that in an ideal fluid, all fluid elements are always in thermodynamic equilibrium, and the second law of thermodynamics further implies that, in such case, the variation of entropy must be zero. However, for a dissipative fluid, this statement is no longer valid. Instead, the occurrence of irreversible thermodynamic processes, due to heat transfer and friction, will lead to an increase of the entropy of the fluid. Therefore, for dissipative fluids, the entropy 4-current is no longer expected to satisfy a continuity equation.

The first efforts to derive an expression for the non-equilibrium entropy 4-current were addressed independently by Eckart [36] and Landau and Lifshitz [37]. In both cases, the equations of motion for the conservation laws, Eqs. (1.34), combined with fundamental thermodynamic relations, were used to obtain the following identity

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} n^\mu \right) = \frac{1}{T} \pi^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{T} \Pi\theta - n^\mu \nabla_\mu \frac{\mu}{T}, \tag{1.41}$$

with $\sigma_{\mu\nu} \equiv \Delta^{\alpha\beta}_{\mu\nu} \partial_\alpha u_\beta$ being the shear tensor. The left-hand side of the above equation was identified as the divergence of the nonequilibrium entropy 4-current, defined as

$$S^\mu_{\mathrm{NS}} \equiv s u^\mu - \frac{\mu}{T} n^\mu. \tag{1.42}$$

Thus, the right-hand side is identified as the entropy production due to dissipative effects. If the dissipative currents are set to zero, one can easily recover the conservation of entropy in an ideal fluid, given by Eq. (1.25). Furthermore, the second law of thermodynamics states that the variation of entropy must be non-negative, leading to the following inequality

$$\frac{1}{T}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{T}\Pi\theta - n^{\mu}\nabla_{\mu}\frac{\mu}{T} \geq 0, \tag{1.43}$$

Then, the simplest form to assure the entropy production is positive definite for any fluid configuration is by assuming that each term must be positive definite separately. Thus, the following *ansatz* is made, leading to the Navier-Stokes equations

$$\Pi \equiv -\zeta\theta, \tag{1.44a}$$

$$n^{\mu} \equiv \kappa_n \nabla^{\mu}\left(\frac{\mu}{T}\right), \tag{1.44b}$$

$$\pi^{\mu\nu} \equiv 2\eta\sigma^{\mu\nu}, \tag{1.44c}$$

where we introduced the bulk viscosity, ζ, the diffusion coefficient, κ_n, and shear viscosity, η. In this case, the entropy production reads

$$\partial_{\mu}S^{\mu} = \frac{1}{2\eta T}\pi_{\mu\nu}\pi^{\mu\nu} + \frac{1}{\zeta T}\Pi^2 - \frac{1}{\kappa_n T}n_{\mu}n^{\mu}. \tag{1.45}$$

Since $n^{\mu}n_{\mu} \leq 0$ and $\pi^{\mu\nu}\pi_{\mu\nu} \geq 0$, the second law of thermodynamics is satisfied as long as ζ, κ_n and η are positive. These transport coefficients dictate the dissipative properties of the fluid, and must be obtained from a microscopic theory, such as kinetic theory.

With the purpose of being the simplest form to ensure the second law of thermodynamics from the Landau-Lifshitz non-equilibrium entropy 4-current, this *ansatz* takes into account only first-order terms in the dissipative currents. Hence, the Navier-Stokes theory is commonly referred to as a *first-order theory*. Although their addition render the solutions rather complicated, other approaches considering higher-order terms were also investigated. In the next section, we explore the Israel-Stewart theory, a *second-order theory*.

Considering the dissipative currents given by the Navier-Stokes equations (1.44), the conservation laws, Eqs. (1.34), can be completely expressed in terms of gradients of the traditional fluid-dynamical variables,

$$\dot{\varepsilon} + (\varepsilon + P - \zeta\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} = 0, \tag{1.46}$$

$$(\varepsilon + P - \zeta\theta)\dot{u}^{\lambda} - \nabla^{\lambda}(P - \zeta\theta) + 2\Delta^{\lambda}_{\nu}\partial_{\mu}(\eta\sigma^{\mu\nu}) = 0, \tag{1.47}$$

$$\dot{n} + n\theta + \partial_{\mu}\left(\kappa_n\nabla^{\mu}\frac{\mu}{T}\right) = 0. \tag{1.48}$$

As we shall demonstrate later in this Chapter, see Section 1.6, the Navier-Stokes equations allows perturbations that propagate with infinite speed and such an acausal behavior leads to unphysical instabilities of the global equilibrium state [38, 39, 44]. This renders the relativistic Navier-Stokes theory unsuitable to describe any fluid in nature and thus one must resort to alternative formulations. So far, the most widespread formalism is due to Israel and Stewart [42] and will be discussed in the following.

1.5 Israel-Stewart theory

The relativistic Navier-Stokes theory is built based on fundamental principles in physics, such as the conservation of net-charge, energy and momentum, and on the second law of thermodynamics. Therefore, it is initially not clear why a theory formulated from such solid foundations can display such fundamental non-physical properties. Naturally, the physical problems that arise from such theory cannot be a product of these properties, and must come from other assumptions made in the derivation process.

The first successful efforts to fix the acausality and instability problems were due to Israel and Stewart's works based on the procedure introduced by Grad for non-relativistic systems [63]. In their works, Israel and Stewart derived fluid dynamics from the second law of thermodynamics [40] and from the relativistic Boltzmann equation [42] using Grad's method of moments generalized for relativistic fluids. Unlike the propositions by Eckart [36] and Landau and Lifshitz [37], in which only terms up to first-order in the dissipative currents were included in the construction of the entropy 4-current, Israel and Stewart further considered the possibility of contributions from second-order terms in the entropy, and therefore this formulation is often referred to as a *second-order theory*.

The second-order terms in the dissipative currents included in the analyses of Israel and Stewart play an essential role in enabling causal signal propagation in a theory, and can be further shown to be an essential feature to its stability and therefore cannot be neglected [47, 48, 53]. This section is dedicated to the derivation of Israel-Stewart theory from the second law of thermodynamics, a phenomenological approach.

1.5.1 Nonequilibrium entropy current

One of the fundamental assumptions of ideal fluid dynamics is that the system must always be in local thermodynamic equilibrium. Naturally, in the presence of dissipation, this assumption no longer holds. Nevertheless, in order to keep using the same variables that are used to describe an ideal fluid (equilibrium variables), we resort to the assumption that a dissipative fluid is in a state that is *close* to local equilibrium. This considerably simplifies the understanding of its dynamics and further simplifies the following calculations.

We then assume that the entropy 4-current S^μ depends only on the conserved currents N^μ and $T^{\mu\nu}$, an assumption previously made when deriving the relativistic Navier-Stokes theory. This is analogous to assuming that S^μ is completely described by the hydrodynamic variables n, ϵ, Π, u^μ, n^μ and $\pi^{\mu\nu}$, and thus it is a function of the type $S^\mu = S^\mu(n, \varepsilon, \Pi, u^\mu, n^\mu, \pi^{\mu\nu})$. The entropy 4-current is then expanded around an equilibrium state in powers of the dissipative currents. Then, Israel and Stewart's approach was based on truncating this series in second order in the dissipative currents. In this case, the entropy 4-current can be written in the following

general form

$$S^\mu = su^\mu - \frac{\mu}{T}n^\mu + Q^\mu + \mathcal{O}(3), \tag{1.49}$$

with Q^μ being the 4-vector that accounts all possible combinations of second order in the dissipative currents. Note that this expansion is constructed so the truncation in first order in the dissipative currents leads to the Landau-Lifshitz entropy, see Eq. (1.41), which is the starting point for the relativistic Navier-Stokes theory. Third-order (or higher) terms in the dissipative currents shall be neglected. In the Landau frame, the explicit form of Q^μ is written as

$$Q^\mu \equiv -\frac{1}{2}u^\mu\left(\delta_0\Pi^2 - \delta_1 n_\alpha n^\alpha + \delta_2\pi_{\lambda\rho}\pi^{\lambda\rho}\right) - \gamma_0\Pi n^\mu - \gamma_1\pi^\mu_\nu n^\nu, \tag{1.50}$$

where δ_0, δ_1, δ_2, γ_0 and γ_1 are scalar functions of the chemical potential and temperature. These functions are transport coefficients and determine certain dissipative properties of the fluid, and must be calculated within a microscopic framework.

Naturally, the entropy 4-current introduced by Israel and Stewart leads to a different entropy production than the one derived in the context of the relativistic Navier-Stokes theory, with several additional terms, including second-order contributions. In this case, the second law of thermodynamics applied to the entropy 4-current considering contributions of second order in the dissipative currents, Eq. (1.49), leads to

$$\partial_\mu S^\mu = \frac{1}{T}\pi^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{T}\Pi\theta - n^\mu\nabla_\mu\frac{\mu}{T} + \partial_\mu Q^\mu \geq 0. \tag{1.51}$$

Hence, the entropy production can be explicitly calculated as

$$\begin{aligned}
&\partial_\mu S^\mu \\
&= \frac{\Pi}{T}\left(-\theta - \frac{1}{2}T\delta_0\Pi\theta - T\gamma_0\partial_\mu n^\mu - \frac{1}{2}Tn^\mu\nabla_\mu\gamma_0 - \frac{1}{2}T\dot{\delta}_0\Pi - T\delta_0\dot{\Pi}\right) \\
&+ n^\mu\left(\nabla_\mu\alpha + \frac{1}{2}\delta_1\theta n_\mu + \delta_1\Delta^\lambda_\mu\dot{n}_\lambda - \gamma_0\nabla_\mu\Pi - \frac{1}{2}\Pi\nabla_\mu\gamma_0 - \gamma_1\Delta^\lambda_\mu\partial_\nu\pi^\nu_\lambda - \frac{1}{2}\pi^\nu_\mu\nabla_\nu\gamma_1 + \frac{1}{2}\dot{\delta}_1 n_\mu\right) \\
&+ \frac{1}{T}\pi^{\mu\nu}\left(\sigma_{\mu\nu} - \frac{1}{2}T\delta_2\Pi\pi_{\mu\nu}\theta - \gamma_1 T\Delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha n_\beta - \frac{1}{2}Tn_\alpha\Delta^{\alpha\beta}_{\mu\nu}\nabla_\beta\gamma_1 - \frac{1}{2}T\dot{\delta}_2\pi_{\mu\nu} - T\delta_2\Delta^{\alpha\beta}_{\mu\nu}\dot{\pi}_{\alpha\beta}\right).
\end{aligned} \tag{1.52}$$

Once again, the simplest way to ensure that the entropy production is always positive for every possible fluid configuration is by imposing that each term on the right-hand side of Eq. (1.52) is positive by itself. This is achieved by assuming that the entropy production is quadratic in the dissipative currents, leading once more to Eq. (1.45). Nevertheless, this *Ansatz* will no longer lead

to constitutive relations for the dissipative currents, but to the so-called Israel-Stewart equations,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \frac{1}{2}\tau_\Pi \Pi\theta - \frac{1}{2}\zeta T \Pi \frac{d}{d\tau}\left(\frac{\tau_\Pi}{\zeta T}\right) - \zeta T \gamma_0 \partial_\mu n^\mu - \frac{1}{2}\zeta T n^\mu \nabla_\mu \gamma_0, \quad (1.53)$$

$$\begin{aligned}\tau_n \Delta^\lambda_\mu \dot{n}_\lambda + n_\mu &= \kappa_n \nabla_\mu \alpha + \frac{1}{2}\tau_n n_\mu \theta + \frac{1}{2} n_\mu \kappa_n \frac{d}{d\tau}\left(\frac{\tau_n}{\kappa_n}\right) \\ &\quad - \frac{1}{2}\kappa_n \pi^\nu_\mu \nabla_\nu \gamma_1 + \kappa_n \gamma_1 \Delta^\lambda_\mu \partial_\nu \pi^\nu_\lambda - \frac{1}{2}\kappa_n \Pi \nabla_\mu \gamma_0 - \kappa_n \gamma_0 \nabla_\mu \Pi, \end{aligned} \quad (1.54)$$

$$\begin{aligned}\tau_\pi \Delta^{\alpha\beta}_{\mu\nu} \dot{\pi}_{\alpha\beta} + \pi_{\mu\nu} &= 2\eta\sigma_{\mu\nu} - \frac{\tau_\pi}{2}\theta\pi_{\mu\nu} - \frac{1}{2}\pi_{\mu\nu}\eta T \frac{d}{d\tau}\left(\frac{\tau_\pi}{\eta T}\right) - 2\eta T \gamma_1 \Delta^{\alpha\beta}_{\mu\nu} \nabla_\beta n_\alpha \\ &\quad + \eta T \Delta^{\alpha\beta}_{\mu\nu} \nabla_\alpha n_\beta, \end{aligned} \quad (1.55)$$

where the relaxation times are identified as

$$\tau_\Pi \equiv T\delta_0\zeta, \quad \tau_n \equiv \kappa_n \delta_1, \quad \tau_\pi \equiv 2T\delta_2\eta. \quad (1.56)$$

We remark that the inclusion of terms of second order in the dissipative currents in the nonequilibrium entropy 4-current leads to equations of motion for the dissipative currents rather than constitutive relations, as it was the case in the relativistic Navier-Stokes theory. In Israel-Stewart theory the dissipative currents satisfy relaxation-type equations, in which the gradients of the primary fluid-dynamical variables act as source terms. The dominant source terms are the ones appearing in the Navier-Stokes relations and define the transport coefficients η, ζ, κ_n, related to the fluid viscosity. The novel transport coefficients, τ_π, τ_n, and τ_Π, are relaxation times, which determine how fast the dissipative currents relax to their respective Navier-Stokes contributions. Finally, γ_0 and γ_1 are coupling parameters, which couple different dissipative currents.

1.6 Linear causality and stability

In order to be suitable to describe realistic physical systems, fluid-dynamical formulations must satisfy two fundamental properties in the linear regime, namely causality and stability. The former dictates that perturbations in a system must propagate with subluminal velocity, while the latter imposes that such perturbations must be exponentially damped with time. In general, in order for these properties to be simultaneously satisfied, the transport coefficients that feature the fluid-dynamical equations cannot assume arbitrary values. In this context, a linear causality and stability analysis consists in constraining the transport coefficients such that the aforementioned properties are both fulfilled.

In order to investigate these properties, we assume a fluid initially in a global equilibrium state with 4-velocity u^μ_0, energy density ε_0 and net-charge n_0. We then perform *small* perturbations (hereon denoted by δ) on all hydrodynamic variables around such state,

$$\begin{aligned} &\varepsilon = \varepsilon_0 + \delta\varepsilon, \quad n = n_0 + \delta n, \quad u^\mu = u^\mu_0 + \delta u^\mu, \\ &\Pi = \delta\Pi, \quad n^\mu = \delta n^\mu, \quad \pi^{\mu\nu} = \delta\pi^{\mu\nu}. \end{aligned} \quad (1.57)$$

We then linearize the fluid-dynamical equations, neglecting all terms of second order (or higher) in perturbations. The conservation laws (1.34) become

$$u_0^\mu \partial_\mu \delta\varepsilon + (\varepsilon_0 + P_0)\partial_\mu \delta u^\mu = 0, \tag{1.58a}$$

$$u_0^\mu \partial_\mu \delta n + n_0 \partial_\mu \delta u^\mu + \partial_\mu \delta n^\mu = 0, \tag{1.58b}$$

$$(\epsilon_0 + P_0) u_0^\mu \partial_\mu \delta u^\lambda - \Delta_0^{\lambda\nu}\partial_\nu(\delta P + \delta\Pi) + \partial_\nu \delta\pi^{\lambda\nu} = 0, \tag{1.58c}$$

where we have defined the projection operator onto the 3-space orthogonal to the fluid background 4-velocity as $\Delta_0^{\mu\nu} \equiv g^{\mu\nu} - u_0^\mu u_0^\nu$. Perturbations on the fluid 4-velocity, net-charge diffusion, shear-stress tensor can be approximated, up to first order in perturbations, to be orthogonal to the 4-velocity,

$$u_0^\mu \delta u_\mu = \mathcal{O}(2) \approx 0, \quad u_0^\mu \delta n_\mu = \mathcal{O}(2) \approx 0, \quad u_0^\mu \delta\pi_{\mu\nu} = \mathcal{O}(2) \approx 0. \tag{1.59}$$

It is practical to express the linearized fluid-dynamical equations in Fourier space. In this thesis, we shall adopt the following convention for the Fourier transform

$$\tilde{\mathcal{X}}(k^\mu) = \int d^4x \, \exp\left(-ix_\mu k^\mu\right) \mathcal{X}(x^\mu), \quad \mathcal{X}(x^\mu) = \int \frac{d^4k}{(2\pi)^4} \, \exp\left(ix_\mu k^\mu\right) \tilde{\mathcal{X}}(k^\mu), \tag{1.60}$$

where $k^\mu = (\omega, \mathbf{k})$, with ω being the frequency and $\mathbf{k}$ the wave vector. It is then convenient to define the covariant variables Ω and κ^μ,

$$\Omega \equiv u_0^\mu k_\mu, \quad \kappa^\mu \equiv \Delta_0^{\mu\nu} k_\nu. \tag{1.61}$$

with Ω being the frequency and κ^μ the wave 4-vector in the local rest frame of the unperturbed fluid. Moreover, we further define the covariant wavenumber in the local rest frame of the unperturbed system, κ, as

$$\kappa^\mu \kappa_\mu = -\kappa^2. \tag{1.62}$$

Then, the linearized fluid-dynamical equations (1.58) in Fourier space read

$$\Omega\delta\tilde{\varepsilon} + (\epsilon_0 + P_0)\kappa_\mu \delta\tilde{u}^\mu = 0, \tag{1.63a}$$

$$\Omega\delta\tilde{n} + n_0 \kappa_\mu \delta\tilde{u}^\mu + \kappa_\mu \delta\tilde{n}^\mu = 0, \tag{1.63b}$$

$$(\varepsilon_0 + P_0)\Omega\delta\tilde{u}^\mu - \kappa^\mu(\delta\tilde{P} + \delta\tilde{\Pi}) + \kappa_\nu \delta\tilde{\pi}^{\mu\nu} = 0. \tag{1.63c}$$

The fluid-dynamical perturbations δu^μ, δn^μ and $\delta\pi^{\mu\nu}$ were all shown to be orthogonal to u_0^μ in the linear regime. It is convenient to further decompose them in terms of their components parallel and orthogonal to κ^μ. Overall, a general 4-vector A^μ that is orthogonal to u_0^μ can be expressed as,

$$A^\mu = A_\parallel \kappa^\mu + A_\perp^\mu, \tag{1.64}$$

where $A_\parallel = -\kappa_\mu A^\mu/\kappa$ is the longitudinal component of the 4-vector (divided by κ), and $A_\perp^\mu = \Delta_\kappa^{\mu\nu} A_\nu$ is its transverse projection. For this purpose, we have introduced the projection operator onto the 2-space orthogonal to u_0^μ and κ^μ,

$$\Delta_\kappa^{\mu\nu} \equiv g^{\mu\nu} - u_0^\mu u_0^\nu + \frac{\kappa^\mu \kappa^\nu}{\kappa^2}. \tag{1.65}$$

Similarly, a traceless second-rank tensor, $B^{\mu\nu}$, that is orthogonal to u_0^μ, can be expressed in the following way,

$$B^{\mu\nu} = B_\parallel \frac{\kappa^\mu \kappa^\nu}{\kappa^2} + \frac{1}{3} B_\parallel \Delta_\kappa^{\mu\nu} + B_\perp^\mu \frac{\kappa^\nu}{\kappa} + B_\perp^\nu \frac{\kappa^\mu}{\kappa} + B_\perp^{\mu\nu}, \tag{1.66}$$

with the respective projections being defined as $B_\parallel \equiv \kappa_\mu \kappa_\nu B^{\mu\nu}/\kappa^2$, $B_\perp^\mu \equiv -\kappa^\lambda \Delta_\kappa^{\mu\nu} B_{\lambda\nu}/\kappa$, and $B_\perp^{\mu\nu} \equiv \Delta_\kappa^{\mu\nu\alpha\beta} B_{\alpha\beta}$. Above, we introduced the double, symmetric, and traceless projection operator,

$$\Delta_\kappa^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\Delta_\kappa^{\mu\alpha}\Delta_\kappa^{\nu\beta} + \Delta_\kappa^{\mu\beta}\Delta_\kappa^{\nu\alpha} - \Delta_\kappa^{\mu\nu}\Delta_\kappa^{\alpha\beta}\right). \tag{1.67}$$

We introduced this decomposition in Ref. [48] and demonstrated that the equations of motion for the longitudinal projections of the perturbations decouple from those for the transverse projections. This considerably simplifies the derivation of the dispersion relations. We explain this procedure in detail in what follows.

We now decompose the linearized equations for the conservation laws in Fourier space, Eqs. (1.63), into their components parallel and transverse to κ^μ. We start with the equations of motion for the longitudinal projections of the perturbations. Equations (1.63a) and (1.63b) are Lorentz scalars and are already expressed solely in terms of longitudinal components, namely $\delta\tilde{\epsilon}$, $\delta\tilde{n}$ and $\delta\tilde{u}_\parallel$. The remaining conservation law, Eq. (1.63c), can be projected into the longitudinal direction by contracting it with κ_μ/κ. This leads to the following set of equations,

$$\Omega\delta\tilde{\epsilon} - (\epsilon_0 + P_0)\kappa\delta\tilde{u}_\parallel = 0, \tag{1.68a}$$

$$\Omega\delta\tilde{n} - n_0\kappa\delta\tilde{u}_\parallel - \kappa\delta\tilde{n}_\parallel = 0, \tag{1.68b}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_\parallel - \kappa(\delta\tilde{P} + \delta\tilde{\Pi} + \delta\tilde{\pi}_\parallel) = 0. \tag{1.68c}$$

An equation of motion containing the transverse components of the perturbations is obtained by projecting Eq. (1.63c) with $\Delta^\nu_{\mu,\kappa}$,

$$(\varepsilon_0 + P_0)\Omega\delta\tilde{u}^\nu_\perp - \kappa\delta\tilde{\pi}^\nu_\perp = 0. \tag{1.69}$$

As already mentioned, we can see that the equations of motion for the longitudinal perturbations decouple from those of the transverse perturbations. Naturally, these longitudinal and transverse projections of the conservation laws – depicted in Eqs. (1.68) and (1.69), respectively – must be supplemented by constitutive or dynamical relations for the dissipative currents from a particular fluid-dynamical formulation. Solving these equations will then lead to a dispersion relation $f(\omega, k) = 0$, whose solutions, $\omega = \omega(k)$, are referred to as the modes of the theory and describe how the original perturbations evolve in spacetime. The solutions related to the longitudinal perturbations will be referred to as longitudinal modes, while the solutions related to the transverse perturbations will be referred to as transverse modes.

The modes of a theory depend, in general, on the thermodynamic and transport properties of the fluid as well as on the properties of the background, i.e., temperature, chemical potential and velocity of the unperturbed fluid. A linear analysis will determine if there are conditions

on the thermodynamic and transport properties of the fluid that may render the perturbations causal and stable. This may be investigated by ensuring that the imaginary part of the modes is positive-definite (given our sign convention for the Fourier transform) and their asymptotic group velocity is smaller than 1. This task will be performed for both fluid-dynamical theories discussed in this chapter, namely Navier-Stokes and Israel-Stewart theories.

1.6.1 Navier-Stokes theory

We first investigate the causality and stability of the relativistic Navier-Stokes theory. In Navier-Stokes theory, the conservation laws are supplemented by constitutive relations, which relate the dissipative currents to gradients of the fluid-dynamical variables. In the linear regime, these can be expressed in the following way in Fourier space,

$$\delta\tilde{\Pi} = i\zeta_0\kappa\delta\tilde{u}_{\parallel}, \tag{1.70a}$$

$$\delta\tilde{n}^{\mu} = i\kappa_{n,0}\kappa^{\mu}\delta\tilde{\alpha}, \tag{1.70b}$$

$$\delta\tilde{\pi}^{\mu\nu} = i\eta_0\left(\kappa^{\mu}\delta\tilde{u}^{\nu} + \kappa^{\nu}\delta\tilde{u}^{\mu} + \frac{2}{3}\Delta_0^{\mu\nu}\kappa\delta\tilde{u}_{\parallel}\right), \tag{1.70c}$$

where ζ_0, $\kappa_{n,0}$, and η_0 denote the transport coefficients calculated at the temperature and chemical potential of the unperturbed fluid. For the sake of simplicity, in the following we neglect any effects due to perturbations of the diffusion 4-current.

Transverse modes

We start with the equations of motion for the transverse perturbations, since these are considerably simpler. We first obtain the partially transverse projection of the constitutive relation satisfied by the shear-stress tensor by projecting Eq. (1.70c) with $\kappa_\nu\Delta^{\lambda}_{\mu,\kappa}/\kappa$,

$$\delta\tilde{\pi}^{\lambda}_{\perp} = i\eta_0\kappa\delta\tilde{u}^{\lambda}_{\perp}. \tag{1.71}$$

Then, inserting Eq. (1.71) in Eq. (1.69), we obtain

$$\left(\Omega - i\tau_\eta\kappa^2\right)\delta\tilde{u}^{\mu}_{\perp} = 0, \tag{1.72}$$

where we have defined the intrinsic hydrodynamic timescale $\tau_\eta = \eta_0/(\varepsilon_0 + P_0)$. This leads to the following dispersion relation,

$$\Omega - i\tau_\eta\kappa^2 = 0. \tag{1.73}$$

Considering perturbations on a static fluid, $u_0^{\mu} = (1,0,0,0)$, we have that $\Omega = \omega$ and $\kappa = k$, leading to the following mode,

$$\omega = i\tau_\eta k^2. \tag{1.74}$$

Since it is a purely imaginary mode, with a positive sign, it describes the exponential damping of the transverse perturbations – hence being stable. Furthermore, it is a hydrodynamic mode,

that is, $\lim_{k\to 0}\omega(k) = 0$, and describes long-lived excitations in the small wavenumber limit. We also note that this mode is typically obtained when solving the diffusion equation[1] and thus implies that the transverse components of the velocity perturbations diffuse into the background fluid. Nevertheless, this diffusion-like dispersion relation leads to nontrivial and pathological consequences in the relativistic regime. In fact, diffusive modes are well known to lead to perturbations that propagate with infinite speed [86] and, as already mentioned, in a relativistic framework, such an acausal behavior leads to intrinsic and unphysical instabilities.

This can be seen analyzing the dispersion relation for perturbations on a moving fluid. For the sake of simplicity, we consider perturbations propagating in the same direction of the velocity of the unperturbed fluid, e.g. $u_0^\mu = \gamma(1, V_0, 0, 0)$ and $k^\mu = (\omega, k, 0, 0)$. In this case, $\Omega = \gamma(\omega - V_0 k)$ and $\kappa^2 = \gamma^2(\omega V_0 - k)^2$, with $\gamma = 1/\sqrt{1 - V_0^2}$. Then, the dispersion relation (1.73) can be rewritten as,

$$\gamma(\omega - V_0 k) - i\tau_\eta \gamma^2(\omega V_0 - k)^2 = 0. \tag{1.75}$$

In this boosted frame, the diffusive-like dispersion relation becomes quadratic in the frequency ω and an additional solution appears when compared to the case of perturbations on a static fluid,

$$\omega_\pm(k) = \frac{1 + 2i\tau_\eta\gamma V_0 k \pm \sqrt{1 + \frac{4i\tau_\eta V_0 k}{\gamma}}}{2i\tau_\eta\gamma V_0^2}. \tag{1.76}$$

In the limit of small wavenumber, $k \to 0$, the modes can be expressed in the following simple form,

$$\omega_+ \sim V_0 k + i\tau_\eta \frac{k^2}{\gamma^3}, \tag{1.77a}$$

$$\omega_- \sim -\frac{i}{\gamma\tau_\eta V_0^2}. \tag{1.77b}$$

We see that the first solution is a boosted version of the diffusion mode, while the other solution is an intrinsically new nonhydrodynamic mode, i.e., a mode for which $\lim_{k\to 0}\omega(k) \neq 0$. Nonhydrodynamic modes are associated with short-lived excitations in the small wavenumber limit and, for this reason, do not usually appear in fluid-dynamical theories – hence the name nonhydrodynamic. It certainly does not appear in the nonrelativistic Navier-Stokes theory, even when considering perturbations on a moving fluid. It is thus interesting that such a mode emerges when boosting the dispersion relation of a relativistic theory. Here, we can see that this happened due to the parabolic nature of the diffusion-like dispersion relation, which is linear in the frequency but quadratic in the wavenumber. Since the Lorentz transformation mixes frequency and wavenumber, a quadratic term in the wavenumber will yield a quadratic term in the frequency when the system is boosted, leading to the additional solution obtained and discussed above. However, the most striking feature of this novel mode is that it is *unstable* –

[1] The diffusion equation for the temperature in a given medium, $\partial_t T = D\nabla^2 T$, can be written in Fourier space as $(\omega - iDk^2)\tilde{T} = 0$, leading to the typical diffusive mode $\omega \sim ik^2$.

that is, its imaginary part is negative-definite, and such unphysical behavior cannot be fixed by tuning the transport coefficients. We remark that this ill-defined instability will always emerge for any diffusion-like dispersion relation due to its parabolic nature. This unphysical instability of the global equilibrium state in relativistic Navier-Stokes theory was already mentioned a few times in this thesis and it renders the theory unsuitable to describe any fluid in nature.

Longitudinal modes

We now discuss the dispersion relations obtained for the longitudinal perturbations. In this case, the constitutive relations satisfied in Navier-Stokes theory become

$$\delta\tilde{\Pi} = i\zeta_0\kappa\delta\tilde{u}_{\parallel}, \tag{1.78}$$

$$\delta\tilde{\pi}_{\parallel} = \frac{4}{3}i\eta_0\kappa\delta\tilde{u}_{\parallel}. \tag{1.79}$$

Therefore, Eqs. (1.68) can be recast as,

$$\Omega\delta\tilde{\epsilon} - \kappa(\epsilon_0 + P_0)\delta\tilde{u}_{\parallel} = 0, \tag{1.80}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_{\parallel} - \kappa\delta\tilde{P} - i\kappa^2\left(\zeta_0 + \frac{4}{3}\eta_0\right)\delta\tilde{u}_{\parallel} = 0. \tag{1.81}$$

These equations can be conveniently cast in the following matrix form

$$\begin{pmatrix} \Omega & -\kappa \\ -c_{\rm s}^2\kappa & \Omega - i\tau_{\rm eff}\kappa^2 \end{pmatrix}\begin{pmatrix} \frac{\delta\tilde{\epsilon}}{\epsilon_0+P_0} \\ \delta\tilde{u} \end{pmatrix} = 0, \tag{1.82}$$

with $c_{\rm s}$ being the speed of sound in the fluid,

$$c_{\rm s}^2 = \left.\frac{\partial P_0}{\partial\epsilon_0}\right|_{s_0,n_0}, \tag{1.83}$$

and $\tau_{\rm eff}$ a new timescale, defined as

$$\tau_{\rm eff} \equiv \frac{\zeta_0 + \frac{4}{3}\eta_0}{\epsilon_0 + P_0}. \tag{1.84}$$

Non-trivial solutions are obtained when the determinant of the matrix on the left-hand side of Eq. (1.82) is equated to zero, leading to the following dispersion relation

$$\Omega^2 - i\tau_{\rm eff}\Omega\kappa^2 - c_{\rm s}^2\kappa^2 = 0. \tag{1.85}$$

Then, the longitudinal modes of Navier-Stokes theory for perturbations on a fluid at rest, $u_0^\mu = (1,0,0,0)$, in which $\Omega = \omega$ and $\kappa = k$, are given by

$$\omega(k) = \frac{i\tau_{\rm eff}k^2 \pm k\sqrt{4c_{\rm s}^2 - \tau_{\rm eff}^2k^2}}{2}. \tag{1.86}$$

In the limit of ideal fluid dynamics, i.e., $\tau_{\rm eff} \to 0$, we recover the dispersion relation of an ideal fluid, $\omega(k) = \pm c_{\rm s}k$, which is the typical dispersion relation of the wave equation in one dimension, and we thus identify $c_{\rm s}$ as the speed of sound.

We now look at the asymptotic behavior of these modes for small and large wave numbers,

$$\omega(k \to 0) = \pm c_s k + i\tau_{\rm eff}\frac{k^2}{2} + \mathcal{O}(k^3) \tag{1.87}$$

$$\omega(k \to \infty) = i\tau_{\rm eff}\frac{k^2}{2}(1 \pm 1). \tag{1.88}$$

In the small wavenumber limit, the real part of the dispersion relation describes a sound wave propagating with velocity given by $c_{\rm s}$, whereas the imaginary part describes the exponential damping of the perturbations. The time scale $\tau_{\rm eff}$ then dictates the intensity with which these modes are damped. In the large wavenumber limit, there is the occurrence of a trivial mode ($\omega = 0$) and a diffusive mode ($\omega \sim ik^2$). The occurrence of this diffusion-like mode for large wave numbers will lead to the same pathologies discussed in the previous subsection for the transverse modes. These can be systematically discussed by analyzing the dispersion relation for a moving background fluid, but we refrain from performing this analysis here, since it would not lead to any new insight. Nevertheless, this discussion was performed in detail in Ref. [87].

We thus conclude that both transverse and longitudinal perturbations of a fluid in global equilibrium described by the Navier-Stokes theory yield parabolic, diffusion-like dispersion relations, that are known to display acausal behavior. As discussed, in a relativistic setting, such a behavior leads to intrinsic and unphysical instabilities and this is the reason that causal and hyperbolic versions of the theory must be developed.

1.6.2 Israel-Stewart theory

We now consider the linear stability analysis of Israel-Stewart theory. In particular, we shall consider a simple limit of the Israel-Stewart theory, in which any dissipation other than due to shear-stress is identically neglected. That is, we set perturbations of bulk viscosity and net-charge diffusion to zero, i.e., $\delta\tilde{n} = 0$, $\delta\tilde{n}^\mu = 0$, and $\delta\tilde{\Pi} = 0$. In this case, the only nontrivial equation of motion is Eq. (1.55), which, in the linear regime, reduces to

$$(i\Omega\tau_\pi + 1)\,\delta\tilde{\pi}^{\mu\nu} = i\eta\left(\kappa^\mu\delta\tilde{u}^\nu + \kappa^\nu\delta\tilde{u}^\mu - \frac{2}{3}\Delta_0^{\mu\nu}\kappa_\lambda\delta\tilde{u}^\lambda\right) + \mathcal{O}(\delta^2), \tag{1.89}$$

which must be coupled with the conservation laws (1.63). Analogously to the previous section, we shall analyze the transverse and longitudinal modes of the theory separately.

Transverse modes

The conservation of momentum (1.69) is coupled with the transverse component of the shear-stress tensor $\delta\tilde{\pi}^\mu_\perp$, which can be obtained by projecting Eq. (1.89) with $\kappa_\mu\Delta^\lambda_{\nu,\kappa}/\kappa$,

$$(i\tau_\pi\Omega + 1)\delta\tilde{\pi}^\lambda_\perp = i\eta\kappa\delta\tilde{u}^\lambda_\perp. \tag{1.90}$$

Then, from Eqs. (1.69) and (1.90), we obtain the dispersion relation associated with the transverse modes

$$\tau_\pi \Omega^2 - i\Omega - \tau_\eta \kappa^2 = 0. \tag{1.91}$$

Considering perturbations on a fluid at rest, the transverse modes read

$$\omega^{\text{shear}}_{T,\pm} = i\frac{1 \pm \sqrt{1 - 4\tau_\eta \tau_\pi k^2}}{2\tau_\pi}. \tag{1.92}$$

We observe the existence of a hydrodynamic and a nonhydrodynamic mode, given by $\omega^{\text{shear}}_{T,-}$ and $\omega^{\text{shear}}_{T,+}$, respectively. This contrasts with what was observed for Navier-Stokes theory, in which there was only a single hydrodynamic mode. In particular, the occurrence of a nonhydrodynamic mode is a consequence of the inclusion of the dynamics of a non-conserved current. In the Israel-Stewart theory, the dissipative currents satisfy equations of motion rather than constitutive relations, as in the case in Navier-Stokes theory, thus yielding nonhydrodynamic modes even for perturbations on a fluid at rest. In particular, such modes act as regulators of the theory [88] and are, in fact, essential to guarantee causality and stability are satisfied in the linear regime.

In the small wavenumber limit, the transverse modes of the Israel-Stewart theory become

$$\omega^{\text{shear}}_{T,+} = \frac{i}{\tau_\pi} - i\tau_\eta k^2 + \mathcal{O}(k^4), \tag{1.93a}$$

$$\omega^{\text{shear}}_{T,-} = i\tau_\eta k^2 + i\tau_\eta^2 \tau_\pi k^4 + \mathcal{O}(k^6). \tag{1.93b}$$

Since the relaxation times are positive-definite transport coefficients, we observe that linear stability is always satisfied in the small wavenumber limit, as the dominant imaginary part of the modes is positive-definite. Furthermore, we observe that the shear relaxation time dictates how fast the perturbations are damped in the system.

In the large wavenumber limit, on the other hand, these modes read

$$\omega_\pm = \frac{i}{2\tau_\pi} \pm \left(\sqrt{\frac{\tau_\eta}{\tau_\pi}}\, k - \frac{1}{8\sqrt{\tau_\eta \tau_\pi^3}\, k} \right) + \mathcal{O}\left(\frac{1}{k^4}\right). \tag{1.94}$$

Therefore, in such limit, the modes are no longer purely imaginary and possess a nonzero real part – that is, the modes become propagating as well as damped. In particular, these modes describe perturbations that propagate subluminally as long as the asymptotic group velocity is smaller than the speed of light, which is ensured if, and only if, the following condition is satisfied [47, 48, 53, 89]

$$\lim_{k\to\infty} \left| \frac{\partial \text{Re}(\omega)}{\partial k} \right| \leq 1 \Longrightarrow \tau_\pi \geq \frac{\eta}{\varepsilon_0 + P_0}. \tag{1.95}$$

Thus, the condition of linear causality imposes a constraint on the shear relaxation time coefficient.

We have shown that the Israel-Stewart theory is always linearly stable for perturbations on a fluid at rest, whereas causality is satisfied as long as the transport coefficients satisfy Eq. (1.95). We now investigate the case of perturbations on a moving fluid, in which case the dispersion relation (1.91) becomes

$$\tau_\pi \gamma^2 (\omega - V_0 k)^2 - i\gamma(\omega - V_0 k) - \tau_\eta \gamma^2 (\omega V_0 - k)^2 = 0. \tag{1.96}$$

We remark that, unlike what was observed for Navier-Stokes theory, perturbations on a moving fluid do not lead to the occurrence of an additional mode. This is due to the fact that the Israel-Stewart equations are hyperbolic and treat time and spacelike derivatives in equal footing, thus leading to dispersion relations in which the terms of higher order in frequency and wavenumber have the same power.

The modes in this case are

$$\omega^{\text{shear}}_{T,\pm} = \frac{i + 2kV_0\gamma(\tau_\pi - \tau_\eta) \pm \sqrt{4\tau_\eta\tau_\pi k^2\gamma^2(1-V_0)^2 - 4ik\tau_\eta V_0\gamma(1-V_0) - 1}}{2\gamma(\tau_\pi - \tau_\eta V_0^2)}. \tag{1.97}$$

For vanishing wavenumber, the modes reduce to

$$\omega^{\text{shear}}_{T,-}(k=0) = 0, \quad \omega^{\text{shear}}_{T,+}(k=0) = \frac{i}{\gamma(\tau_\pi - \tau_\eta V_0^2)}. \tag{1.98}$$

Therefore, in this regime, stability is guaranteed as long as $\tau_\pi > \tau_\eta V_0^2$. The theory must be stable for *any* value of the background fluid velocity. In particular, the strongest constraint is then obtained when the background fluid velocity is maximal, leading to the stability condition $\tau_\pi > \tau_\eta$, which is exactly the causality condition (1.95) obtained for perturbations on a fluid at rest. In the small wavenumber limit, these modes become

$$\omega^{\text{shear}}_{T,+} = \frac{i}{\gamma(\tau_\pi - V_0^2\tau_\eta)} + \frac{[\tau_\pi + (2 - V_0^2)\tau_\eta]}{\tau_\pi - V_0^2\tau_\eta} V_0\, k - i\tau_\eta k^2 + \mathcal{O}(k^4), \tag{1.99a}$$

$$\omega^{\text{shear}}_{T,-} = V_0 k + i\tau_\eta k^2 + \mathcal{O}(k^4). \tag{1.99b}$$

We remark that these modes are the analogue of Eq. (1.93) for perturbations on a moving fluid. As a matter of fact, taking $V_0 = 0$, one immediately recovers the modes for perturbations on a fluid at rest, given in the aforementioned equations.

We thus conclude that perturbations on a moving fluid lead to stability conditions that are identical to the causality conditions obtained for perturbations on a fluid at rest [47, 48, 53]. In the linear regime, these two properties are intrinsically connected and thus can only be satisfied simultaneously.

Longitudinal modes

For the sake of completeness, we now perform a causality and stability analysis for the longitudinal modes of Israel-Stewart theory. In particular, in the absence of bulk viscous pressure

and net-charge diffusion, Eqs. (1.68) reduce to

$$\Omega\delta\tilde{\epsilon} - \kappa(\epsilon_0 + P_0)\delta\tilde{u}_\parallel = 0, \tag{1.100}$$

$$(\epsilon_0 + P_0)\Omega\delta\tilde{u}_\parallel - \kappa(\delta\tilde{P} + \delta\tilde{\pi}_\parallel) = 0. \tag{1.101}$$

The longitudinal projection of the shear-stress tensor is obtained by projecting Eq. (1.89) with $\kappa_\mu\kappa_\nu/\kappa^2$, thus leading to

$$\delta\tilde{\pi}_\parallel = \frac{4\eta}{3(i\tau_\pi\Omega + 1)} i\kappa\delta\tilde{u}_\parallel. \tag{1.102}$$

These equations can be written in the following matrix form

$$\begin{pmatrix} \Omega & -\kappa \\ -c_{\rm s}^2\kappa(i\Omega\tau_\pi + 1) & \Omega(i\Omega\tau_\pi + 1) - i\frac{4}{3}\tau_\eta\kappa^2 \end{pmatrix} \begin{pmatrix} \frac{\delta\tilde{\epsilon}}{\epsilon_0 + P_0} \\ \delta\tilde{u}_\parallel \end{pmatrix} = 0, \tag{1.103}$$

leading to the following dispersion relation when the determinant of the matrix on the left-hand side equates to zero,

$$i\hat{\tau}_\pi\hat{\Omega}^3 + \hat{\Omega}^2 - \frac{4}{3}i\hat{\kappa}^2\hat{\Omega} - i\hat{\tau}_\pi c_{\rm s}^2\hat{\kappa}^2\hat{\Omega} - c_{\rm s}^2\hat{\kappa}^2 = 0. \tag{1.104}$$

The solutions of Eq. (1.104) are, in general, rather intricate and thus shall not be displayed here in their complete form. Instead, as was done in the previous sections, we once again resort to analyzing these modes in the asymptotic limits of wavenumber. In the small wavenumber limit, the longitudinal modes become

$$\omega^{\pm}_{\rm sound} = \pm c_{\rm s}\hat{k} + \frac{2}{3}i\hat{k}^2 + \mathcal{O}\left(\hat{k}^3\right), \tag{1.105}$$

$$\omega_{\rm shear} = \frac{i}{\hat{\tau}_\pi} - \frac{4}{3}i\hat{k}^2 + \mathcal{O}\left(\hat{k}^4\right). \tag{1.106}$$

In the small wavenumber limit, the longitudinal modes of Israel-Stewart theory are identical to the corresponding solutions obtained for Navier-Stokes, Eq. (1.87), with the addition of a novel nonhydrodynamic mode. Once again, since the relaxation times are positive-definite quantities, linear stability is always satisfied in this regime.

In the large wavenumber limit, the longitudinal modes become

$$\omega^{\pm}_{\rm sound} = \pm\sqrt{\frac{4 + 3c_{\rm s}^2\hat{\tau}_\pi}{3\hat{\tau}_\pi}}\hat{k} + \frac{2i}{\hat{\tau}_\pi\left(4 + 3c_{\rm s}^2\hat{\tau}_\pi\right)} + \mathcal{O}\left(\frac{1}{\hat{k}}\right), \tag{1.107}$$

$$\omega_{\rm shear} = \frac{3c_{\rm s}^2}{4 + 3c_{\rm s}^2\hat{\tau}_\pi}i + \mathcal{O}\left(\frac{1}{\hat{k}}\right). \tag{1.108}$$

Linear causality is satisfied as long as the real part of the sound modes is smaller than 1, thus leading to the following constraint [47, 48, 53]

$$\hat{\tau}_\pi \geq \frac{4}{3(1 - c_{\rm s}^2)}. \tag{1.109}$$

In particular, considering the equation of state for an ultrarelativistic gas, $c_{\mathrm{s}}^2 = 1/3$, the causality condition becomes simply

$$\hat{\tau}_\pi \geq 2 \Longrightarrow \tau_\pi \geq \frac{2\eta}{\varepsilon_0 + P_0}. \tag{1.110}$$

Similarly to what was observed in the previous subsection, considering longitudinal perturbations on a moving fluid, one obtains a stability condition that is identical to the causality condition (1.110). This discussion will not be further developed in this thesis, since it will not lead to any additional insight, and the reader is referred to Ref. [87] for a thorough analysis.

1.7 Discussion

In this chapter, we have investigated the phenomenological derivation of relativistic dissipative fluid-dynamical theories. First, we developed a set of thermodynamic relations to employ in a macroscopic description of a system in equilibrium. We then started the study of relativistic fluid dynamics with the simplest possible case of ideal fluids. In this case, the system is always in local equilibrium, i.e., each fluid element is a thermodynamic system in equilibrium. We showed that the entropy is conserved for ideal fluids and its evolution can be completely described using solely the conservation laws and an equation of state.

We then extended this discussion to account also for dissipation in the fluid in the form of heat and friction between fluid elements. In this case, the system can no longer be assumed to be in local equilibrium and the conserved currents must be appropriately corrected by the inclusion of novel nonequilibrium effects. Nevertheless, we assumed that the system is *close* to a (fictitious) equilibrium state, such that it can be described in terms of a reduced number of degrees of freedom. Unlike the ideal case, in order to describe the dynamics of dissipative fluids, the conservation laws must be supplemented by relations for the dissipative currents. In particular, in this chapter we investigated the derivation of Navier-Stokes and Israel-Stewart theories.

The presence of dissipative currents in the system generates entropy. Therefore, unlike the ideal case, the entropy is not conserved in dissipative fluids. However, there is not an exact form for the entropy 4-current out of equilibrium. We first considered a nonequilibrium entropy 4-current that contains terms up to first order in the dissipative currents. The entropy production was then assumed to be positive-definite by imposing the simplest possible *ansatz* in which it is quadratic in the dissipative currents, thus satisfying the second law of thermodynamics. In this case, we obtain the relativistic Navier-Stokes theory of fluid dynamics, a set of parabolic equations that express the dissipative currents as gradients of temperature, chemical potential and 4-velocity. Nevertheless, this theory is acausal and intrinsically unstable and cannot be used to describe any fluid in nature.

The pathologies displayed by Navier-Stokes theory can be attributed to the definition of

the nonequilibrium entropy 4-current. In order to address the aforementioned issues, Israel and Stewart further included terms of second order in the dissipative currents in the entropy 4-current. Then, when taking the simplest *ansatz* that renders the entropy production positive-definite, i.e., assuming that each term is quadratic, one obtains the Israel-Stewart equations of dissipative fluid dynamics. In this approach, the dissipative currents satisfy hyperbolic equations of motion rather than constitutive relations – that is, they are independent degrees of freedom that evolve according to their own dynamics. The inclusion of second order terms in the entropy 4-current naturally leads to relaxation equations, with the appearance of timescales that are crucial for causality and stability. Unlike the case of Navier-Stokes theory, the Israel-Stewart equations can be rendered causal with an appropriate choice of transport coefficients.

In order to understand the minimal requirements that fluid-dynamical theories must satisfy in order to be applicable for practical purposes, we resort to a linear stability and causality analysis.We showed that the Navier-Stokes equations yield hydrodynamic modes that display a diffusion-like behavior at large values of wavenumber. In particular, the corresponding parabolic dispersion relations always admit a new unstable and acausal solution for perturbations on a moving fluid, regardless of the values of the transport coefficients [38]. These pathologies forbid the application of Navier-Stokes theory in the description of realistic systems. On the other hand, the Israel-Stewart theory was shown to be compatible with linear causality as long as the transport coefficients are consistently constrained according to Eq. (1.110), whereas stability is always satisfied for perturbations on a fluid at rest. When considering perturbations on a moving fluid, we obtain stability conditions that are identical to the aforementioned causality constraints for perturbations on a fluid at rest. Furthermore, we remark that the constraints obtained when analyzing the evolution of the longitudinal degrees of freedom supersede those obtained for the transverse degrees of freedom. Therefore, the Israel-Stewart theory passes such essential tests and is thus suitable to be applicable to numerical fluid-dynamical schemes as long as the transport coefficients are consistently constrained. Finally, we remark that nonlinear constraints on the causality of such formalism have been explored in Refs. [90–92].

2 Relativistic Kinetic Theory

In the previous Chapter, causal and stable relativistic fluid-dynamical equations were derived phenomenologically, using the properties of the Lorentz transformation, the conservation of charge, energy and momentum, and the second law of thermodynamics. Nevertheless, such a derivation procedure is never complete since it does not provide expressions for the equation of state nor the several transport coefficients that appear in the equations of motion. For this purpose, a microscopic derivation of fluid dynamics is required, i.e., one must understand how the fluid-dynamical equations emerge as the long-time, long-distance limit of a given microscopic dynamics. In this thesis, this task will be performed for relativistic dilute gases, that can be microscopically described in terms of the relativistic Boltzmann equation. The primary goal of this chapter is to understand how the Boltzmann equation is derived and what is its domain of applicability. We then start our discussion on how fluid dynamics emerges from kinetic theory – a discussion that has a long history of developments and will be continued to be addressed in Chapter 3.

2.1 Derivation of the Boltzmann equation

Naturally, describing the motion of each individual particle in a gas is an unfeasible and often useless task. In this context, we introduce the distribution function, $f(\mathbf{x}, \mathbf{p}, t)$, a density of particles in phase space – that is, the number of particles lying withing a volume $d^3\mathbf{x}$ around $\mathbf{x}$ whose momenta lie in a momentum-space element $d^3\mathbf{p}$ around $\mathbf{p}$ at a time t. The goal is then to determine how collisions between particles in a dilute system change its momentum distribution, which will eventually lead to the Boltzmann equation.

The total number of particles in an infinitesimal volume in phase space at a given time instant t can be obtained by simply integrating the distribution function over the entire phase space,[1]

$$dN = f(\mathbf{x}, \mathbf{p}, t)\, \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}. \tag{2.1}$$

The goal of this section is to obtain an equation to describe the dynamics of the single-particle distribution function, which shall be described by the means of the Boltzmann equation throughout this thesis. For this purpose, we follow Ref. [59].

[1] The factor $1/(2\pi)^3$ is a requirement from quantum mechanics, in particular from Heisenberg uncertainty principle. A quantum particle occupies a volume $h^3 = (2\pi\hbar)^3$ in phase space, and thus in order to correctly count the number of accessible states, it is necessary to take into account this factor. Furthermore, if spin degrees of freedom are taken into account, even more states are accessible to the system, and one must include a multiplying degeneracy factor g_s as well [59]. However, throughout this thesis, we shall always treat particles of zero spin, such that the degeneracy factor reduces to unity, $g_s = 1$.

The total number of particles inside a volume in phase space changes dynamically as particles travel inwards or outwards this region, either after scatterings with other particles in the medium or under the action of external forces. The infinitesimal change in the number of particles in the time interval dt is $dN(t+dt) - dN(t)$. The phase space is defined by the set of coordinates (x_i, p_i). At a time $t + dt$, these coordinates can be expressed as

$$x' = x(t+dt) = x(t) + \frac{dx(t)}{dt}dt = x(t) + \frac{p}{p^0}dt, \tag{2.2a}$$

$$p' = p(t+dt) = p(t) + \frac{dp(t)}{dt}dt = p(t) + Fdt. \tag{2.2b}$$

To obtain the second equalities, we have used that

$$\frac{d\mathbf{x}}{dt} = \frac{1}{\gamma}\frac{d\mathbf{x}}{d\tau} = \frac{1}{\gamma}\frac{\mathbf{p}}{m} = \frac{m}{p^0}\frac{\mathbf{p}}{m} = \frac{\mathbf{p}}{p^0}, \quad \frac{d\mathbf{p}}{dt} = \frac{1}{\gamma}\frac{d\mathbf{p}}{d\tau} = \frac{\mathbf{K}}{\gamma} = \mathbf{F}, \tag{2.3}$$

where the energy of a particle is given by $p^0 = \gamma m$. Furthermore, $\mathbf{K}$ is the external covariant Lorentz force (commonly referred to as the Minkowski force), whereas $\mathbf{F}$ is its non-relativistic counterpart. Then, the Jacobian of the transformation $(x, p) \to (x', p')$, defined in Eqs. (2.2), for a two-dimensional phase space, is

$$\mathcal{J}(t) = \begin{vmatrix} 1 & \frac{1}{p^0}dt \\ \frac{\partial F}{\partial x}dt & 1 + \frac{\partial F}{\partial p}dt \end{vmatrix} = \left(1 + \frac{\partial F}{\partial p}dt\right) + \mathcal{O}(dt^2). \tag{2.4}$$

This result can be straightforwardly generalized to three dimensions in both space and momentum spaces, leading to

$$d^3\mathbf{x}' \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} = \left(1 + \frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{F}dt\right) d^3\mathbf{x} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} + \mathcal{O}(dt^2). \tag{2.5}$$

In the absence of external forces, the Jacobian of the transformation reduces simply to an identity, in which case the volume in phase space does not change with time, $d^3\mathbf{x}\, d^3\mathbf{p} = d^3\mathbf{x}'\, d^3\mathbf{p}'$.

The rate of change in the number of particles within a volume element in phase space $d^3\mathbf{x}d^3\mathbf{p}$ can thus be expressed as

$$\begin{aligned}
&dN(t+dt) - dN(t) \\
&= f\left(\mathbf{x} + \frac{d\mathbf{x}}{dt}dt, \mathbf{p} + \frac{d\mathbf{p}}{dt}dt, t + dt\right) \mathrm{d}^3\mathbf{x}' \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3} - f(\mathbf{x}, \mathbf{p}, t)\, \mathrm{d}^3\mathbf{x} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \\
&\overset{(2.5)}{=} \left[f\left(\mathbf{x} + \frac{d\mathbf{x}}{dt}dt, \mathbf{p} + \frac{d\mathbf{p}}{dt}dt, t + dt\right)\left(1 + \frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{F}dt\right) - f(\mathbf{x}, \mathbf{p}, t)\right] \mathrm{d}^3\mathbf{x} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \\
&\overset{(2.3)}{=} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathbf{p}}{p^0} + \frac{\partial}{\partial \mathbf{p}} \cdot (f\mathbf{F})\right] dt\, \mathrm{d}^3\mathbf{x} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3},
\end{aligned} \tag{2.6}$$

where we have expanded the term $f\left(\mathbf{x} + \frac{d\mathbf{x}}{dt}dt, \mathbf{p} + \frac{d\mathbf{p}}{dt}dt, t + dt\right)$ in a Taylor series in powers of dt,

$$f\left(\mathbf{x} + \frac{d\mathbf{x}}{dt}dt, \mathbf{p} + \frac{d\mathbf{p}}{dt}dt, t + dt\right) = f(\mathbf{x}, \mathbf{p}, t) + \frac{\partial f}{\partial t}dt + \left(\frac{\partial f}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt}\right) dt + \mathcal{O}(dt^2). \tag{2.7}$$

It is convenient to rewrite Eq. (2.6) as

$$\frac{dN}{d\tau} = \gamma \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathbf{p}}{p^0} + \frac{\partial}{\partial \mathbf{p}} \cdot (f\mathbf{F}) \right] \mathrm{d}^3\mathbf{x} \, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \tag{2.8}$$

The change in the total particle number is a Lorentz invariant quantity, since all observers count the same number of particles, as well as the proper time, $d\tau$. In addition, the volume element in phase space, $\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$, is an invariant. Hence, the term multiplying it in the above equation must also be an invariant itself. The first two terms inside square brackets can be immediately expressed in a manifestly covariant form

$$\gamma \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \frac{\mathbf{p}}{p^0} \right) = \frac{\gamma p^\mu}{p^0} \frac{\partial f}{\partial x^\mu} = \frac{p^\mu}{m} \frac{\partial f}{\partial x^\mu} \equiv \frac{p^\mu}{m} \partial_\mu f, \tag{2.9}$$

The next step is to work out the last term inside square brackets. The on-shell condition, $p_\mu p^\mu = m^2$ implies that the Minkowski force is orthogonal to the 4-momentum,

$$p_\mu K^\mu = p_\mu \frac{dp^\mu}{d\tau} = \underbrace{\frac{d(p_\mu p^\mu)}{d\tau}}_{=0} - p^\mu \frac{dp_\mu}{d\tau} = -p^\mu \frac{dp_\mu}{d\tau} = 0, \tag{2.10}$$

since the particle rest mass is constant. This constrains the time and space components of K^μ to

$$K^0 p_0 = \mathbf{K} \cdot \mathbf{p}. \tag{2.11}$$

At this point, we consider p^0 and $\mathbf{p}$ to be independent variables [59], in which case

$$\frac{\partial (f\mathbf{F})}{\partial \mathbf{p}} \to \frac{\partial (f\mathbf{F})}{\partial p^0} \frac{\partial p^0}{\partial \mathbf{p}} + \frac{\partial (f\mathbf{F})}{\partial \mathbf{p}} = \frac{\mathbf{p}}{p^0} \frac{\partial (f\mathbf{F})}{\partial p^0} + \frac{\partial (f\mathbf{F})}{\partial \mathbf{p}}. \tag{2.12}$$

The second equality is obtained using the mass-shell constraint. It then follows that

$$\begin{aligned}
\gamma \frac{\partial (f\mathbf{F})}{\partial \mathbf{p}} &\to \gamma \left[\frac{\mathbf{p}}{p^0} \frac{\partial (f\mathbf{F})}{\partial p^0} + \frac{\partial (f\mathbf{F})}{\partial \mathbf{p}} \right] \\
&= \gamma \left[\frac{\mathbf{p}}{p^0} \frac{\partial}{\partial p^0} \left(\frac{mf\mathbf{K}}{p^0} \right) + \frac{\partial}{\partial \mathbf{p}} \left(\frac{mf\mathbf{K}}{p^0} \right) \right] \\
&= \gamma m \left[\frac{1}{p^0} \frac{\partial}{\partial p^0} \left(\frac{f\mathbf{K} \cdot \mathbf{p}}{p^0} \right) + \frac{1}{p^0} \frac{\partial}{\partial \mathbf{p}} (f\mathbf{K}) \right] \\
&\overset{(2.11)}{=} \frac{\gamma m}{p^0} \left[\frac{\partial (fK^0)}{\partial p^0} + \frac{\partial}{\partial \mathbf{p}} (f\mathbf{K}) \right] \\
&= \frac{\partial (fK^\mu)}{\partial p^\mu}.
\end{aligned} \tag{2.13}$$

This is a scalar invariant. In obtaining this result, we have once again employed the assumption of treating p^0 and $\mathbf{p}$ as independent variables. Eq. (2.8) can then be recast as

$$\frac{dN}{dt} = \frac{1}{\gamma} \left[\frac{p^\mu}{m} \partial_\mu f + \frac{\partial (fK^\mu)}{\partial p^\mu} \right] \mathrm{d}^3\mathbf{x} \, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} = \left[p^\mu \partial_\mu f + m \frac{\partial (fK^\mu)}{\partial p^\mu} \right] \mathrm{d}^3\mathbf{x} \, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 p^0}. \tag{2.14}$$

Without loss of generality, the rate of change in the particle number inside a volume slab in phase space can be decomposed in terms of two contributions, accounting for incoming particles with momenta lying outside (inside) the volume element in phase space being scattered to momenta states that lie within (outside) this region. The first process defines the *gain term*, whereas the latter defines the *loss term*. In this case, we have

$$\frac{dN}{dt} = \underbrace{\frac{dN^+}{dt}}_{\text{gain term}} - \underbrace{\frac{dN^-}{dt}}_{\text{loss term}} . \tag{2.15}$$

At this point, we take a set of assumptions regarding the system:

1. Only binary elastic collisions are relevant. This is a sufficiently reasonable approximation for a dilute system, in which case all three-body (and beyond) interactions can be systematically disregarded as they become increasingly improbable.

2. The momenta of the two interacting particles are uncorrelated prior to and after the collision whenever the distance between them is larger than the typical interaction range. This implies that the system has no memory. In this case, the two-particle distribution function can be factorized as a product of single-particle distribution functions as follows

$$f_2(t, \mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2) = f(t, \mathbf{x}_1, \mathbf{p}_1)\, f(t, \mathbf{x}_2, \mathbf{p}_2) \equiv f_{\mathbf{p}_1}\, f_{\mathbf{p}_2}. \tag{2.16}$$

   This is the assumption of molecular chaos, also known as *Stoßzahlansatz*.

3. The time between collisions, $\tau_{\rm s}$, is typically much larger than the duration of a collision, $\tau_{\rm coll}$. The distribution function varies slowly over a time interval $\tau_{\rm meso}$ for which $\tau_{\rm coll} \ll \tau_{\rm meso} \ll \tau_{\rm s}$.

Having proposed these assumptions, the next step is to obtain an expression for the collision term. At this point, it can be inferred that the gain term must be proportional to the density of particles with incoming momenta p^μ and p'^μ, given by $f(t, x, \mathbf{p}) f(t, x, \mathbf{p}')$, weighted by the probability per time for these particles to be scattered into outgoing momenta states k^μ and k'^μ, defined as the so-called Lorentz-invariant transition rate, $W_{pp' \to kk'}$. Conversely, the loss term is proportional to $f(t, x, \mathbf{k}) f(t, x, \mathbf{k}') W_{kk' \to pp'}$. The net-rate of collisions is then obtained by integrating these contributions over *all* possible momenta $\mathbf{p}'$, $\mathbf{k}$ and $\mathbf{k}'$ for which the total 4-momentum of the system is conserved, as well as over space, $d^3\mathbf{x}$. As a matter of fact, such constraint is imprinted in the transition rate, as shall be demonstrated.

Binary collisions are symmetric under time-reversal and parity transformations. The first lead the outgoing (incoming) momenta into the incoming (outgoing) momenta, whose spatial components have opposite signs, i.e., $k \to \bar{k} = (k^0, -\mathbf{k})$, and the corresponding transition rate for such processes is then $W_{\bar{p}\bar{p}' \to \bar{k}\bar{k}'}$. On top of that, under parity transformations, the transition rate $W_{pp' \to kk'}$ becomes $W_{\bar{p}\bar{p}' \to \bar{k}\bar{k}'}$. Therefore, simultaneously applying the aforementioned operations,

we conclude that $W_{pp'\to kk'} = W_{kk'\to pp'} \equiv W_{pp'\leftrightarrow kk'}$. That is, the occurrence of a process $pp' \to kk'$ is equally probable to that of a process $kk' \to pp'$. Hence,

$$\frac{dN}{dt} = \frac{1}{2} d^3\mathbf{x} \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \int \frac{d^3\mathbf{p}'}{(2\pi)^3 p'^0} \frac{d^3\mathbf{k}}{(2\pi)^3 k'^0} \frac{d^3\mathbf{k}'}{(2\pi)^3 k^0} W_{kk'\leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \right), \tag{2.17}$$

where the factor $1/2$ accounts for the fact that a final momenta state pp' cannot be distinguished from $p'p$ – therefore, by including this factor we avoid over-counting any redundant momenta states.

Finally, comparing Eqs. (2.14) and (2.17), we obtain the Boltzmann equation

$$p^\mu \partial_\mu f + m \frac{\partial (f K^\mu)}{\partial p^\mu} = \frac{1}{2} \int \frac{d^3\mathbf{p}'}{(2\pi)^3 p'^0} \frac{d^3\mathbf{k}}{(2\pi)^3 k'^0} \frac{d^3\mathbf{k}'}{(2\pi)^3 k^0} W_{kk'\leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \right). \tag{2.18}$$

The right-hand side of this equation is usually referred to as *collision term*, defined

$$C[f] = \frac{1}{2} \int \frac{d^3\mathbf{p}'}{(2\pi)^3 p'^0} \frac{d^3\mathbf{k}}{(2\pi)^3 k'^0} \frac{d^3\mathbf{k}'}{(2\pi)^3 k^0} W_{kk'\leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \right), \tag{2.19}$$

such that the Boltzmann equation can be expressed in a more condensed form

$$p^\mu \partial_\mu f + m \frac{\partial (f K^\mu)}{\partial p^\mu} = C[f]. \tag{2.20}$$

When taking into account quantum statistics, the collision term should be consistently revised. In the case of fermions, Pauli's exclusion principle dictates that each momentum state can only be occupied by a single particle. Therefore, in a collision between particles with incoming momenta $(\mathbf{k}, \mathbf{k}')$ and outgoing momenta $(\mathbf{p}, \mathbf{p}')$, the number of particles of particles with momentum $\mathbf{p}$ must be corrected by a factor $(1 - f_{\mathbf{k}})(1 - f_{\mathbf{k}'})$, which account for the number of vacant, and thus accessible, momenta states. For bosons, on the other hand, the corresponding correction factor is $(1 + f_{\mathbf{k}})(1 + f_{\mathbf{k}'})$, which accounts for the apparent attraction between bosons. The collision term (2.19) then becomes

$$C[f] = \frac{1}{2} \int dP' dK dK' W_{kk'\leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right), \tag{2.21}$$

where $\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, with $a = -1$ ($a = 1$) for bosons (fermions), while $a = 0$ corresponds to classical particles, in which case Eq. (2.19) is recovered. Above, we expressed the Lorentz-invariant volume element as

$$dK \equiv \frac{d^3\mathbf{k}}{(2\pi)^3 k^0}, \tag{2.22}$$

a notation that will be adopted throughout this thesis.

Transition rate and cross section

The transition rate is a scalar invariant function of the momenta exchanged in a binary collision. In particular, there are 10 possible scalars that can be constructed from contractions

of p^μ, p'^μ, k^μ and k'^μ. However, the on-shell condition satisfied by each particle provide 4 constraints, reducing the number of parameters to 6. Furthermore, the conservation of the total 4-momentum must be imprinted in the transition rate (as a matter of fact, collisions that violate the conservation of momentum are not possible), providing 4 additional constraints. Therefore, we conclude that the transition rate can be completely described by 2 scalars, for which there are numerous possible choices. One particularly convenient choice is to adopt the following set of Lorentz-invariant variables with a fixed sum,

$$s = (p^\mu + p'^\mu)(p_\mu + p'_\mu), \tag{2.23a}$$

$$t = (p^\mu - k^\mu)(p_\mu - k_\mu) \tag{2.23b}$$

known as the Mandelstam variables [93][2].

In order to better appreciate the physical meaning of the Mandelstam variables, it is convenient to evaluate them in the center-of-mass reference frame. In this case, the total 3-momentum of the system is zero,

$$\mathbf{p} + \mathbf{p}' = \mathbf{k} + \mathbf{k}' = 0. \tag{2.24}$$

Together with the on-shell condition and the conservation of the total 4-momentum, this implies that

$$p^0 = p'^0 = k^0 = k'^0, \quad |\mathbf{p}| = |\mathbf{p}'| = |\mathbf{k}| = |\mathbf{k}'|. \tag{2.25}$$

Therefore, the s variable becomes simply

$$s = 2m^2 + 2p^\mu p'_\mu = 2m^2 + 2(p^0)^2 - 2\underbrace{\mathbf{p}\cdot\mathbf{p}'}_{=-|\mathbf{p}|^2} = (2p^0)^2 = (p^0 + p'^0)^2, \tag{2.26}$$

where we have used Eqs. (2.24) and (2.25) to obtain the second and third equalities, respectively. Therefore, s is the square of the total energy in the center-of-mass frame.

The t parameter, on the other hand, is related to the scattering angle Θ. First, we note that $t = 2m^2 - 2p^\mu k_\mu$, where the last term can be written as

$$p^\mu k_\mu = p^0 k_0 - \mathbf{p}\cdot\mathbf{k} \overset{(2.25)}{=} (p^0)^2 - |\mathbf{p}|^2\cos\Theta = (p^0)^2 - \left[(p^0)^2 - m^2\right]\cos\Theta, \tag{2.27}$$

where we have used the on-shell condition to obtain the last equality and defined the scattering angle in the center-of-mass frame,

$$\cos\Theta = \left.\frac{\mathbf{p}\cdot\mathbf{k}}{|\mathbf{p}||\mathbf{k}|}\right|_{\mathrm{CM}} = \frac{(p^\mu - p'^\mu)(k_\mu - k'_\mu)}{(p^\mu - p'^\mu)(p_\mu - p'_\mu)}, \tag{2.28}$$

[2] In general, there are three Mandelstam variables, with the remaining one being defined as $u = (p^\mu - k^\mu)(p_\mu - k_\mu)$. In particular, they satisfy the relation $s + t + u = 4m^2$, where m is the mass of the particle considered. Therefore, only two of these variables are linearly independent. In particular, since only two are required to completely describe the transition rate, we choose s and t, which will be justified in the following.

with the subscript 'CM' denoting the scalar products are evaluated in the center-of-mass frame. In particular,

$$\cos\Theta = 1 + \frac{2t}{s - 4m^2}. \tag{2.29}$$

We thus conclude that the parameter t is associated with the scattering angle in the center-of-mass frame.

The transition rate can thus be completely characterized using the aforementioned Mandelstam variables s and t. On top of that, it must also encompass the conservation of total 4-momentum in a binary collision. In general, it can be written as [60]

$$W_{kk'\leftrightarrow pp'} = (2\pi)^6 s\sigma(s,\Theta)\,\delta^{(4)}(p^\mu + p'^\mu - k^\mu - k'^\mu), \tag{2.30}$$

where σ is the cross section – a Lorentz scalar with the dimension of area that is determined from the microscopic interaction between the constituents of the gas.

2.2 Microscopic origin of the conservation laws

The main goal of this Chapter is to understand how a macroscopic description of a gas can be constructed from the Boltzmann equation, derived in the previous section. As discussed in the previous Chapter, the cornerstone of a fluid-dynamical description are the continuity equations that emerge due to the conservation of the fundamental charges (in this Chapter, the number of particles) and energy-momentum. Our starting point is thus to demonstrate how the macroscopic conservation laws emerge from the microscopic picture discussed so far.

Let $\varphi_{\mathbf{k}}$ be an arbitrary function of the incoming momentum $\mathbf{k}$. From the definition of the collision operator, Eq. (2.19), we have

$$\begin{aligned}
\int dK\varphi_{\mathbf{k}}\, C[f] &= \int dKdK'dPdP' W_{kk'\leftrightarrow pp'}\left(f_{\mathbf{p}}f_{\mathbf{p}'} - f_{\mathbf{k}}f_{\mathbf{k}'}\right)\varphi_{\mathbf{k}} \\
&= \int dKdK'dPdP' W_{k'k\leftrightarrow pp'}\left(f_{\mathbf{p}}f_{\mathbf{p}'} - f_{\mathbf{k}'}f_{\mathbf{k}}\right)\varphi_{\mathbf{k}'} \quad (k\leftrightarrow k') \\
&= \int dKdK'dPdP' W_{pp'\leftrightarrow kk'}\left(f_{\mathbf{k}}f_{\mathbf{k}'} - f_{\mathbf{p}}f_{\mathbf{p}'}\right)\varphi_{\mathbf{p}} \quad (k\leftrightarrow p,\;\; k'\leftrightarrow p') \\
&= \int dKdK'dPdP' W_{pp'\leftrightarrow kk'}\left(f_{\mathbf{k}'}f_{\mathbf{k}} - f_{\mathbf{p}'}f_{\mathbf{p}}\right)\varphi_{\mathbf{p}'} \quad (k\leftrightarrow p',\;\; k'\leftrightarrow p).
\end{aligned} \tag{2.31}$$

Note that by exchanging the momenta involved in a binary collision in the equations above, we get the same integrals in different forms [94]. Furthermore, it it crucial to note that, naturally, $W_{kk'\leftrightarrow pp'} = W_{k'k\leftrightarrow pp'} = W_{pp'\leftrightarrow kk'} = W_{p'p\leftrightarrow k'k}$. Then, adding these equations, we have

$$\int dK\varphi_{\mathbf{k}}\, C[f] = \frac{1}{4}\int dKdK'dPdP' W_{kk'\leftrightarrow pp'}\left(f_{\mathbf{p}}f_{\mathbf{p}'} - f_{\mathbf{k}}f_{\mathbf{k}'}\right)\left(\varphi_{\mathbf{k}} + \varphi_{\mathbf{k}'} - \varphi_{\mathbf{p}} - \varphi_{\mathbf{p}'}\right). \tag{2.32}$$

Thus, if $\varphi_{\mathbf{k}}$ is a quantity that is conserved in a binary collision, i.e., if $\varphi_{\mathbf{k}} + \varphi_{\mathbf{k}'} = \varphi_{\mathbf{p}} + \varphi_{\mathbf{p}'}$, the integral above vanishes. In binary collisions, we have solely two conserved quantities, namely 1

and k^μ, due to the conservation of the number of particles and energy-momentum in collisions. This leads to the following universal relations, valid for any interaction,

$$\int dK\, C[f] = 0, \tag{2.33}$$

$$\int dK\, k^\mu C[f] = 0. \tag{2.34}$$

Therefore, multiplying the Boltzmann equation (2.18) by 1 and k^ν and integrating over momentum, it follows from the equations above that

$$\partial_\mu \int dK\, k^\mu\, f_{\mathbf{k}} = 0, \tag{2.35a}$$

$$\partial_\mu \int dK\, k^\mu k^\nu\, f_{\mathbf{k}} = 0. \tag{2.35b}$$

These are continuity equations that describe the conservation of particle number, energy, and momentum. They are the macroscopic correspondence of the conservation laws in microscopic collisions.

In the context of relativistic kinetic theory, the particle 4-current and the energy-momentum tensor are identified as the first and second moments of the single-particle distribution function, respectively,

$$N^\mu = \int dK\, k^\mu\, f_{\mathbf{k}}, \tag{2.36a}$$

$$T^{\mu\nu} = \int dK\, k^\mu k^\nu\, f_{\mathbf{k}}. \tag{2.36b}$$

The conservation equations (2.35) can thus be recast in their usual form,

$$\partial_\mu N^\mu = 0, \tag{2.37a}$$

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.37b}$$

The 4-momentum, k^μ, can be decomposed in terms of u^μ as $k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}$, where $E_{\mathbf{k}} \equiv u_\mu k^\mu$ is the energy in the local rest frame of the fluid, and $k^{\langle\mu\rangle} \equiv \Delta^{\mu\nu} k_\nu$ is the orthogonal projection of the 4-momentum. Then, Eqs. (2.36) can be cast in the following form

$$N^\mu = \langle E_{\mathbf{k}} \rangle\, u^\mu + \left\langle k^{\langle\mu\rangle} \right\rangle, \tag{2.38a}$$

$$T^{\mu\nu} = \left\langle E_{\mathbf{k}}^2 \right\rangle u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu}\langle \Delta_{\mathbf{kk}} \rangle + \left\langle E_{\mathbf{k}} k^{\langle\mu\rangle} \right\rangle u^\nu + \left\langle E_{\mathbf{k}} k^{\langle\nu\rangle} \right\rangle u^\mu + \left\langle k^{\langle\mu} k^{\nu\rangle} \right\rangle, \tag{2.38b}$$

where we have adopted the notation

$$\langle \cdots \rangle \equiv \int dK\, (\cdots)\, f_{\mathbf{k}}, \tag{2.39}$$

following Ref. [2] and defined

$$\Delta_{\mathbf{kk}} \equiv \Delta_{\mu\nu} k^\mu k^\nu = m^2 - E_{\mathbf{k}}^2, \tag{2.40}$$

$$k^{\langle\mu\rangle} \equiv \Delta^\mu_\nu k^\nu, \tag{2.41}$$

$$k^{\langle\mu} k^{\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta. \tag{2.42}$$

Comparing Eqs. (2.38) with Eqs. (1.33), it is then possible to identify the fluid-dynamical fields as the following moments of $f_{\mathbf{k}}$,

$$n = \langle E_{\mathbf{k}} \rangle \, , \, n^{\mu} = \left\langle k^{\langle \mu \rangle} \right\rangle \, , \, \varepsilon = \langle E_{\mathbf{k}}^2 \rangle, \; P_0 + \Pi = -\frac{1}{3} \langle \Delta_{\mathbf{k}\mathbf{k}} \rangle, \; h^{\mu} = \left\langle E_{\mathbf{k}} k^{\langle \mu \rangle} \right\rangle \, , \, \pi^{\mu\nu} = \left\langle k^{\langle \mu} k^{\nu \rangle} \right\rangle . \tag{2.43}$$

2.3 Equilibrium solution

The Boltzmann equation has a stationary and homogeneous solution, for which the single-particle distribution function satisfies

$$k^{\mu} \partial_{\mu} f_{\mathbf{k}} = 0. \tag{2.44}$$

In order for this solution to be realized, the collision term must vanish,

$$\int dK' dP' dP \; W_{kk' \leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right) = 0. \tag{2.45}$$

Naturally, the collision term is zero when particles are not interacting, but this will not be the case considered here. In particular, particles in the system are still colliding, however collisions between particles with incoming momenta $\mathbf{k}$, $\mathbf{k}'$ and outgoing momenta $\mathbf{p}$, $\mathbf{p}'$ are equally probable and cancel each other. This regime is called the detailed balance condition, and it is satisfied by a single-particle distribution denoted by $f_{0\mathbf{k}}$. Thus, from Eq. (2.21), we have

$$f_{0\mathbf{p}} f_{0\mathbf{p}'} \tilde{f}_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}'} = f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'}. \tag{2.46}$$

Taking the natural logarithm on both sides, we have

$$\ln \left(\frac{f_{0\mathbf{k}}}{\tilde{f}_{0\mathbf{k}}} \right) + \ln \left(\frac{f_{0\mathbf{k}'}}{\tilde{f}_{0\mathbf{k}'}} \right) = \ln \left(\frac{f_{0\mathbf{p}}}{\tilde{f}_{0\mathbf{p}}} \right) + \ln \left(\frac{f_{0\mathbf{p}'}}{\tilde{f}_{0\mathbf{p}'}} \right). \tag{2.47}$$

The left- and right-hand side of the equation above comprise quantities regarding the incoming and outgoing momenta states, respectively. Therefore, we conclude that $\ln(f_{0\mathbf{k}}/\tilde{f}_{0\mathbf{k}})$ can only depend on quantities that are conserved in binary collisions, namely $1, k^{\mu}$, as well as on the usual thermodynamic variables. Hence, it is generally given by

$$\ln \left(\frac{f_{0\mathbf{k}}}{\tilde{f}_{0\mathbf{k}}} \right) = \alpha - \beta_{\mu} k^{\mu}. \tag{2.48}$$

In particular, β_{μ} is a 4-vector with dimension of inverse momentum, and it must have the following form $\beta_{\mu} = u_{\mu}/T$, with u_{μ} being a 4-velocity (that defines the local rest frame of the system). In particular, the minus sign in the equality above is merely a convention such that T is identified as the temperature, whereas α is identified as the thermal potential. Hence,

$$f_{0\mathbf{k}} = \frac{1}{e^{\beta E_{\mathbf{k}} - \alpha} + a}. \tag{2.49}$$

Taking $a = 0$, one simply recovers the usual Maxwell-Boltzmann distribution, whereas taking $a = 1$ ($a = -1$), one obtains the Fermi-Dirac and Bose-Einstein distributions.

This solution is a fixed point of the Boltzmann equation and defines the equilibrium state of a dilute gas. Any isolated dilute gas will approach this solution, after some characteristic time scale determined from the interactions, encoded in the cross section. Equilibrium states are known to exist in nature, but are rather nontrivial to obtain microscopically without resorting to thermodynamic or statistical arguments. The Boltzmann equation thus provides a microscopic description that is consistent with thermodynamic principles. In the next section, we discuss yet another fundamental thermodynamic conjecture that naturally arises from the Boltzmann equation: the second law of thermodynamics.

2.4 Entropy production

In kinetic theory, the entropy 4-current is defined as [59]

$$S^\mu = -\int dK k^\mu \left(\frac{\tilde{f}_{\mathbf{k}}}{a} \ln \tilde{f}_{\mathbf{k}} + f_{\mathbf{k}} \ln f_{\mathbf{k}} \right). \tag{2.50}$$

Therefore, an equation of motion for the entropy production can be directly derived from the Boltzmann equation,

$$\partial_\mu S^\mu = -\int dK C[f_{\mathbf{k}}] \ln \left(\frac{f_{\mathbf{k}}}{\tilde{f}_{\mathbf{k}}} \right). \tag{2.51}$$

In Eq. (2.47), we have shown that, in equilibrium, $\ln(f_{0\mathbf{k}}/\tilde{f}_{0\mathbf{k}}) = -\beta_\mu u^\mu + \alpha$, and is a quantity conserved in binary collisions. Therefore, from the conservation theorem (2.32), it follows that

$$\partial_\mu S^\mu_{\text{eq}} = 0. \tag{2.52}$$

This result was obtained in the previous chapter in the context of hydrodynamics, see Eq. (1.25), and now it has been recovered from microscopic calculations.

For a general single-particle distribution function, the condition above is not necessarily satisfied and the entropy production must be calculated in detail. Following the same reasoning adopted in deriving Eq. (2.31), we have

$$\begin{aligned}
\partial_\mu S^\mu &= -\frac{1}{4}\int dK dK' dP dP' W_{kk'\leftrightarrow pp'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right) \\
&\times \left[\ln\left(\frac{f_{\mathbf{k}}}{\tilde{f}_{\mathbf{k}}} \right) + \ln\left(\frac{f_{\mathbf{k}'}}{\tilde{f}_{\mathbf{k}'}} \right) - \ln\left(\frac{f_{\mathbf{p}}}{\tilde{f}_{\mathbf{p}}} \right) - \ln\left(\frac{f_{\mathbf{p}'}}{\tilde{f}_{\mathbf{p}'}} \right) \right] \\
&= -\frac{1}{4}\int dK dK' dP dP' W_{kk'\leftrightarrow pp'} f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} \left(1 - \frac{f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'}}{f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'}} \right) \ln\left(\frac{f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'}}{f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'}} \right).
\end{aligned} \tag{2.53}$$

The distribution function is defined as a non-negative quantity, cf. Eq. (2.1). Moreover, the integrand involves a function of the type $(1-x)\ln x$, which is non-positive for all $x \geq 0$. Therefore, we conclude that

$$\partial_\mu S^\mu \geq 0, \tag{2.54}$$

thus recovering the second law of thermodynamics from purely microscopic arguments. This is a remarkable result and is essentially the only existing microscopic realization of the second law of thermodynamics.

2.5 Ideal fluid dynamics

As discussed in Chapter 1, ideal fluid dynamics is based on the assumption of *local* equilibrium, i.e., the fluid elements are assumed to be thermodynamic systems in equilibrium. In the context of kinetic theory, a state of local equilibrium is achieved by a single-particle distribution function given by Eq. (2.49), with the 4-velocity, $u^\mu(x^\lambda)$, the temperature, $T(x^\lambda)$, and the thermal potential, $\alpha(x^\lambda)$, being *locally* defined, i.e., functions of spacetime. Then, the conserved currents of an ideal fluid are obtained by taking $f_{\mathbf{k}} \to f_{0\mathbf{k}}$ in Eqs. (2.38),

$$N^\mu = \langle k^\mu \rangle_0 \,, \quad T^{\mu\nu} = \langle k^\mu k^\nu \rangle_0 \,. \tag{2.55}$$

Above, we denoted momentum integrals over the local equilibrium distribution function with the following notation,

$$\langle \cdots \rangle_0 \equiv \int dK \, (\cdots) \, f_{0\mathbf{k}}. \tag{2.56}$$

We remark that all momentum integrals of $f_{0\mathrm{k}}$ must depend solely on the scalars, T and α, the 4-vector, u^μ, and the metric tensor, $g^{\mu\nu}$. Thus, the tensor structure of the conserved currents in equilibrium, Eqs. (2.55), is determined solely in terms of combinations of u^μ and $g^{\mu\nu}$. Then, the local equilibrium particle 4-current must be proportional to u^μ,

$$N^\mu_{\mathrm{eq}} = n_0 u^\mu, \tag{2.57}$$

with n_0 being identified as the local particle density,

$$n_0 = u_\mu N^\mu_{\mathrm{eq}} = \langle E_{\mathbf{k}} \rangle_0 \,. \tag{2.58}$$

We note that n_0 is a Lorentz scalar and can be calculated in any reference frame. This implies that it cannot depend on the 4-velocity and is a function of the thermodynamic variables, $n_0 = n_0(T, \alpha)$.

The energy-momentum tensor is a second-rank tensor and, thus, in local equilibrium, can only depend on a linear combination of $u^\mu u^\nu$ and $g^{\mu\nu}$. It then becomes,

$$T^{\mu\nu}_{\mathrm{eq}} = \varepsilon_0 \, u^\mu u^\nu - \Delta^{\mu\nu} P_0, \tag{2.59}$$

with ε_0 being identified as the local energy density and P_0 the thermodynamic pressure,

$$\varepsilon_0 = u_\mu u_\nu T^{\mu\nu}_{\rm eq} = \left\langle E^2_{\mathbf{k}} \right\rangle_0 , \tag{2.60}$$

$$P_0 = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}_{\rm eq} = -\frac{1}{3}\langle \Delta_{\mathbf{kk}} \rangle_0 . \tag{2.61}$$

We note that both ε_0 and P_0 are Lorentz scalars and thus are functions solely of the thermodynamic variables, $\varepsilon_0 = \varepsilon_0(T, \alpha)$ and $P_0 = P_0(T, \alpha)$. These relations guarantee the existence of an equation of state, i.e., a relation $P_0 = P_0(n_0, \varepsilon_0)$. As a matter of fact, for ultrarelativistic gases, the mass of the particles can be neglected (when compared to the temperature), leading to the very simple relation $P_0 = \varepsilon_0/3$,

$$P_0 = -\frac{1}{3}\langle m^2 - E^2_{\mathbf{k}} \rangle_0 \overset{m\to 0}{\approx} \frac{1}{3}\langle E^2_{\mathbf{k}} \rangle_0 = \frac{\varepsilon_0}{3}. \tag{2.62}$$

These general expressions for the conserved currents in local thermodynamic equilibrium were already obtained in the previous chapter, see Eqs. (1.16) and (1.15). They lead to the relativistic Euler equations (1.21). We now understood how these expression arise from kinetic theory and also obtained a microscopic expression for the equation of state.

2.6 Chapman-Enskog method

For the past decades, relativistic fluid dynamics has been successfully used in the description of cold atomic gases [32,95] as well as of the quark-gluon plasma created in ultra-relativistic heavy-ion collisions [96–98]. In order to obtain an accurate description of these systems, it is indispensable to take into account dissipative effects in the differential equations that govern the relativistic fluid-dynamical evolution. This has motivated a general discussion on the derivation of relativistic dissipative fluid dynamics from an underlying microscopic theory [58, 98, 99] and, furthermore, in establishing the domain of applicability of such fluid-dynamical formulations. In particular, this issue has been widely investigated for relativistic dilute gases using the Boltzmann equation as a starting point [58].

Chapman-Enskog theory [59–61] is the most widespread method to derive fluid dynamics from the Boltzmann equation. In this approach, a particular solution for the single-particle distribution function is found in the form of a gradient expansion. Different truncations of this expansion lead to distinct fluid-dynamical theories: a truncation at zeroth order leads to the Euler equations (ideal fluid dynamics), while at first order it yields the Navier-Stokes equations [37]. Unlike its non-relativistic counterpart, the relativistic Navier-Stokes theory is ill-defined, as its acausal nature renders the global equilibrium state unstable [39, 100], as discussed in the previous Chapter. Moreover, higher-order truncations of the Chapman-Enskog expansion lead to Burnett and super-Burnett theories, which suffer from the so-called Bobylev instability even in the non-relativistic regime [101]. These problems lead to the conclusion that the traditional Chapman-Enskog method does not yield fluid-dynamical formulations that can actually be

employed for practical purposes.[3] Nevertheless, it is an important development in kinetic theory and serves as an introductory formalism to derive a hydrodynamic theory.

It is a perturbative approach to obtain approximate solutions of the Boltzmann equation, following Hilbert's [105] proposal to convert the Boltzmann equation into a perturbative problem. The hydrodynamic regime is expected to emerge when the single-particle distribution function varies extremely slowly in space and time when compared to the mean-free path of the particles. This implies that the left-hand side of the Boltzmann equation, which involves spacetime derivatives of $f_{\mathbf{k}}$, becomes very small and, thus, $f_{\mathbf{k}}$ must approach a form which minimizes the magnitude of the collision term. As proposed by Hilbert, this can be accomplished from a mathematical point of view by introducing a book-keeping parameter ϵ on the left-hand side of the Boltzmann equation,

$$\epsilon \left(E_{\mathbf{k}} D f_{\mathbf{k}} + k^{\langle \mu \rangle} \nabla_{\mu} f_{\mathbf{k}} \right) = C[f], \tag{2.63}$$

where we have decomposed the 4-derivative into its parallel and orthogonal components with respect to the 4-velocity, $\partial_{\mu} = u_{\mu} D + \nabla_{\mu}$. Then, the single-particle distribution function is expressed as a power series in ϵ,

$$f_{\mathbf{k}} = \sum_{i=0}^{\infty} \epsilon^{i} f_{\mathbf{k}}^{(i)}, \tag{2.64}$$

where $f_{\mathbf{k}}^{(i)}$ denotes the i-th order term in the parameter ϵ. Therefore, in order to determine the expansion coefficients and, thus, reconstruct the single-particle distribution function, the Boltzmann equation must be solved iteratively, order by order in ϵ. The parameter ϵ is then set to unity, thus leading to a perturbative solution of the Boltzmann equation. In particular, at each order in ϵ, the Boltzmann equation becomes

$$0 = \mathcal{C}^{(0)}, \quad \text{(zeroth order)} \tag{2.65a}$$

$$E_{\mathbf{k}} \left[D f_{\mathbf{k}} \right]^{(0)} + k^{\langle \mu \rangle} \nabla_{\mu} f_{\mathbf{k}}^{(0)} = \mathcal{C}^{(1)}, \quad \text{(first order)} \tag{2.65b}$$

$$\vdots$$

$$E_{\mathbf{k}} \left[D f_{\mathbf{k}} \right]^{(n-1)} + k^{\langle \mu \rangle} \nabla_{\mu} f_{\mathbf{k}}^{(n-1)} = \mathcal{C}^{(n)}. \quad (n\text{-th order}) \tag{2.65c}$$

where $\mathcal{C}^{(i)}$ denotes the i-th order contribution to the collision term and $\left[D f_{\mathbf{k}} \right]^{(i)}$ is the contribution of $D f_{\mathbf{k}}$ at i-th order, which should not be confused with the comoving derivative of the i-th order contribution of $f_{\mathbf{k}}$, denoted by $D f_{\mathbf{k}}^{(i)}$, i.e.,

$$\left[D f_{\mathbf{k}} \right]^{(i)} \neq D f_{\mathbf{k}}^{(i)}. \tag{2.66}$$

In particular, a crucial feature of the Chapman-Enskog approach is to also expand the comoving derivative of the single-particle distribution function in orders of ϵ,

$$D f_{\mathbf{k}} = \sum_{i=0}^{\infty} \epsilon^{i} \left[D f_{\mathbf{k}} \right]^{(i)}. \tag{2.67}$$

[3] In Ref. [102], a possibly *convergent* generalization of the Chapman-Enskog expansion was proposed. However, its convergence has only been investigated in Bjorken flow [103] within the relaxation time approximation [104].

Throughout the remaining of this chapter, we shall work the first two truncations of the Chapman-Enskog series and their implications to the single-particle distribution function as well as their connections to formulations of relativistic fluid dynamics.

2.6.1 Zeroth-order solution

At zeroth order in ϵ, we obtain the following integral equation,

$$\mathcal{C}^{(0)} = \frac{1}{2}\int dK'dPdP' W_{kk'\leftrightarrow pp'}\left(f^{(0)}_{\mathbf{p}} f^{(0)}_{\mathbf{p}'} \tilde{f}^{(0)}_{\mathbf{k}} \tilde{f}^{(0)}_{\mathbf{k}'} - f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'}\right) = 0. \tag{2.68}$$

In general, this term is zero if, and only if, the integrand is zero, which is exactly the detailed balance condition,

$$f^{(0)}_{\mathbf{p}} f^{(0)}_{\mathbf{p}'} \tilde{f}^{(0)}_{\mathbf{k}} \tilde{f}^{(0)}_{\mathbf{k}'} = f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'}. \tag{2.69}$$

As already discussed, the solution for this equation is the local equilibrium single-particle distribution function, as was shown in Eq. (2.49),

$$f^{(0)}_{\mathbf{k}} \equiv f_{0\mathbf{k}} = \left[\exp\left(\beta k^{\mu} u_{\mu} - \alpha\right) + a\right]^{-1}. \tag{2.70}$$

Here, α, β and u^{μ} are *a priori* arbitrary functions of spacetime.

Thus, the zeroth-order solution of the Chapman-Enskog expansion is the local equilibrium distribution function. As demonstrated in Sec. 2.5, replacing this solution into the kinetic expression for the conserved currents leads to the relativistic Euler equations (1.21), in which the evolution of the system is entirely described using local conservation laws and an equation of state. However, we remark that such truncation does not provide a solution for the Boltzmann equation. In fact, even though the detailed balance condition leads to a vanishing collision term, the left-hand side of the Boltzmann equation is not zero. Therefore, in order to obtain a more precise description, it is necessary to consider higher-order truncations for the Chapman-Enskog expansion.

2.6.2 First-order solution

The first-order Chapman-Enskog equation is given by,

$$E_{\mathbf{k}}\left[Df_{\mathbf{k}}\right]^{(0)} + k^{\langle\mu\rangle}\nabla_{\mu} f^{(0)}_{\mathbf{k}} = \mathcal{C}^{(1)}, \tag{2.71}$$

with $\mathcal{C}^{(1)}$ being the first order correction in ϵ of the collision term. It is obtained by linearizing the collision term around the zeroth-order solution, i.e., the local equilibrium solution, leading to

$$\begin{aligned}\mathcal{C}^{(1)} = \frac{1}{2}\int dK'dPdP' W_{kk'\leftrightarrow pp'} f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'} \\ \times\left(\frac{f^{(1)}_{\mathbf{p}}}{f^{(0)}_{\mathbf{p}}} + \frac{f^{(1)}_{\mathbf{p}'}}{f^{(0)}_{\mathbf{p}'}} + \frac{\tilde{f}^{(1)}_{\mathbf{k}}}{\tilde{f}^{(0)}_{\mathbf{k}}} + \frac{\tilde{f}^{(1)}_{\mathbf{k}'}}{\tilde{f}^{(0)}_{\mathbf{k}'}} - \frac{f^{(1)}_{\mathbf{k}}}{f^{(0)}_{\mathbf{k}}} - \frac{f^{(1)}_{\mathbf{k}'}}{f^{(0)}_{\mathbf{k}'}} - \frac{\tilde{f}^{(1)}_{\mathbf{p}}}{\tilde{f}^{(0)}_{\mathbf{p}}} - \frac{\tilde{f}^{(1)}_{\mathbf{p}'}}{\tilde{f}^{(0)}_{\mathbf{p}'}}\right),\end{aligned} \tag{2.72}$$

where we used the detailed balance condition, $f^{(0)}_{\mathbf{p}} f^{(0)}_{\mathbf{p}'} \tilde{f}^{(0)}_{\mathbf{k}} \tilde{f}^{(0)}_{\mathbf{k}'} = f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'}$. Since $\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, at first order in ϵ, we have that $\tilde{f}^{(1)}_{\mathbf{p}} = -a f^{(1)}_{\mathbf{k}}$. Thus, the term inside parentheses in the collision term can be expressed as

$$f^{(0)}_{\mathbf{k}} f^{(0)}_{\mathbf{k}'} \tilde{f}^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}'} \left(\frac{f^{(1)}_{\mathbf{p}}}{f^{(0)}_{\mathbf{p}}} + \frac{f^{(1)}_{\mathbf{p}'}}{f^{(0)}_{\mathbf{p}'}} - a\frac{f^{(1)}_{\mathbf{k}}}{\tilde{f}^{(0)}_{\mathbf{k}}} - a\frac{f^{(1)}_{\mathbf{k}'}}{\tilde{f}^{(0)}_{\mathbf{k}'}} - \frac{f^{(1)}_{\mathbf{k}}}{f^{(0)}_{\mathbf{k}}} - \frac{f^{(1)}_{\mathbf{k}'}}{f^{(0)}_{\mathbf{k}'}} + a\frac{f^{(1)}_{\mathbf{p}}}{\tilde{f}^{(0)}_{\mathbf{p}}} + a\frac{f^{(1)}_{\mathbf{p}'}}{\tilde{f}^{(0)}_{\mathbf{p}'}} \right)$$
$$= f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'} \right), \tag{2.73}$$

where we have defined $\phi_{\mathbf{p}} \equiv f^{(1)}_{\mathbf{p}} / [f^{(0)}_{\mathbf{p}} \tilde{f}^{(0)}_{\mathbf{p}}]$. After these simplifications, the first-order contribution to the collision term in the Chapman-Enskog expansion is given by,

$$\mathcal{C}^{(1)} = \frac{1}{2} \int dK' dP dP' W_{kk' \leftrightarrow pp'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'} \right) \equiv f_{0\mathbf{k}} \hat{L} \phi_{\mathbf{k}}, \tag{2.74}$$

where $\hat{L}$ is the *linearized* collision operator, a self-adjoint operator [62] (see Appendix A), defined as implied. In particular, the linearized collision term has five degenerate zero modes, corresponding to the conserved quantities – as a matter of fact, $\hat{L} 1 = 0$ and $\hat{L}\, k^{\mu} = 0$.

The first-order solution of the Boltzmann equation reads

$$E_{\mathbf{k}} \left[D f_{\mathbf{k}} \right]^{(0)} + k^{\langle \mu \rangle} \nabla_{\mu} f_{0\mathbf{k}} = f_{0\mathbf{k}} \hat{L} [\phi_{\mathbf{k}}]. \tag{2.75}$$

At this point, it is crucial to note that the zeroth order contribution of $D f_{\mathbf{k}}$ should not be simply identified as the comoving derivative of the equilibrium distribution function,

$$D f_{0\mathbf{k}} = \left(D\alpha - E_{\mathbf{k}} D\beta - \beta k^{\langle \mu \rangle} D u_{\mu} \right) f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{2.76}$$

It follows from the conservation laws that, at leading order in ϵ,

$$D\alpha = \frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{D_{20}} \theta + \mathcal{O}\left(\epsilon^2\right), \tag{2.77a}$$

$$D\beta = \frac{(\varepsilon_0 + P_0) J_{10} - n_0 J_{20}}{D_{20}} \theta + \mathcal{O}\left(\epsilon^2\right), \tag{2.77b}$$

$$D u^{\mu} = \frac{1}{\varepsilon_0 + P_0} \nabla^{\mu} P + \mathcal{O}\left(\epsilon^2\right), \tag{2.77c}$$

where we have defined the thermodynamics integrals

$$J_{mn} = \frac{(-1)^n}{(2n+1)!!} \int dK E^{m-2n}_{\mathbf{k}} (\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^n f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{2.78}$$

Terms of second order in ϵ in Eqs. (2.77) contain either the product between dissipative currents (which are *at least* of first order in ϵ, since they are identically zero in equilibrium) and gradients of the primary fluid-dynamical variables or derivatives of the dissipative currents. We then identify $[D f_{\mathbf{k}}]^{(0)}$ as the first order contribution in ϵ stemming from the comoving derivative of the equilibrium single-particle distribution function. That is,

$$[D f_{\mathbf{k}}]^{(0)} = [D f_{0\mathbf{k}}]^{(1)} = \left\{ \left[\frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{D_{20}} - E_{\mathbf{k}} \frac{(\varepsilon_0 + P_0) J_{10} - n_0 J_{20}}{D_{20}} \right] \theta \right.$$
$$\left. - \frac{\beta}{\varepsilon_0 + P_0} k_{\langle \mu \rangle} \nabla^{\mu} P \right\} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{2.79}$$

Furthermore, the space derivative of the equilibrium distribution function is

$$\nabla_\mu f_{0\mathbf{k}} = \left(\nabla_\mu \alpha - E_{\mathbf{k}} \nabla_\mu \beta - \beta k^{\langle\nu\rangle} \nabla_\mu u_\nu\right) f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{2.80}$$

Therefore, one can show that

$$\begin{aligned} k^{\langle\mu\rangle} \nabla_\mu f_{0\mathbf{k}} = \Bigg[& k^{\langle\mu\rangle} \nabla_\mu \alpha - E_{\mathbf{k}} k^{\langle\mu\rangle} \left(\frac{n_0}{\varepsilon_0 + P_0} \nabla_\mu \alpha - \frac{\beta}{\varepsilon_0 + P_0} \nabla_\mu P \right) \\ & - \beta \left(k^{\langle\mu} k^{\nu\rangle} \sigma_{\mu\nu} + \frac{1}{3} (\Delta_{\alpha\beta} k^\alpha k^\beta) \theta \right) \Bigg] f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}, \end{aligned} \tag{2.81}$$

where we have used the identities $k^{\langle\mu\rangle} k^{\langle\nu\rangle} = k^{\langle\mu} k^{\nu\rangle} + \Delta^{\mu\nu} (\Delta_{\alpha\beta} k^\alpha k^\beta)/3$ and $\nabla_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \theta (\Delta_{\alpha\beta} k^\alpha k^\beta)/3$, with $\omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ being the vorticity tensor, to obtain the last two equalities. The Boltzmann equation at first order in ϵ (2.75) thus becomes

$$\left(\mathcal{A}_{\mathbf{k}} \theta + \mathcal{B}_{\mathbf{k}} k^{\langle\mu\rangle} \nabla_\mu \alpha - \beta k^{\langle\nu} k^{\nu\rangle} \sigma_{\mu\nu}\right) f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} = f_{0\mathbf{k}} \hat{L}[\phi_{\mathbf{k}}], \tag{2.82}$$

where we have introduced the coefficients

$$\mathcal{A}_{\mathbf{k}} = E_{\mathbf{k}} \left[\frac{(\varepsilon_0 + P_0) J_{20} - n_0 J_{30}}{D_{20}} - E_{\mathbf{k}} \frac{(\varepsilon_0 + P_0) J_{10} - n_0 J_{20}}{D_{20}} \right] - \frac{\beta}{3} (\Delta_{\alpha\beta} k^\alpha k^\beta), \tag{2.83}$$

$$\mathcal{B}_{\mathbf{k}} = 1 - E_{\mathbf{k}} \frac{n_0}{\varepsilon_0 + P_0}. \tag{2.84}$$

In particular, multiplying both sides of Eq. (2.82) by 1 and k_λ and making use of the self-adjointness of the linearized collision operator and its zero modes, it follows that

$$\int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \left(\mathcal{A}_{\mathbf{k}} \theta + \mathcal{B}_{\mathbf{k}} k^{\langle\mu\rangle} \nabla_\mu \alpha - \beta k^{\langle\nu} k^{\nu\rangle} \sigma_{\mu\nu}\right) = 0, \tag{2.85a}$$

$$\int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} k_\alpha \left(\mathcal{A}_{\mathbf{k}} \theta + \mathcal{B}_{\mathbf{k}} k^{\langle\mu\rangle} \nabla_\mu \alpha - \beta k^{\langle\nu} k^{\nu\rangle} \sigma_{\mu\nu}\right) = 0, \tag{2.85b}$$

which one can explicitly verify by employing the orthogonality of the basis formed by the irreducible momenta $\{1, k^{\langle\mu\rangle}, k^{\langle\mu} k^{\nu\rangle}, \cdots\}$ [2, 60], the definition of the thermodynamic integrals J_{nq} (2.78) and the thermodynamic relation (1.10).

Therefore, we conclude that the approximation taken in the identification of $[D f_{\mathbf{k}}]^{(0)}$ is indispensable for the consistency with the vanishing eigenvalues of the linearized collision operator, and thus it cannot be taken differently [106]. In this context, this implies that spacetime derivatives must be replaced by timelike ones, thus rendering the Chapman-Enskog series a gradient expansion i.e., a power series of spacelike gradients [58, 107].

The next step is to solve Eq. (2.82) for $\phi_{\mathbf{k}}$. This is an inhomogeneous equation, with the most general solution being of the following form,

$$\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{\mathrm{hom}} + \phi_{\mathbf{k}}^{\mathrm{part}}, \tag{2.86}$$

where $\phi_{\mathbf{k}}^{\mathrm{hom}}$ is the solution of the homogeneous equation, i.e., when the source term is set to zero, $\hat{L} \phi_{\mathbf{k}} = 0$, whereas $\phi_{\mathbf{k}}^{\mathrm{part}}$ is a solution of the inhomogeneous equation (2.82). It can readily be seen

that, given the zero modes of the linearized collision operator, the solution to the homogeneous equation must be a linear combination of the eigenfunctions 1 and k^μ. Therefore, its most general form is given by

$$\phi_{\mathbf{k}}^{\text{hom}} = \varphi + \varphi_\mu k^\mu = \varphi_0 + \varphi_1 E_{\mathbf{k}} + \varphi_{2,\mu} k^{\langle\mu\rangle}, \tag{2.87}$$

where the coefficients φ_0, φ_1 and $\varphi_{2,\mu}$ must be determined from the matching conditions. At this point, we emphasize that there is no unique form to define these coefficients. In fact, such an arbitrariness can be interpreted as different definitions of temperature, thermal potential and 4-velocity.

Inhomogeneous solution

Obtaining solutions for the inhomogeneous equations, on the other hand, is a rather non-trivial task, as it involves inverting the linearized collision operator. However, given the existence of the aforementioned zero modes, the inversion procedure can only be performed in the subspace orthogonal to the kernel of $\hat{L}$ [58, 106]. Furthermore, since $\hat{L}$ is a linear operator, the particular solution has the form

$$\phi_{\mathbf{k}}^{\text{part}} = \mathcal{S}_{\mathbf{k}}\theta + \mathcal{T}_{\mathbf{k}} k^{\langle\mu\rangle}\nabla_\mu\alpha + \mathcal{V}_{\mathbf{k}} k^{\langle\nu} k^{\nu\rangle}\sigma_{\mu\nu}, \tag{2.88}$$

where the functions $\mathcal{S}_{\mathbf{k}}$, $\mathcal{T}_{\mathbf{k}}$ and $\mathcal{V}_{\mathbf{k}}$ depend on momentum through $E_{\mathbf{k}}$. Substituting this solution in Eq. (2.82), we have

$$\begin{aligned}&\left(\mathcal{A}_{\mathbf{k}}\theta + \mathcal{B}_{\mathbf{k}} k^{\langle\mu\rangle}\nabla_\mu\alpha - \beta k^{\langle\nu} k^{\nu\rangle}\sigma_{\mu\nu}\right) f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \\ &= \theta f_{0\mathbf{k}}\hat{L}\left[\mathcal{S}_{\mathbf{k}}\right] + \nabla_\mu\alpha f_{0\mathbf{k}}\hat{L}\left[\mathcal{T}_{\mathbf{k}} k^{\langle\mu\rangle}\right] + \sigma_{\mu\nu} f_{0\mathbf{k}}\hat{L}\left[\mathcal{V}_{\mathbf{k}} k^{\langle\nu} k^{\nu\rangle}\right].\end{aligned} \tag{2.89}$$

These functions are then expanded using a complete basis of functions of $E_{\mathbf{k}}$, defined as $\mathcal{P}_{n,\mathbf{k}}^{(\ell)}$, with $n, \ell = 0, 1, 2, \ldots$,

$$\mathcal{S}_{\mathbf{k}} = \sum_{n=0}^{\infty} s_n \mathcal{P}_{n,\mathbf{k}}^{(0)}, \quad \mathcal{V}_{\mathbf{k}} = \sum_{n=0}^{\infty} v_n \mathcal{P}_{n,\mathbf{k}}^{(1)}, \quad \mathcal{T}_{\mathbf{k}} = \sum_{n=0}^{\infty} t_n \mathcal{P}_{n,\mathbf{k}}^{(2)}. \tag{2.90}$$

The scalar, vector and tensor contributions to the inhomogeneous solution $\phi_{\mathbf{k}}^{\text{part}}$ can be obtained by multiplying Eq. (2.89) by $\mathcal{P}_{r,\mathbf{k}}^{(0)}$, $k_{\langle\alpha\rangle}\mathcal{P}_{r,\mathbf{k}}^{(1)}$ and $k_{\langle\alpha} k_{\beta\rangle}\mathcal{P}_{r,\mathbf{k}}^{(2)}$, respectively. One then obtains

$$\sum_n S_{rn} s_n = A_r, \quad \sum_n V_{rn} v_n = B_r, \quad \sum_n T_{rn} t_n = C_r, \tag{2.91}$$

where we have defined

$$S_{rn} = \int dK\, f_{0\mathbf{k}} \mathcal{P}_{r,\mathbf{k}}^{(0)} \hat{L}\left[\mathcal{P}_{n,\mathbf{k}}^{(0)}\right], \tag{2.92a}$$

$$V_{rn} = \int dK\, f_{0\mathbf{k}} k_{\langle\mu\rangle} \mathcal{P}_{r,\mathbf{k}}^{(1)} \hat{L}\left[k^{\langle\mu\rangle}\mathcal{P}_{n,\mathbf{k}}^{(1)}\right], \tag{2.92b}$$

$$T_{rn} = \int dK\, f_{0\mathbf{k}} k_{\langle\mu} k_{\nu\rangle} \mathcal{P}_{r,\mathbf{k}}^{(2)} \hat{L}\left[k^{\langle\mu} k^{\nu\rangle}\mathcal{P}_{n,\mathbf{k}}^{(2)}\right], \tag{2.92c}$$

and

$$A_r = \int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathcal{P}^{(0)}_{r,\mathbf{k}} \mathcal{A}_{\mathbf{k}}, \tag{2.93a}$$

$$B_r = \int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathcal{P}^{(1)}_{r,\mathbf{k}} (\Delta_{\alpha\beta} k^\alpha k^\beta) \mathcal{B}_{\mathbf{k}}, \tag{2.93b}$$

$$C_r = -\frac{2}{3}\beta \int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathcal{P}^{(2)}_{r,\mathbf{k}} (\Delta_{\alpha\beta} k^\alpha k^\beta)^2. \tag{2.93c}$$

The sums in Eqs. (2.91) can be inverted by acting with the corresponding inverse matrix on the left-hand, thus leading to

$$s_m = \sum_n [S^{-1}]_{mn} A_n, \quad v_m = \sum_n [V^{-1}]_{mn} B_n, \quad t_m = \sum_n [T^{-1}]_{mn} C_n. \tag{2.94}$$

Therefore, the functions $\mathcal{S}_{\mathbf{k}}$, $\mathcal{V}_{\mathbf{k}}$ and $\mathcal{T}_{\mathbf{k}}$ can be expressed as

$$\mathcal{S}_{\mathbf{k}} = \sum_n \sum_m [S^{-1}]_{nm} A_m \mathcal{P}^{(0)}_{n,\mathbf{k}}, \quad \mathcal{V}_{\mathbf{k}} = \sum_n \sum_m [V^{-1}]_{nm} B_m \mathcal{P}^{(1)}_{n,\mathbf{k}}, \quad \mathcal{T}_{\mathbf{k}} = \sum_n \sum_m [T^{-1}]_{nm} C_m \mathcal{P}^{(2)}_{n,\mathbf{k}}. \tag{2.95}$$

Thus, the solution of the inhomogeneous equation is

$$\begin{aligned} \phi^{\text{part}}_{\mathbf{k}} = &\sum_n \sum_m [S^{-1}]_{nm} A_m \mathcal{P}^{(0)}_{n,\mathbf{k}} \theta + k^{\langle\mu\rangle} \nabla_\mu \alpha \sum_n \sum_m [V^{-1}]_{nm} B_m \mathcal{P}^{(1)}_{n,\mathbf{k}} \\ &+ k^{\langle\nu} k^{\nu\rangle} \sigma_{\mu\nu} \sum_n \sum_m [T^{-1}]_{nm} C_m \mathcal{P}^{(2)}_{n,\mathbf{k}}. \end{aligned} \tag{2.96}$$

Homogeneous solution

The final step to obtain a solution to the first-order truncation of the Boltzmann equation is to determine the coefficients of the homogeneous solution, $\phi^{\text{hom}}_{\mathbf{k}}$. This task is performed by imposing matching conditions.

The free parameters appearing in the homogeneous solution (2.87) can be interpreted as corrections to the local equilibrium fields, α and $\beta_\mu = u_\mu/T$. In order to understand this, let us consider what happens when we consider small variations of these fields in the local equilibrium distribution function,

$$f_{0\mathbf{k}}(\alpha + \delta\alpha, \beta_\mu - \delta\beta_\mu) - f_{0\mathbf{k}}(\alpha, \beta_\mu) = f_{0\mathbf{k}}(\alpha, \beta_\mu) \tilde{f}_{0\mathbf{k}}(\alpha, \beta_\mu) \left(\delta\alpha + \delta\beta_\mu k^\mu\right) + \mathcal{O}(2), \tag{2.97}$$

where $\mathcal{O}(2)$ denotes terms of at least second order in $\delta\alpha$ and $\delta\beta_\mu$. We note that the right-hand side of the above equation has exactly the same form as the nonequilibrium correction due to the homogeneous solution, $f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi^{\text{hom}}_{\mathbf{k}} = f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} (\varphi + \varphi_\mu k^\mu)$, with the correspondence $\delta\alpha \leftrightarrow \varphi$ and $\delta\beta_\mu \leftrightarrow \varphi_\mu$. Thus, the homogeneous solution can be interpreted as a correction to the thermal potential, temperature, and 4-velocity that were arbitrarily introduced in the equilibrium state. Indeed, the local equilibrium solution satisfies the zeroth-order Chapman-Enskog equation for *any* choice of α and β_μ, reflecting a certain level of arbitrariness in selecting such an equilibrium

state. The free parameters of the homogeneous solution emerge as a consequence of such arbitrariness.

As discussed in Section 1.3.1, this issue can be fixed by imposing matching conditions, which specify the reference fictitious local equilibrium state of a dissipative system. The Landau matching conditions, defined in Eqs. (1.37) and (1.39), can be written in terms of integrals of the nonequilibrium distribution function as,

$$\int dK E_{\mathbf{k}} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}} = 0, \quad \int dK E_{\mathbf{k}}^2 f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}} = 0, \quad \int dK E_{\mathbf{k}} k^{\langle\mu\rangle} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}} = 0, \tag{2.98}$$

with $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{\text{hom}} + \phi_{\mathbf{k}}^{\text{part}}$. Replacing Eqs. (2.87) and (2.96) into the above expressions and performing some algebra, one can show that

$$\varphi_0 = \frac{J_{20}\langle E_{\mathbf{k}}^2 \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}} - J_{30}\langle E_{\mathbf{k}} \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}}}{D_{20}}\theta, \tag{2.99a}$$

$$\varphi_1 = \frac{J_{20}\langle E_{\mathbf{k}} \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}} - J_{10}\langle E_{\mathbf{k}}^2 \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}}}{D_{20}}\theta, \tag{2.99b}$$

$$\varphi_2^{\mu} = \frac{1}{3}\frac{\nabla^{\mu}\alpha\langle E_{\mathbf{k}} \mathcal{V}_{\mathbf{k}}(\Delta_{\alpha\beta}k^{\alpha}k^{\beta})\rangle_{\tilde{0}}}{J_{31}}, \tag{2.99c}$$

where we have introduced the thermodynamic integrals

$$D_{mn} = J_{m+1,n}J_{m-1,n} - (J_{mn})^2, \tag{2.100}$$

with J_{mn} being defined in Eq. (2.78), and employed the notation

$$\langle \cdots \rangle_{\tilde{0}} = \int dK \, (\cdots) \, f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{2.101}$$

In particular, using Eqs. (2.95), one can show that

$$\langle E_{\mathbf{k}}^2 \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}} = \sum_n \sum_m [S^{-1}]_{nm} A_m \mathcal{J}_{2n}^{(0)}, \tag{2.102}$$

$$\langle E_{\mathbf{k}} \mathcal{S}_{\mathbf{k}}\rangle_{\tilde{0}} = \sum_n \sum_m [S^{-1}]_{nm} A_m \mathcal{J}_{1n}^{(0)}, \tag{2.103}$$

$$\langle E_{\mathbf{k}} \mathcal{V}_{\mathbf{k}}(\Delta_{\alpha\beta}k^{\alpha}k^{\beta})\rangle_{\tilde{0}} = -3\sum_n \sum_m [V^{-1}]_{nm} B_m \mathcal{J}_{1n}^{(1)}, \tag{2.104}$$

where we have introduced the following thermodynamic integrals,

$$\mathcal{J}_{mn}^{(\ell)} = \frac{(-1)^{\ell}}{(2\ell+1)!!}\left\langle \left(\Delta_{\alpha\beta}k^{\alpha}k^{\beta}\right)^{\ell} E_{\mathbf{k}}^m \mathcal{P}_{n\mathbf{k}}^{(\ell)} \right\rangle_{\tilde{0}}. \tag{2.105}$$

Therefore, the homogeneous solution reads

$$\phi_{\mathbf{k}}^{\text{hom}} = \sum_n \sum_m [S^{-1}]_{nm} A_m \theta \left(\frac{J_{20}\mathcal{J}_{2n}^{(0)} - J_{30}\mathcal{J}_{1n}^{(0)}}{D_{20}} - \frac{J_{10}\mathcal{J}_{2n}^{(0)} - J_{20}\mathcal{J}_{1n}^{(0)}}{D_{20}} E_{\mathbf{k}} \right)$$
$$- k_{\langle\mu\rangle}\nabla^{\mu}\alpha \sum_n \sum_m [V^{-1}]_{nm} B_m \frac{\mathcal{J}_{1n}^{(1)}}{J_{31}}. \tag{2.106}$$

Finally, combining Eqs. (2.96) and (2.106), we obtain the complete solution of the Boltzmann equation at first order in ϵ,

$$\phi_{\mathbf{k}} = \tilde{\mathcal{S}}_{\mathbf{k}}\theta + \tilde{\mathcal{V}}_{\mathbf{k}} k_{\langle\mu\rangle}\nabla^{\mu}\alpha + \mathcal{T}_{\mathbf{k}} k^{\langle\mu}k^{\nu\rangle}\sigma_{\mu\nu}, \tag{2.107}$$

where

$$\tilde{\mathcal{S}}_{\mathbf{k}} = \sum_n \sum_m [S^{-1}]_{nm} A_m \left(\mathcal{P}^{(0)}_{n\mathbf{k}} + \frac{J_{20}\mathcal{J}^{(0)}_{2n} - J_{30}\mathcal{J}^{(0)}_{1n}}{D_{20}} - \frac{J_{10}\mathcal{J}^{(0)}_{2n} - J_{20}\mathcal{J}^{(0)}_{1n}}{D_{20}} E_{\mathbf{k}} \right), \tag{2.108a}$$

$$\tilde{\mathcal{V}}_{\mathbf{k}} = \sum_m [V^{-1}]_{nm} B_m \left(\mathcal{P}^{(1)}_{n\mathbf{k}} - \frac{\mathcal{J}^{(1)}_{1n}}{J_{31}} \right), \tag{2.108b}$$

whereas $\mathcal{T}_{\mathbf{k}}$ has been defined in Eq. (2.95).

2.6.3 Connection with hydrodynamics: Navier-Stokes theory

The fluid-dynamical quantities can be expressed as integrals of the single-particle distribution function, as shown in Eqs. (2.43). This leads to the following expressions for the dissipative currents at first order,

$$\Pi = -\frac{1}{3}\int dK\, (m^2 - E_{\mathbf{k}}^2) f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}}, \tag{2.109}$$

$$n^{\mu} = \int dK\, k^{\langle\mu\rangle} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}}, \tag{2.110}$$

$$\pi^{\mu\nu} = \int dK\, k^{\langle\mu}k^{\nu\rangle} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}}. \tag{2.111}$$

As already discussed, the zeroth-order expressions for these currents vanish. Then, using Eq. (2.107), we derive the relativistic Navier-Stokes constitutive relations,

$$\Pi = -\zeta\theta, \tag{2.112a}$$

$$n^{\mu} = \kappa\nabla^{\mu}\alpha, \tag{2.112b}$$

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{2.112c}$$

where we have introduced the transport coefficients

$$\zeta = \sum_n \sum_m [S^{-1}]_{nm} A_m \left[\frac{1}{3}(m^2\mathcal{J}^{(0)}_{0n} - \mathcal{J}^{(0)}_{2n}) + \frac{\mathcal{J}^{(0)}_{2n}}{3}\frac{(m^2 G_{20} - G_{22})}{D_{20}} - \frac{\mathcal{J}^{(0)}_{1n}}{3} m^2 G_{30} \right], \tag{2.113}$$

$$\kappa = \sum_n \sum_m [V^{-1}]_{nm} B_m \left(\frac{J_{21}}{J_{31}}\mathcal{J}^{(1)}_{1n} - \mathcal{J}^{(1)}_{0n} \right), \tag{2.114}$$

$$\eta = \sum_n \sum_m [T^{-1}]_{nm} C_m \mathcal{J}^{(2)}_{0n}, \tag{2.115}$$

where we have defined

$$G_{mn} = J_{m0}J_{n0} - J_{m-1,0}J_{n+1,0}, \tag{2.116}$$

with the thermodynamic integrals J_{mn} being defined in Eq. (2.78).

The transport coefficients depicted above are, in general, rather intricate functions of the temperature and chemical potential. Once these quantities are computed (which typically involves assuming certain simplifications), one obtains the relativistic Navier-Stokes equations (2.112). At this point, we remark that in the phenomenological derivation of Navier-Stokes theory from the second law of thermodynamics, see Sec. 1.4, the coefficients ζ, κ and η were introduced in a purely *ad hoc* form. However, in order to obtain expressions for these functions, it is necessary to resort to microscopic calculations.

As previously mentioned, the Navier-Stokes equations possess a pathological mathematical structure that enables signals to propagate faster than the speed of light, thus being an acausal formalism. In addition, such behavior further causes small perturbations around a global equilibrium state to grow exponentially, and therefore the Navier-Stokes theory is also (linearly) unstable. According to what we have anticipated at the beginning of this section, the Chapman-Enskog method yields formulations that cannot be employed for practical purposes, and thus we must resort to alternative approaches to solve the Boltzmann equation and, in addition, derive fluid-dynamical theories that are consistent with the requirements of causality and stability. For that purpose, we shall employ the so-called method of moments. In particular, since such formalism is the central object of study of this thesis, we shall dedicate the entirety of next chapter to its derivation and to demonstrate how transient fluid-dynamical theories – which are consistent with nonlinear causality and linear stability – can be consistently derived from a microscopic approach.

3 Method of moments I: Microscopic derivation of fluid dynamics

In the previous chapter, we demonstrated how it is possible to obtain relativistic fluid dynamics from the Boltzmann equations using the Chapman-Enskog method. In this case, the single-particle distribution function is expanded perturbatively in gradients, and the Boltzmann equation is then solved order by order. We showed that a truncation at zeroth order yields the Euler equations (ideal fluid dynamics), whereas a truncation at first order leads to the relativistic Navier-Stokes equations. However, we have shown in Sec. 1.6.1 that such theory is linearly acausal and unstable, thus being unsuitable to describe realistic fluids. Therefore, we concluded that in order to obtain fluid-dynamical theories from the Boltzmann equation that can be employed for practical purposes, one must resort to different approaches.

In this Chapter, we develop the method of moments to investigate the fluid-dynamical limit of the Boltzmann equation. The method, originally proposed by Grad for nonrelativistic gases [63], consists in expanding the single-particle distribution function in terms of its moments. Unlike the Chapman-Enskog expansion, the method of moments corresponds to a truncation in degrees of freedom and not on a small parameter. Israel and Stewart [42] developed the first relativistic generalization of the method of moments, proposing an expansion for the single-particle distribution function using a basis of 4-momenta, $1, k^\mu, k^\mu k^\nu, \cdots$. Then, truncating the so-called moment expansion at second rank, one can express the distribution function solely in terms of the degrees of freedom that appear in the conserved currents N^μ and $T^{\mu\nu}$ – an approach known as the 14-moment approximation. The truncated moment expansion is then used to compute the first three moments of the distribution function, ultimately leading to the Israel-Stewart equations of fluid dynamics, a set of coupled hyperbolic partial differential equations for the dissipative currents. As shown in Sec. 1.6.2, the Israel-Stewart theory does not display the same unphysical problems of the Navier-Stokes theory and can be linearly causal and stable as long as the transport coefficients satisfy a set of constraints [45, 47, 48, 53, 64].

However, since the basis of 4-momenta $1, k^\mu, k^\mu k^\nu, \cdots$ is not orthogonal, the coefficients of the moment expansion cannot be straightforwardly obtained. In this context, in Ref. [2], the authors proposed a generalization of the method of moments, introducing an expansion of the single-particle distribution function in terms of a complete and orthogonal basis of irreducible momenta, $1, k^{\langle\mu\rangle}, k^{\langle\mu}k^{\nu\rangle}, \cdots$. In this case, the expansion coefficients can be obtained in exact form. In particular, the method of moments consists in converting the problem of solving an integro-differential equation for the single-particle distribution function by solving a set of coupled partial differential equations for its irreducible moments.

In what follows, we first outline the generalization of the method of moments developed

in Ref. [2] and obtain an expansion for the single-particle distribution function in terms of its irreducible moments. Then, we explicitly obtain the equations of motion satisfied by the first five moments, which will be later used to estimate the single-particle distribution function. Next, we derive a set of third-order fluid-dynamical equations, i.e., that contains terms of third-order in Knudsen number, neglected in traditional formulations of transient fluid dynamics. In particular, in order to obtain third-order fluid dynamics, we show that it is necessary to include nonhydrodynamic currents of rank 3 and 4 in the moment expansion. Nonetheless, we show that setting these terms to zero, one immediately recovers the traditional Israel-Stewart-like equations. The moment expansion is then truncated employing a minimal truncation scheme, which consists in including solely the least amount of degrees of freedom in order to take into account the aforementioned third-order contributions. This implies in extending the 14-moment approximation by the inclusion of irreducible moments of rank 3 and 4, which effectively increases the number of degrees of freedom to 30. We then investigate the causality and stability of this novel theory in the linear regime and show that the constraints satisfied by the Israel-Stewart theory remain being valid and novel conditions for the third-order transport coefficients appear. Last, we investigate the solutions of the third-order theory in a highly symmetric flow profile known as Bjorken flow [103] and compare them with semi-analytical solutions of the Boltzmann equation.

3.1 Outline

Fluid dynamics is expected to work when the system is close to a state of local thermodynamic equilibrium. For this reason, it is convenient to decompose the single-particle distribution function into an equilibrium and a non-equilibrium contribution,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right), \tag{3.1}$$

where $f_{0\mathbf{k}} = \exp\left(\alpha - \beta E_{\mathbf{k}}\right)$ is the usual Maxwell-Boltzmann equilibrium distribution function [85], with $\alpha = \mu/T$ (referred to as the thermal potential, where μ is the chemical potential and T is the temperature, both being functions of spacetime), $\beta = 1/T$, $E_{\mathbf{k}} = u_\mu k^\mu$ being the energy of the particle in the local rest frame of the fluid. The temperature and chemical potential are defined via matching conditions [67]. Furthermore, $\delta f_{\mathbf{k}}$ denotes the deviation from local equilibrium of the distribution function, with $\phi_{\mathbf{k}} \equiv \delta f_{\mathbf{k}}/f_{0\mathbf{k}}$.

In the method of moments, the non-equilibrium distribution function is expanded using a complete and orthogonal basis of irreducible[1] momenta [2],

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}, \tag{3.2}$$

[1] with respect to Lorentz transformations that leave the fluid 4-velocity unchanged, i.e., $\Lambda^\mu{}_\nu u^\nu = u^\mu$ [60]. In summary, such irreducible tensors are symmetric, traceless and orthogonal to the fluid 4-velocity in every index.

where $A^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} A^{\nu_1\cdots\nu_\ell}$ denotes the irreducible projection of an arbitrary tensor $A^{\nu_1\cdots\nu_\ell}$, with

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} = \sum_{q=0}^{[\ell/2]} \frac{C(\ell,q)}{\mathcal{N}_{\ell,q}} \sum_{\mathcal{P}^\ell_\mu \mathcal{P}^\ell_\nu} \Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2q-1}\mu_{2q}}\Delta_{\nu_1\nu_2}\cdots\Delta_{\nu_{2q-1}\nu_{2q}}\Delta^{\mu_{2q+1}}_{\nu_{2q+1}}\cdots\Delta^{\mu_\ell}_{\nu_\ell}, \tag{3.3}$$

being the 2ℓ-index projection operator symmetric under the exchange of μ- and ν-type indices, traceless and orthogonal to the 4-velocity in every index [58,60]. It is constructed in terms of the 2-index projection operator onto the 3-space orthogonal to the fluid 4-velocity, $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. The first sum runs up to the largest integer less than or equal to $\ell/2$, while the second sum accounts for all possible permutations of the indices. The factors $C(\ell,q)$ and $\mathcal{N}_{\ell,q}$ are defined as

$$C(\ell,q) = (-1)^q \frac{(\ell!)^2}{(2\ell)!}\frac{(2\ell-2q)!}{q!(\ell-q)!(\ell-2q)!}, \qquad \mathcal{N}_{\ell,q} = \frac{1}{(\ell-2q)!}\left(\frac{\ell!}{2^q q!}\right)^2. \tag{3.4}$$

The former is essential to ensure the traceless property of the projection operator,

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} g^{\nu_i\nu_j} = 0, \qquad \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} g_{\mu_i\mu_j} = 0, \qquad 1 \leq (i,j) \leq \ell, \tag{3.5}$$

while the latter is simply the inverse of the total number of permutations given in the second sum in Eq. (3.3), in order to avoid over-counting any particular term.

We remark that, in contrast to the basis chosen by Israel and Stewart, the irreducible momenta satisfy the following orthogonality condition [58, 60]

$$\int dK F(E_{\mathbf{k}}) k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} k_{\langle\nu_1}\cdots k_{\nu_m\rangle} = \frac{\ell!\,\delta_{\ell m}}{(2\ell+1)!!}\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}\int dK F(E_{\mathbf{k}}) \left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^m, \tag{3.6}$$

with $F(E_{\mathbf{k}})$ being an arbitrary function of $E_{\mathbf{k}}$.

Following Ref. [2], the expansion coefficients $\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}}$ are functions of $E_{\mathbf{k}}$ and are further expanded using a complete basis of orthogonal functions, $P^{(\ell)}_{\mathbf{k}n}$,

$$\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}} = \sum_{n=0}^{\infty} \Phi^{\langle\mu_1\cdots\mu_\ell\rangle}_n P^{(\ell)}_{\mathbf{k}n}. \tag{3.7}$$

For the sake of convenience[2], $P^{(\ell)}_{\mathbf{k}n}$ are taken as polynomials of $E_{\mathbf{k}}$,

$$P^{(\ell)}_{\mathbf{k}n} = \sum_{r=0}^{n} a^{(\ell)}_{nr} E^r_{\mathbf{k}}. \tag{3.8}$$

Without loss of generality, we set $P^{(\ell)}_{\mathbf{k}0} = 1$, so that $a^{(\ell)}_{00} = 1$ and all remaining coefficients can be obtained using the Gram-Schmidt orthogonalization procedure, see Appendix E of Ref. [2] and Ref. [58] for details. In particular, these functions are constructed to satisfy the following orthogonality condition

$$\int dK \omega^{(\ell)} P^{(\ell)}_{\mathbf{k}n} P^{(\ell)}_{\mathbf{k}m} = \delta_{mn}, \tag{3.9}$$

[2] For massless particles, they reduce to the associated Laguerre polynomials [58].

with the weight $\omega^{(\ell)}$ being

$$\omega^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{(2\ell+1)!!} \left(\Delta_{\alpha\beta} k^\alpha k^\beta\right)^\ell f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{3.10}$$

From the orthogonality condition (3.6), and using Eqs. (3.2)-(3.9), the expansion coefficients in Eq. (3.7) can be expressed as

$$\Phi_n^{\langle \mu_1 \cdots \mu_\ell \rangle} = \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{r=0}^{n} a_{nr}^{(\ell)} \rho_r^{\mu_1 \cdots \mu_\ell}, \tag{3.11}$$

with $\rho_r^{\mu_1 \cdots \mu_\ell}$ being the irreducible moments of the non-equilibrium distribution function,

$$\rho_r^{\mu_1 \cdots \mu_\ell} = \int dK E_{\mathbf{k}}^r k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}. \tag{3.12}$$

Finally, we have

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{H}_{n\mathbf{k}}^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \right). \tag{3.13}$$

where we have defined the coefficients

$$\mathcal{H}_{n\mathbf{k}}^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{r=n}^{N_\ell} a_{rn}^{(\ell)} P_{\mathbf{k}r}^{(\ell)}. \tag{3.14}$$

We note that some of these irreducible moments correspond to the fluid-dynamical variables introduced in the previous Chapters. For instance, we identify,

$$\Pi = -\frac{1}{3}(m^2 \rho_0 - \rho_2), \quad n^\mu = \rho_0^\mu, \quad h^\mu = \rho_1^\mu, \quad \pi^{\mu\nu} = \rho_0^{\mu\nu}. \tag{3.15}$$

We note that in the Landau picture, adopted throughout this thesis, $\rho_1 = 0$, $\rho_2 = 0$, and $\rho_1^\mu = 0$. Thus, the bulk viscous pressure reduces to $\Pi = -m^2 \rho_0 / 3$, and hence vanishes in the massless limit.

We have expressed the single-particle distribution function completely in terms of the irreducible moments of $\delta f_{\mathbf{k}}$, which are the only terms *a priori* unknown in the expression above. The single-particle distribution function can thus be reconstructed from its irreducible moments, with the *exact* solution for $f_{\mathbf{k}}$ being obtained when all sums are taken to infinity. Although unfeasible in practice, this limit can be consistently reproduced by including a sufficiently large number of moments in the moment expansion. As more terms are included in the expansion, one expects to obtain values of the distribution function which are in better agreement with the *exact* solution of the Boltzmann equation, at least up to a given energy scale. It is then necessary to determine the time evolution of the irreducible moments in order to obtain the single-particle distribution function itself, thus effectively solving the Boltzmann equation. In general, this task requires the inclusion of moments of arbitrarily high ranks and not just those that appear in fluid-dynamical theories.

The method of moments provides a suitable approach to obtain fluid dynamics from a microscopic perspective. In a fluid-dynamical description, the evolution of the system is given in terms of a reduced number of degrees freedom, and thus it is not necessary to know the single-particle distribution function, $f_{\mathbf{k}}$ – one is only required to know a small subset of its moments, the so-called hydrodynamic variables. Therefore, it should be possible to truncate the moment expansion of the distribution function, with different truncation schemes providing a wide range of effective descriptions of the fluid-dynamical regime. We remark that in this case we would not have a perturbative series in terms of a small parameter, but rather a truncation in degrees of freedom. In order to move forward, we essentially need to derive the equations of motion for these moments and understand how their truncation affect the fluid-dynamical description of the system. The derivation of differential equations for the irreducible moments is pursued in detail throughout the following sections.

3.2 Equations of motion

In this section, we derive the equations of motion satisfied by the irreducible moments of the nonequilibrium single-particle distribution function, $\delta f_{\mathbf{k}}$. These equations will later be used to provide an estimate of $f_{\mathbf{k}}$ in the fluid-dynamical limit. The equations of motion for the moments of rank 0, 1, and 2 have been derived for the first time in Ref. [2] and are reproduced below. First, the equation of motion for the scalar moment is

$$\dot{\rho}_r = C_{r-1} + \alpha_r^{(0)}\theta + r\rho_{r-1}^{\mu}\dot{u}_\mu - \nabla_\mu \rho_{r-1}^{\mu} + \frac{G_{3r}}{D_{20}}\partial_\mu n^\mu + \left[(r-1)\rho_{r-2}^{\mu\nu} + \frac{G_{3r}}{D_{20}}\pi^{\mu\nu}\right]\sigma_{\mu\nu} + \frac{1}{3}\left[(r-1)m^2\rho_{r-2} - (r+2)\rho_r - 3\frac{G_{2r}}{D_{20}}\Pi\right]\theta. \tag{3.16}$$

The moments of rank 1 satisfy the following equation of motion

$$\dot{\rho}_r^{\langle\mu\rangle} = C_{r-1}^{\langle\mu\rangle} + \alpha_r^{(1)}\nabla^\mu\alpha_0 + r\dot{u}_\nu\rho_{r-1}^{\mu\nu} - \Delta_\alpha^\mu\nabla_\beta\rho_{r-1}^{\alpha\beta} + \rho_r^{\langle\nu}\omega^{\mu\rangle}{}_\nu + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu} - (r+3)\rho_r^\mu\right]\theta + (r-1)\sigma_{\alpha\beta}\rho_{r-2}^{\mu\alpha\beta} + +\frac{1}{3}\left[rm^2\rho_{r-1} - (r+3)\rho_{r+1}\right]\dot{u}^\mu - \frac{1}{3}\nabla^\mu\left(m^2\rho_{r-1} - \rho_{r+1}\right) + \frac{1}{5}\left[(2r-2)m^2\rho_{r-2}^{\nu} - (2r+3)\rho_r^\nu\right]\sigma_\mu^\nu + \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta_\nu^\mu\partial_\lambda\pi^{\nu\lambda}\right), \tag{3.17}$$

while the irreducible moments of rank 2 satisfy

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = C_{r-1}^{\langle\mu\nu\rangle} + 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}\left[(r-1)m^4\rho_{r-2} - (2r+3)m^2\rho_r + (r+4)\rho_{r+2}\right]\sigma^{\mu\nu} + \frac{2}{5}\left[rm^2\rho_{r-1}^{\langle\mu} - (r+5)\rho_{r+1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right) + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda + \frac{2}{7}\left[(2r-2)m^2\rho_{r-2}^{\lambda\langle\mu} - (2r+5)\rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu\nu} - (r+4)\rho_r^{\mu\nu}\right]\theta + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda + (r-1)\rho_{r-2}^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda}. \tag{3.18}$$

In Chapter 4, as well as in Appendix D, we discuss in detail how these equations are derived.

A fundamental part of this thesis was to calculate the equations of motion for higher-rank tensors, which were simply unknown before. Our starting point was to calculate the equations of motion for the third and fourth rank tensors, which we display below. Later in this thesis, we shall derive equations of motion for moments of arbitrary rank. For the irreducible moments of rank 3, we obtain

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\lambda\rangle} &= C_{r-1}^{\langle\mu\nu\lambda\rangle} + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu\nu\lambda} - (r+5)\rho_r^{\mu\nu\lambda}\right]\theta + \frac{3}{7}\left[rm^2\rho_{r-1}^{\langle\mu\nu} - (r+7)\rho_{r+1}^{\langle\mu\nu}\right]\dot{u}^{\lambda\rangle} \\ &+ \frac{6}{35}\sigma^{\langle\mu\nu}\left[(r-1)m^4\rho_{r-2}^{\lambda\rangle} - (2r+5)m^2\rho_r^{\lambda\rangle} + (r+6)\rho_{r+2}^{\lambda\rangle}\right] - \frac{3}{7}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\lambda\rangle} - \rho_{r+1}^{\nu\lambda\rangle}\right) \\ &+ 3\rho_r^{\alpha\langle\mu\nu}\omega^{\lambda\rangle}{}_\alpha + \frac{1}{3}\sigma_\alpha^{\langle\mu}\left[m^2(2r-2)\rho_{r-2}^{\nu\lambda\rangle\alpha} - (2r+7)\rho_r^{\nu\lambda\rangle\alpha}\right] + r\dot{u}_\alpha\rho^{\mu\nu\lambda\alpha} - \Delta^{\mu\nu\lambda}_{\alpha\beta\sigma}\nabla_\gamma\rho_{r-1}^{\alpha\beta\sigma\gamma} \\ &+ (r-1)\sigma_{\alpha\beta}\rho_{r-2}^{\mu\nu\lambda\alpha\beta}. \end{aligned} \tag{3.19}$$

Finally, the irreducible moments of rank 4 satisfy the following equations of motion

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\alpha\beta\rangle} &= C_{r-1}^{\langle\mu\nu\alpha\beta\rangle} + r\dot{u}_\lambda\rho_r^{\mu\nu\alpha\beta\lambda} + 4\dot{u}^{\langle\mu}\left[\frac{rm^2}{9}\rho_{r-1}^{\nu\alpha\beta\rangle} - \left(1+\frac{r}{9}\right)\rho_{r+1}^{\nu\alpha\beta\rangle}\right] \\ &+ \frac{4}{21}\sigma^{\langle\mu\nu}\left[(r-1)m^4\rho_{r-2}^{\alpha\beta\rangle} - (2r+7)m^2\rho_r^{\alpha\beta\rangle} + (r+8)\rho_{r+2}^{\alpha\beta\rangle}\right] + 4\rho_r^{\lambda\langle\mu\nu\alpha}\omega^{\beta\rangle}{}_\lambda \\ &\frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu\nu\alpha\beta} - (r+6)\,\rho_r^{\mu\nu\alpha\beta}\right]\theta - \frac{4}{9}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\alpha\beta\rangle} - \rho_{r+1}^{\nu\alpha\beta\rangle}\right) - \Delta^{\mu\nu\alpha\beta}_{\sigma\gamma\psi\rho}\nabla_\lambda\rho_{r-1}^{\sigma\gamma\psi\rho\lambda} \\ &+ \frac{4}{11}\sigma_\lambda^{\langle\mu}\left[(2r-2)m^2\rho_{r-2}^{\nu\alpha\beta\rangle\lambda} - (2r+9)\rho_r^{\nu\alpha\beta\rangle\lambda}\right] + (r-1)\sigma_{\lambda\sigma}\rho_{r-2}^{\mu\nu\alpha\beta\lambda\sigma}. \end{aligned} \tag{3.20}$$

In deriving these equations, we have used the following identities

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle} = k^{\langle\mu}k^{\nu\rangle} + \frac{1}{3}\Delta^{\mu\nu}\Delta_{\mathbf{kk}}, \tag{3.21a}$$

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\alpha\rangle} = k^{\langle\mu}k^\nu k^{\alpha\rangle} + \frac{1}{5}\Delta_{\mathbf{kk}}\Delta^{(\mu\nu}k^{\alpha)}, \tag{3.21b}$$

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\alpha\rangle}k^{\langle\beta\rangle} = k^{\langle\mu}k^\nu k^\alpha k^{\beta\rangle} + \frac{6}{7}\Delta_{\mathbf{kk}}\Delta^{(\mu\nu}k^{\alpha}k^{\beta)} - \frac{3}{35}\Delta_{\mathbf{kk}}\Delta^{\mu(\nu}\Delta^{\alpha\beta)}, \tag{3.21c}$$

$$k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\alpha\rangle}k^{\langle\beta\rangle}k^{\langle\rho\rangle} = k^{\langle\mu}k^\nu k^\alpha k^\beta k^{\rho\rangle} + \frac{10}{9}\Delta_{\mathbf{kk}}\Delta^{(\mu\nu}k^{\alpha}k^{\beta}k^{\rho)} - \frac{15}{63}\Delta^2_{\mathbf{kk}}\Delta^{(\mu\nu}\Delta^{\alpha\beta}k^{\rho)}, \tag{3.21d}$$

$$\begin{aligned}k^{\langle\mu\rangle}k^{\langle\nu\rangle}k^{\langle\alpha\rangle}k^{\langle\beta\rangle}k^{\langle\rho\rangle}k^{\langle\sigma\rangle} &= k^{\langle\mu}k^\nu k^\alpha k^\beta k^\rho k^{\sigma\rangle} + \frac{15}{11}\Delta_{\mathbf{kk}}\Delta^{(\mu\nu}k^{\alpha}k^{\beta}k^{\rho}k^{\sigma)} \\ &- \frac{45}{99}\Delta^2_{\mathbf{kk}}\Delta^{(\mu\nu}\Delta^{\alpha\beta}k^{\rho}k^{\sigma)} + \frac{15}{693}\Delta^3_{\mathbf{kk}}\Delta^{(\mu\nu}\Delta^{\alpha\beta}\Delta^{\rho\sigma)}, \end{aligned} \tag{3.21e}$$

where we remind the reader that $\Delta_{\mathbf{kk}} = \Delta^{\mu\nu}k_\mu k_\nu$. Moreover, the parentheses in the indices denote all possible permutations in them. We further used the equations of motion for the thermal potential, inverse temperature and 4-velocity, that stem from the conservation laws, Eqs. (1.11),

$$\dot{\alpha}_0 = \frac{1}{D_{20}}\left\{-J_{30}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{20}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\}, \tag{3.22a}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}}\left\{-J_{20}\left(n_0\theta + \partial_\mu n^\mu\right) + J_{10}\left[\left(\varepsilon_0 + P_0 + \Pi\right)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right]\right\}, \tag{3.22b}$$

$$\dot{u}^\mu = \frac{1}{\varepsilon_0 + P_0}\left(\nabla^\mu P - \Pi\dot{u}^\mu + \nabla^\mu\Pi - \Delta^\mu_\nu\partial_\lambda\pi^{\nu\lambda}\right), \tag{3.22c}$$

and introduced the following thermodynamic variables

$$\alpha_r^{(0)} \equiv (1-r)I_{r1} - I_{r0} - \frac{1}{D_{20}}\left[(\varepsilon_0 + P_0)G_{2r} - n_0 G_{3r}\right], \tag{3.23a}$$

$$\alpha_r^{(1)} \equiv J_{r+1,1} - h_0^{-1} J_{r+2,1}, \tag{3.23b}$$

$$\alpha_r^{(2)} \equiv I_{r+2,1} + (r-1)I_{r+2,2}. \tag{3.23c}$$

We have also employed the notation for the thermodynamic quantities introduced in Eqs. (2.78), (2.100) and (2.116), but are reproduced below for convenience,

$$I_{mn} = \frac{(-1)^n}{(2n+1)!!}\int dK E_{\mathbf{k}}^{m-2n}(\Delta^{\alpha\beta}k_\alpha k_\beta)^n f_{0\mathbf{k}}, \tag{3.24a}$$

$$J_{mn} = \frac{(-1)^n}{(2n+1)!!}\int dK E_{\mathbf{k}}^{m-2n}(\Delta^{\alpha\beta}k_\alpha k_\beta)^n f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}, \tag{3.24b}$$

$$G_{mn} = J_{m0}J_{n0} - J_{m-1,0}J_{n+1,0}, \tag{3.24c}$$

$$D_{mn} = J_{m+1,n}J_{m-1,n} - (J_{mn})^2, \tag{3.24d}$$

and defined the generalized collision term

$$C_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} C[f], \tag{3.25}$$

following the notation of Ref. [2]. As already stated, in Chapter 4 and Appendix D, we discuss in detail how equations of motion for the irreducible moments are derived.

The equations of motion for the irreducible moments up to rank 2, Eqs. (3.16)-(3.18), all have the same structure: the dominant terms are the so-called Navier-Stokes terms, which are given by irreducible projections of first order derivatives of temperature, chemical potential, and 4-velocity, i.e., $\nabla^\mu\alpha$, θ and $\sigma^{\mu\nu}$. These type of terms are of first-order in Knudsen number and appear as the dominant contribution in a gradient expansion of these irreducible tensors [58, 60]. The remaining terms in these equations of motion are at least of second order in a gradient expansion.

The equations of motion for the irreducible moments of rank 3 and 4, Eqs. (3.19) and (3.20), respectively, on the other hand, display qualitative differences from the scenario depicted above. First, such equations do not contain any term that is of first order in gradients of the primary hydrodynamic variables. This happens because it is not possible to construct irreducible tensors of rank higher than 2 solely from first-order derivatives of temperature, chemical potential, and 4-velocity. Therefore, in these equations of motion, the dominant contribution in a gradient expansion is at least of second order in Knudsen number. Thus, in the same way that gradients of T, μ and u^μ act as source terms for the dissipative currents appearing in N^μ and $T^{\mu\nu}$, irreducible moments of rank 1 and 2 and derivatives thereof act as the dominant source terms for the irreducible moments of rank 3 and 4.

We finally note that the irreducible moments of rank 3 and 4 only appear in the equations of motion for ρ_r, ρ_r^μ and $\rho_r^{\mu\nu}$ multiplied by a term of first order in gradients or being differentiated.

Thus, such contributions would be at least of third order in a gradient expansion. This is the reason why these terms are usually neglected in the derivation of the so-called second-order theories [2], since such formulations only include contributions to the conserved currents that are up to second order in Knudsen number. In this thesis, one of our goals is to derive a third-order theory of fluid dynamics and, for this purpose, these contributions cannot be ignored. Thus, as argued in Ref. [72], we shall include degrees of freedom that can be matched to irreducible tensors of rank 3 and 4 and will incorporate such corrections. We remark that irreducible moments of rank 5 or higher are at least of third order in a gradient expansion (appearing as corrections of fourth order or higher in the dynamics of the particle diffusion 4-current and the shear-stress tensor) and will not contribute to a third-order formulation.

The next step is to use the equations of motion for the irreducible moments of the nonequilibrium distribution function, Eqs. (3.16)–(3.20), to obtain a closed set of equations of motion for the dissipative currents. This task will be carefully performed in the following section.

3.3 Microscopic derivation of transient fluid dynamics

In the method of moments, fluid dynamics is derived as a truncation in degrees of freedom from the expansion of $f_{\mathbf{k}}$ in terms of its irreducible moments. This truncation procedure can be systematically improved by including more terms in the moment expansion. Then, one is able to describe the system using a reduced – and, in particular, *finite* – number of degrees of freedom. In this section, we detail how these truncation procedures are constructed and explain the 14-moment approximation developed by Israel and Stewart [42], as well as its extension that we developed to obtain a set of third-order fluid-dynamical equations [72].

3.3.1 Israel-Stewart theory

Currently, second-order theories of fluid dynamics are the most widely employed in the description of relativistic viscous fluids, since such theories can be constructed to be linearly causal and stable around global equilibrium [38, 45–48, 53]. We remark that the most traditional second-order theory is due to Israel and Stewart [40], developed in the 1970s for applications in cosmology, but several additional formulations have been developed ever since [2, 108–115]. Furthermore, nonlinear constraints on the causality of these theories have been thoroughly investigated over the last years [43, 90, 91, 116].

In this section, we detail the 14-moment approximation developed by Israel and Stewart [40]. In their original work, Israel and Stewart [40] truncated the expansion of $\phi_{\mathbf{k}}$ at second order in momenta,

$$\phi_{\mathbf{k}}^{\mathrm{IS}} = \lambda + \lambda^{\mu} k_{\mu} + \lambda^{\mu\nu} k_{\mu} k_{\nu} + \mathcal{O}(k^3). \tag{3.26}$$

In this truncated expansion, there is a total of 14 degrees of freedom, which can be matched to the number of independent components of the particle 4-current, N^{μ}, and the energy-momentum

tensor, $T^{\mu\nu}$. This procedure is usually referred to as the 14-moment approximation. This approach can also be implemented using the *complete* basis of irreducible tensors introduced in the previous section [2]. In this case, one expresses $\phi_{\mathbf{k}}$ up to second order in momenta as

$$\phi^{\rm IS}_{\mathbf{k}} = \lambda_{\mathbf{k}} + \lambda^{\langle\mu\rangle}_{\mathbf{k}} k_{\langle\mu\rangle} + \lambda^{\langle\mu\nu\rangle}_{\mathbf{k}} k_{\langle\mu} k_{\nu\rangle} + \mathcal{O}(k^3). \tag{3.27}$$

The coefficients $\lambda_{\mathbf{k}}$, $\lambda^{\langle\mu\rangle}_{\mathbf{k}}$, and $\lambda^{\langle\mu\nu\rangle}_{\mathbf{k}}$ now carry a momentum dependence and are written in terms of orthogonal polynomials so that only terms that are of second-order or less in momentum remain,

$$\lambda_{\mathbf{k}} = \Phi_0 + P^{(0)}_{\mathbf{k}1}\Phi_1 + P^{(0)}_{\mathbf{k}2}\Phi_2, \tag{3.28a}$$

$$\lambda^{\langle\mu\rangle}_{\mathbf{k}} = \Phi^{\langle\mu\rangle}_0 + P^{(1)}_{\mathbf{k}1}\Phi^{\langle\mu\rangle}_1, \tag{3.28b}$$

$$\lambda^{\langle\mu\nu\rangle}_{\mathbf{k}} = \Phi^{\langle\mu\nu\rangle}_0. \tag{3.28c}$$

This approximation corresponds to truncating the expansion defined in (3.7) using $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$ and corresponds to an equivalent way to express the 14-moment approximation proposed by Israel and Stewart.

As already stated, the truncation above is not motivated by an expansion in terms of a small parameter, as occurs in the Chapman-Enskog expansion [61]. It is a truncation in degrees of freedom and one simply stops when the number of degrees of freedom in the expansion becomes identical to the number of degrees of freedom expected in the fluid-dynamical theory (in the case of second-order fluid dynamics, 14 fields). For this reason, we included three terms in the expansion of the scalar coefficient ($\ell = 0$), since one of them is mapped onto the bulk viscous pressure, while the other two are determined from the matching conditions that define the temperature and chemical potential. For the 4-vector coefficient ($\ell = 1$), we included two terms, since one is mapped onto the particle diffusion 4-current and the other onto the energy diffusion 4-current – one of these currents (here, the energy diffusion) is traditionally eliminated by matching conditions that define the fluid 4-velocity. Finally, for $\ell = 2$, we included only one term in the expansion, that is mapped onto the shear-stress tensor. This truncation procedure is usually referred to as a *minimal truncation scheme*.

Equations of motion

The next step is to derive relations between the irreducible moments of the nonequilibrium distribution function and the coefficients of the truncated moment expansion. In other words, our goal is to obtain relations between $\Phi^{\mu_1\cdots\mu_\ell}_r$ and the irreducible moments $\rho^{\mu_1\cdots\mu_\ell}_r$. From Eqs. (3.6)-(3.13), one can show that

$$\rho^{\mu_1\cdots\mu_\ell}_r = (-1)^\ell \ell! \sum_{n=0}^{N_\ell}\sum_{m=0}^{n} \Phi^{\langle\mu_1\cdots\mu_\ell\rangle}_n a^{(\ell)}_{nm} J_{r+m+2\ell,\ell}. \tag{3.29}$$

Therefore, given the truncation scheme adopted ($N_0 = 2$, $N_1 = 1$, and $N_2 = 0$, with all higher-rank contributions being identically neglected), it follows that the irreducible moments of rank 0,

1, and 2 can all be expressed in terms of the dissipative currents,

$$\rho_r = \gamma_r^{\Pi}\Pi, \tag{3.30a}$$
$$\rho_r^{\mu} = \gamma_r^{n} n^{\mu}, \tag{3.30b}$$
$$\rho_r^{\mu\nu} = \gamma_r^{\pi}\pi^{\mu\nu}, \tag{3.30c}$$

where we have introduced the following thermodynamic coefficients

$$\gamma_r^{\Pi} = \mathcal{A}_{\Pi} J_{r,0} + \mathcal{A}_{\Pi} J_{r+1,0} + \mathcal{C}_{\Pi} J_{r+2,0}, \tag{3.31a}$$
$$\gamma_r^{n} = -\frac{J_{41}J_{r+2,1}}{D_{31}} + \frac{J_{31}J_{r+3,1}}{D_{31}}, \tag{3.31b}$$
$$\gamma_r^{\pi} = \frac{J_{r+4,2}}{J_{4,2}}, \tag{3.31c}$$

with

$$\mathcal{A}_{\Pi} = -\frac{3}{m^2}\frac{D_{30}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \tag{3.32a}$$
$$\mathcal{B}_{\Pi} = -\frac{3}{m^2}\frac{G_{23}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \tag{3.32b}$$
$$\mathcal{C}_{\Pi} = -\frac{3}{m^2}\frac{D_{20}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}. \tag{3.32c}$$

In particular, we remark that there is an infinite number of equations of motion for the irreducible moments, labeled by the subindices r, cf. Eqs. (3.16)-(3.20). Therefore, there is an ambiguity in the choice of the dynamic variable of the theory. That is, one has the freedom to take any particular value for r and construct the theory around the corresponding irreducible moment [113]. Following Ref. [113], we choose $r = 0$, which corresponds to the exact equations of motion for the bulk viscous pressure ($\Pi = -m^2\rho_0/3$), the particle diffusion 4-current ($n^{\mu} = \rho_0^{\mu}$), and the shear-stress tensor ($\rho_0^{\mu\nu} = \pi^{\mu\nu}$), cf. Eq. (3.15).

A relation for the generalized collision term in terms of the irreducible moments is still required. Here, we adopt a rather simplified prescription for the collision term, known as the relaxation time approximation [104, 117], in which the single-particle distribution function is assumed to relax to the equilibrium distribution with a timescale τ_R. Assuming an energy-independent relaxation time, this approximation yields

$$C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}\delta f_{\mathbf{k}}, \tag{3.33}$$

In this case, the generalized collision term becomes

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\frac{\rho_r^{\mu_1\cdots\mu_\ell}}{\tau_R}. \tag{3.34}$$

We remark that such prescription for the collision term is only consistent with the conservation of energy and momentum as long as one imposes Landau matching conditions. A generalization

of the relaxation time approximation for which the conservation laws are fulfilled for arbitrary matching conditions was first addressed in Ref. [117].

We are now in position to obtain a closed set of equations of motion for the dissipative currents, Π, n^μ and $\pi^{\mu\nu}$. For this purpose, we insert the relations in Eqs. (3.30) and (3.34) into the equations of motion for the irreducible moments, Eqs. (3.16)-(3.20), for $r = 0$. We then obtain, for the bulk viscous pressure,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \tau_{\Pi n} n_\mu \nabla^\mu P_0 - \ell_{\Pi n}\partial_\mu n^\mu - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi n} n^\mu \nabla_\mu \alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}. \quad (3.35)$$

For the particle diffusion, we have

$$\begin{aligned}\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = {} & \kappa_n \nabla^\mu \alpha_0 - \tau_n n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta + \tau_{n\Pi}\Pi\nabla^\mu P_0 - \tau_{n\pi}\pi^{\mu\nu}\nabla_\nu P_0 - \ell_{n\Pi}\nabla^\mu\Pi \\ & + \ell_{n\pi}\Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi}\Pi\nabla^\mu\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0. \end{aligned} \quad (3.36)$$

The equation of motion for the shear-stress tensor reads

$$\begin{aligned}\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = {} & 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\langle\mu}_\lambda \sigma^{\nu\rangle\lambda} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle} P_0 \\ & + \ell_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0. \end{aligned} \quad (3.37)$$

We remark that these equations are identical to the ones obtained in Ref. [2]. These equations are the most simple version of the Denicol-Niemi-Molnár-Rischke (DNMR) equations [2]. They are very similar to the usual Israel-Stewart theory, discussed in Chapter 1 but contain additional nonlinear terms that do not appear in the phenomenological derivation of the equations. This approach is investigated in Appendix C.

The viscosity coefficients appearing in the fluid-dynamical equations are given by,

$$\zeta = -\frac{3}{m^2}\tau_\Pi \left\{ I_{11} - I_{00} - \frac{1}{D_{20}}\left[(\varepsilon_0 + P_0)G_{20} - n_0 G_{30}\right]\right\}, \quad (3.38a)$$

$$\kappa_n = \tau_n \left(J_{2,1} - h_0^{-1} J_{2,1}\right), \quad (3.38b)$$

$$\eta = \tau_\pi \left(I_{2,1} - I_{2,2}\right). \quad (3.38c)$$

The remaining transport coefficients can be found in [2]. We remark that, since we are employing the relaxation time approximation for the collision term, all relaxation times are identical to the relaxation time introduced in the approximation, $\tau_\Pi = \tau_n = \tau_\pi = \tau_R$.

3.3.2 Third-order fluid dynamics

Naturally, one may also consider the derivation of third-order relativistic fluid-dynamical theories. As a matter of fact, several authors have already investigated this topic, using different frameworks e.g., a gradient expansion [118], a phenomenological description using the second law of thermodynamics [119, 120] and kinetic theory, using a method inspired in the Chapman-Enskog expansion [1]. In particular, the latter formulation was shown to be in good agreement

with solutions of the relativistic Boltzmann equation [1, 121]. These studies were performed assuming the highly symmetric Bjorken flow scenario [103], where solutions of the Boltzmann equation can actually be obtained without resorting to complex numerical schemes [119, 122].

Recently, the third-order formalism developed in Ref. [1] was shown to be linearly acausal and unstable [50], presenting the same pathology originally observed in Navier-Stokes theory [39]. We proposed an *ad hoc* modification to this theory in Ref. [50], in order to address the aforementioned problem. In this chapter, we derive a more fundamental version of this framework from kinetic theory using the traditional method of moments [2], outlined in Sec. 3.1. We shall demonstrate that, in order to obtain equations that include all terms that are asymptotically of third order in gradients, it is necessary to include novel degrees of freedom that correspond to irreducible tensors of rank 3 and 4, while traditional fluid-dynamical theories developed only account for irreducible tensors of rank 0, 1, and 2. Finally, we show that solutions of this formulation are in good agreement with solutions of the relativistic Boltzmann equation assuming a Bjorken flow scenario.

As previously mentioned, the moment expansion can be in principle improved by the systematic inclusion of additional degrees of freedom. In this subsection, we consider a truncation beyond the traditional hydrodynamic limit by including moments of rank higher than 2, i.e., nonhydrodynamic[3] currents. In particular, these terms are at least of second order in Knudsen number and appear in the moment equations as terms of third order. We then investigate how we can interpret these extended fluid-dynamical theories.

In order to accommodate these novel nonhydrodynamic degrees of freedom, it is necessary to propose a new minimal truncation for the expansion of $\phi_{\mathbf{k}}$, which then takes the following form

$$\phi_{\mathbf{k}}^{\mathrm{3rd-order}} = \lambda_{\mathbf{k}} + \lambda_{\mathbf{k}}^{\langle\mu\rangle} k_{\langle\mu\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} k_{\langle\mu} k_{\nu\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\rangle} k_{\langle\mu} k_{\nu} k_{\alpha\rangle} + \lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\beta\rangle} k_{\langle\mu} k_{\nu} k_{\alpha} k_{\beta\rangle} + \mathcal{O}\left(k^5\right). \quad (3.39)$$

We remark that the truncation of the expansion of the distribution function in momenta of rank 3 has also been studied in the context of relativistic shock waves [123]. The expansion coefficients $\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle}$ are given by,

$$\lambda_{\mathbf{k}} = \Phi_0 + P_{\mathbf{k}1}^{(0)}\Phi_1 + P_{\mathbf{k}2}^{(0)}\Phi_2, \quad (3.40a)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\rangle} = \Phi_0^{\langle\mu\rangle} + P_{\mathbf{k}1}^{(1)}\Phi_1^{\langle\mu\rangle}, \quad (3.40b)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} = \Phi_0^{\langle\mu\nu\rangle}, \quad (3.40c)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\rangle} = \Phi_0^{\langle\mu\nu\alpha\rangle}, \quad (3.40d)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\nu\alpha\beta\rangle} = \Phi_0^{\langle\mu\nu\alpha\beta\rangle}. \quad (3.40e)$$

The expressions for the expansion coefficients of rank 0, 1, and 2 are identical to the ones used in the 14-moment approximation, see Eqs. (3.28). The expansion coefficients of rank 3 and 4

[3] Here, the term *nonhydrodynamic* denotes degrees of freedom that do not appear in the conserved currents.

are new and guarantee that irreducible moments of rank 3 and 4 can be introduced as novel dynamical variables. This is the minimal truncation scheme for a linearly causal and stable third-order theory [50]. We note that these new terms in the expansion of $\phi_{\mathbf{k}}$ increase the number of independent fields from 14 to 30, since each tensor contributes with $2\ell+1$ degrees of freedom, with ℓ being the rank of the respective tensor, given that they are symmetric, traceless and orthogonal to u^μ in all indices.

Equations of motion

We have truncated the expansion for the nonequilibrium distribution function in its irreducible moments imposing that it can be determined in terms of 30 degrees of freedom, instead of the 14 degrees of freedom present in the traditional second-order hydrodynamic equation. Following the same steps oulined in the previous section, this procedure will naturally lead to a closed set of equations of motion for such extended set of variables. We once again adopt a minimal truncation, which implies taking $N_0 = 2$, $N_1 = 1$, and $N_2 = N_3 = N_4 = 0$ in Eq. (3.29), with all higher-rank contributions being identically neglected. The irreducible moments are given by

$$\rho_r = \gamma_r^{\Pi}\Pi, \tag{3.41a}$$
$$\rho_r^{\mu} = \gamma_r^{n} n^{\mu}, \tag{3.41b}$$
$$\rho_r^{\mu\nu} = \gamma_r^{\pi}\pi^{\mu\nu}, \tag{3.41c}$$
$$\rho_r^{\mu\nu\alpha} = \gamma_r^{\Omega}\Omega^{\mu\nu\alpha}, \tag{3.41d}$$
$$\rho_r^{\mu\nu\alpha\beta} = \gamma_r^{\Theta}\Theta^{\mu\nu\alpha\beta}, \tag{3.41e}$$

where γ_r^{Π}, γ_r^{n} and γ_r^{π} have been introduced in Eqs. (3.31), and we have defined $\Omega^{\mu\nu\alpha} \equiv \rho_0^{\mu\nu\alpha}$ and $\Theta^{\mu\nu\alpha\beta} \equiv \rho_0^{\mu\nu\alpha\beta}$, as well as the following thermodynamic coefficients

$$\gamma_r^{\Omega} = \frac{J_{r+6,3}}{J_{6,3}}, \quad \gamma_r^{\Theta} = \frac{J_{r+8,4}}{J_{8,4}}. \tag{3.42}$$

In order to obtain a closed set of equations of motion for the dissipative currents, Π, n^μ and $\pi^{\mu\nu}$, and, more importantly, for the novel fields $\Omega^{\mu\nu\lambda}$ and $\Theta^{\mu\nu\alpha\beta}$, we follow Refs. [64, 113] and take $r = 0$ in the moment equations, Eqs. (3.16)-(3.20). We further employ the relations (3.30) and the relaxation time approximation, Eq. (3.34). We then obtain, for the bulk viscous pressure,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \tau_{\Pi n} n_\mu \nabla^\mu P_0 - \ell_{\Pi n}\partial_\mu n^\mu - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi n} n^\mu \nabla_\mu \alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}. \tag{3.43}$$

For the particle diffusion, we have

$$\begin{aligned}\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = {}& \kappa_n \nabla^\mu \alpha_0 - \tau_n n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta + \tau_{n\Pi}\Pi\nabla^\mu P_0 - \tau_{n\pi}\pi^{\mu\nu}\nabla_\nu P_0 - \ell_{n\Pi}\nabla^\mu\Pi \\ & + \ell_{n\pi}\Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta} - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi}\Pi\nabla^\mu\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0 - \tau_n\gamma^{\Omega}_{-2}\Omega^{\mu\alpha\beta}\sigma_{\alpha\beta}.\end{aligned} \tag{3.44}$$

The equation of motion for the shear-stress tensor reads

$$\begin{aligned}\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}P_0 \\ &+ \tau_\pi\left(-\gamma^{\Omega}_{-1}\Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\Omega^{\alpha\beta\lambda} + \lambda_{\pi\Omega}\Omega^{\mu\nu\lambda}\nabla_\lambda\alpha_0 + \tau_{\pi\Omega}\dot{u}_\alpha\Omega^{\mu\nu\alpha} - \gamma^{\Theta}_{-2}\Theta^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\right) \\ &+ \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0. \end{aligned} \tag{3.45}$$

We remark that these equations are identical to the ones obtained in the previous section, with the exception of the third-order corrections that are considered in this section. The expressions for the novel transport coefficients of the theory are explicitly listed in Appendix B. The coefficients that already appeared in the second-order version of this formulation can be found in Ref. [2].

The dissipative currents are now coupled to novel degrees of freedom, which satisfy their own equations of motion. First, the equation of motion for $\Omega^{\mu\nu\alpha}$ is derived substituting Eq. (3.41d) into Eq. (3.19) for $r = 0$, leading to

$$\begin{aligned}\tau_\Omega \dot{\Omega}^{\langle\mu\nu\alpha\rangle} + \Omega^{\mu\nu\alpha} &= \frac{3}{7}\eta_\Omega\nabla^{\langle\mu}\pi^{\nu\alpha\rangle} + \delta_{\Omega\Omega}\Omega^{\mu\nu\alpha}\theta + \ell_{\Omega n}\sigma^{\langle\mu\nu}n^{\alpha\rangle} + 3\tau_\Omega\Omega^{\lambda\langle\mu\nu}\omega^{\alpha\rangle}{}_\lambda + \tau_{\Omega\Omega}\sigma_\lambda^{\langle\mu}\Omega^{\nu\alpha\rangle\lambda} \\ &+ \lambda_{\Omega\pi}\pi^{\langle\mu\nu}\nabla^{\alpha\rangle}\alpha + \tau_{\Omega\pi}\pi^{\langle\mu\nu}\nabla^{\alpha\rangle}P - 3\tau_\Omega\gamma_1^\pi\pi^{\langle\mu\nu}\dot{u}^{\alpha\rangle} + \lambda_{\Omega\Theta}\Theta^{\mu\nu\alpha\beta}\nabla_\beta\alpha \\ &+ \tau_{\Omega\Theta}\Theta^{\mu\nu\alpha\beta}\dot{u}_\beta - \tau_\Omega\gamma^{\Theta}_{-1}\Delta^{\mu\nu\alpha}_{\lambda\sigma\rho}\nabla_\beta\Theta^{\lambda\sigma\rho\beta}. \end{aligned} \tag{3.46}$$

Finally, substituting Eq. (3.41e) into Eq. (3.20) for $r = 0$, one obtains an equation of motion for $\Theta^{\mu\nu\alpha\beta}$,

$$\begin{aligned}\tau_\Theta\dot{\Theta}^{\langle\mu\nu\alpha\beta\rangle} + \Theta^{\mu\nu\alpha\beta} &= \delta_{\Theta\Theta}\Theta^{\mu\nu\alpha\beta}\theta + 4\tau_\Theta\Theta^{\lambda\langle\mu\nu\alpha}\omega^{\beta\rangle}{}_\lambda + \tau_{\Theta\Theta}\sigma_\lambda^{\langle\mu}\Theta^{\nu\alpha\beta\rangle\lambda} + \ell_{\Theta\pi}\sigma^{\langle\mu\nu}\pi^{\alpha\beta\rangle} + \\ &+ \ell_{\Theta\Omega}\nabla^{\langle\mu}\Omega^{\nu\alpha\beta\rangle} + \tau_{\Theta\Omega}\dot{u}^{\langle\mu}\Omega^{\nu\alpha\beta\rangle} + \lambda_{\Theta\Omega}\Omega^{\langle\mu\nu\alpha}\nabla^{\beta\rangle}\alpha_0, \end{aligned} \tag{3.47}$$

where all transport coefficients in these equations are listed in Appendix B. Once again, we emphasize that, since we employ the relaxation time approximation, all relaxation times are identical $\tau_\Pi = \tau_n = \tau_\pi = \tau_\Omega = \tau_\Theta = \tau_R$.

Furthermore, in the derivation of these equations we have used Eq. (3.22c) to express gradients of the thermodynamic pressure in terms of the time derivative of the fluid 4-velocity, further omitting fourth-order terms. On top of that, we have also used the covariant version of the Gibbs-Duhem equation,

$$\nabla_\mu\beta_0 = \frac{1}{\varepsilon_0 + P_0}\left(n_0\nabla_\mu\alpha_0 - \beta_0\nabla_\mu P_0\right). \tag{3.48}$$

3.3.3 Linear causality and stability

The next step is to verify whether the third-order theory derived in the previous section satisfies causality and stability in the linear regime. For the sake of simplicity, we consider a simplified version of this theory, in which any dissipation due to bulk viscous pressure and net-charge diffusion are identically neglected. Similarly to what was done in Sec. 1.6, we consider small perturbations on a system initially in a global equilibrium state. In the present case, we

must extend the previous linear stability analysis to also consider perturbations in the third-order currents $\rho^{\mu\nu\alpha}$ and $\Theta^{\mu\nu\alpha\beta}$,

$$\varepsilon = \varepsilon_0 + \delta\varepsilon, \;\; u^\mu = u_0^\mu + \delta u^\mu, \;\; \pi^{\mu\nu} = \delta\pi^{\mu\nu}, \;\; \Omega^{\mu\nu\alpha} = \delta\Omega^{\mu\nu\alpha}, \;\; \Theta^{\mu\nu\alpha\beta} = \delta\Theta^{\mu\nu\alpha\beta}. \tag{3.49}$$

The linearized equations of motion for the dissipative currents become

$$\tau_\pi D_0 \delta\pi^{\mu\nu} + \delta\pi^{\mu\nu} = \eta\left(\nabla_0^\mu \delta u^\nu + \nabla_0^\nu \delta u^\mu - \frac{2}{3}\Delta_0^{\mu\nu}\partial_\lambda \delta u^\lambda\right) - \tau_\pi \nabla_\alpha^0 \delta\Omega^{\alpha\mu\nu}, \tag{3.50}$$

$$\begin{aligned}\tau_\Omega D_0 \delta\Omega^{\mu\nu\lambda} + \delta\Omega^{\mu\nu\lambda} = \eta_\Omega \Bigg[&\frac{1}{7}\left(\nabla_0^\lambda \delta\pi^{\mu\nu} + \nabla_0^\nu \delta\pi^{\mu\lambda} + \nabla_0^\mu \delta\pi^{\nu\lambda}\right) \\ &- \frac{2}{35}\left(\Delta_0^{\mu\nu}\nabla_\alpha^0 \delta\pi^{\lambda\alpha} + \Delta_0^{\mu\lambda}\nabla_\alpha^0 \delta\pi^{\nu\alpha} + \Delta_0^{\nu\lambda}\nabla_\alpha^0 \delta\pi^{\mu\alpha}\right)\Bigg].\end{aligned} \tag{3.51}$$

We remark that the equation of motion for $\delta\Theta^{\mu\nu\alpha\beta}$ is of second order in perturbations and hence does not contribute to this linear analysis. In Fourier space, the above equations can be expressed as

$$(i\Omega\tau_\pi + 1)\,\delta\tilde{\pi}^{\mu\nu} = i\eta\left(\kappa^\mu \delta\tilde{u}^\nu + \kappa^\nu \delta\tilde{u}^\mu - \frac{2}{3}\Delta^{\mu\nu}\kappa_\lambda \delta\tilde{u}^\lambda\right) - i\tau_\pi \kappa_\alpha \delta\tilde{\Omega}^{\alpha\mu\nu}, \tag{3.52}$$

$$\begin{aligned}(i\Omega\tau_\Omega + 1)\,\delta\tilde{\Omega}^{\mu\nu\lambda} = i\eta_\Omega \Bigg[&\frac{1}{7}\left(\kappa^\lambda \delta\tilde{\pi}^{\mu\nu} + \kappa^\nu \delta\tilde{\pi}^{\mu\lambda} + \kappa^\mu \delta\tilde{\pi}^{\nu\lambda}\right) \\ &- \frac{2}{35}\left(\Delta^{\mu\nu}\kappa_\alpha \delta\tilde{\pi}^{\lambda\alpha} + \Delta^{\mu\lambda}\kappa_\alpha \delta\tilde{\pi}^{\nu\alpha} + \Delta^{\nu\lambda}\kappa_\alpha \delta\tilde{\pi}^{\mu\alpha}\right)\Bigg].\end{aligned} \tag{3.53}$$

Note that, on the right-hand side of Eq. (3.52), only the projection $\kappa_\alpha \delta\tilde{\rho}^{\alpha\mu\nu}$ appears. In order to obtain this projection, we must contract Eq. (3.53) with κ_μ,

$$(i\Omega\tau_\Omega + 1)\,\kappa_\mu \delta\tilde{\Omega}^{\mu\nu\lambda} = -\frac{i}{7}\eta_\Omega \kappa^2 \delta\tilde{\pi}^{\nu\lambda} + \frac{3i}{35}\eta_\Omega\left(\kappa_\alpha \kappa^\nu \delta\tilde{\pi}^{\lambda\alpha} + \kappa_\alpha \kappa^\lambda \delta\tilde{\pi}^{\nu\alpha}\right) - \frac{2i}{35}\eta_\Omega \Delta^{\nu\lambda}\kappa_\alpha \kappa_\beta \delta\tilde{\pi}^{\alpha\beta}, \tag{3.54}$$

where we recall that $\Omega = u_0^\mu k_\mu$ and $\kappa^\mu = \Delta^{\mu\nu}k_\nu$ are the frequency and wave 4-vector in the local rest frame of the unperturbed fluid, see Eq. (1.61).

Once again, the linear stability analysis of the third-order theory shall be divided in the study of its transverse and longitudinal modes, employing the procedure introduced in Sec. 1.6.

3.3.3.1 Transverse modes

We start the analysis by looking the transverse modes of the theory. First, we compute the transverse component of Eq. (3.54), which is obtained by the following projection

$$\left(-\frac{\kappa_\mu}{\kappa}\Delta^\alpha_{\nu,\kappa}\right)\kappa_\lambda \delta\tilde{\Omega}^{\mu\nu\lambda} = -\frac{8i}{35}\frac{\eta_\Omega \kappa^2}{i\Omega\tau_\Omega + 1}\delta\tilde{\pi}^\alpha_\perp. \tag{3.55}$$

Inserting this equation in the partially transverse projection of Eq. (3.52), we obtain

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\frac{\hat{\eta}_\Omega\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega + 1} + 1\right)\frac{\delta\tilde{\pi}^\mu_\perp}{\varepsilon_0 + P_0} - i\hat{\kappa}\delta\tilde{u}^\mu_\perp = 0. \tag{3.56}$$

This equation is then coupled with the transverse projection of the conservation of energy and momentum, Eq. (1.69), which is reproduced below for convenience,

$$\hat{\Omega}\delta\tilde{u}^{\mu}_{\perp} - \hat{\kappa}\frac{\delta\tilde{\pi}^{\mu}_{\perp}}{\varepsilon_0 + P_0} = 0. \tag{3.57}$$

Therefore, the equations that describe the transverse degrees of freedom of the third-order theory can be written in the following matrix form

$$\begin{pmatrix} i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\frac{\hat{\eta}_\Omega\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega+1} + 1 & -i\hat{\kappa} \\ -\hat{\kappa} & \hat{\Omega} \end{pmatrix} \begin{pmatrix} \frac{\delta\tilde{\pi}^{\mu}_{\perp}}{\varepsilon_0+P_0} \\ \delta\tilde{u}^{\mu}_{\perp} \end{pmatrix} = 0. \tag{3.58}$$

The dispersion relation associated with the transverse modes is obtaining equating the determinant of the matrix on the left-hand side to zero, leading to

$$\hat{\Omega}\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{8}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\Omega\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega+1} + 1\right) - i\hat{\kappa}^2 = 0. \tag{3.59}$$

Considering perturbations on a fluid at rest, in which case $\Omega = \omega$ and $\kappa = k$, the dispersion relation becomes

$$\hat{\omega}\left(i\hat{\tau}_\pi\hat{\omega} + \frac{8}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\Omega\hat{k}^2}{i\hat{\omega}\hat{\tau}_\Omega+1} + 1\right) - i\hat{k}^2 = 0. \tag{3.60}$$

As it was done in the previous analysis, we shall look at the asymptotic form of these modes. In the small wave number limit, these modes can be written as

$$\hat{\omega}^{\text{shear}}_{T,-} = i\hat{k}^2 + i\hat{\tau}_\pi\left(1 - \frac{8\hat{\eta}_\Omega}{35}\right)\hat{k}^4 + \mathcal{O}(\hat{k}^6), \tag{3.61}$$

$$\hat{\omega}^{\text{shear}}_{T,+} = \frac{i}{\hat{\tau}_\pi} + i\left[1 - \frac{8\hat{\tau}_\pi\hat{\eta}_\Omega}{35\left(\hat{\tau}_\pi - \hat{\tau}_\Omega\right)}\right]\hat{k}^2 + \mathcal{O}(\hat{k}^4), \tag{3.62}$$

$$\hat{\omega}^{\text{shear}}_{T,\text{new}} = \frac{i}{\hat{\tau}_\Omega} - \frac{8i\hat{\tau}_\pi\hat{\eta}_\Omega}{35\left(\hat{\tau}_\pi - \hat{\tau}_\Omega\right)}\hat{k}^2 + \mathcal{O}(\hat{k}^4). \tag{3.63}$$

The inclusion of the nonconserved current defined as $\Omega^{\mu\nu\lambda}$ leads to the occurrence of an additional nonhydrodynamic mode already for perturbations on a static background fluid, as compared to the second-order limit of the theory, see Sec. 1.6.2. In particular, this new mode behaves as $\hat{\omega} \sim i/\hat{\tau}_\Omega$ in the small wave number limit.

In the large wave number limit, the transverse modes of the third-order theory become

$$\hat{\omega}^{\text{shear}}_{T,-} = \frac{35i}{35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\eta}_\Omega} - 9800i\hat{\tau}_\pi\hat{\eta}_\Omega\frac{35\hat{\tau}_\Omega + \hat{\tau}_\pi\left(8\hat{\eta}_\Omega - 35\right)}{\left(35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\eta}_\Omega\right)^4}\frac{1}{\hat{k}^2} + \mathcal{O}\left(\frac{1}{\hat{k}^4}\right), \tag{3.64}$$

$$\hat{\omega}^{\text{shear}}_{T,+} = \hat{k}\sqrt{\frac{35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\eta}_\Omega}{35\hat{\tau}_\pi\hat{\tau}_\Omega}} + i\frac{35\hat{\tau}_\Omega^2 + 8\hat{\tau}_\pi^2\hat{\eta}_\Omega + 8\hat{\tau}_\pi\hat{\tau}_\Omega\hat{\eta}_\Omega}{2\hat{\tau}_\pi\hat{\tau}_\Omega\left(35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\tau}_\Omega\right)} + \mathcal{O}\left(\frac{1}{\hat{k}^2}\right), \tag{3.65}$$

$$\hat{\omega}^{\text{shear}}_{T,\text{new}} = -\hat{k}\sqrt{\frac{35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\eta}_\Omega}{35\hat{\tau}_\pi\hat{\tau}_\Omega}} + i\frac{35\hat{\tau}_\Omega^2 + 8\hat{\tau}_\pi^2\hat{\eta}_\Omega + 8\hat{\tau}_\pi\hat{\tau}_\Omega\hat{\eta}_\Omega}{2\hat{\tau}_\pi\hat{\tau}_\Omega\left(35\hat{\tau}_\Omega + 8\hat{\tau}_\pi\hat{\tau}_\Omega\right)} + \mathcal{O}\left(\frac{1}{\hat{k}^2}\right). \tag{3.66}$$

In the small wave number limit, the transverse modes are purely imaginary, whereas at large values of $\hat{k}$ they become propagating, i.e., their real parts are no longer zero. Therefore, it is

now necessary to impose constraints on the linear causality of this theory. The asymptotic group velocity must be subluminal [89], and we thus obtain

$$\lim_{\hat{k}\to\infty}\left|\frac{\partial \mathrm{Re}(\hat{\omega})}{\partial \hat{k}}\right| \leq 1 \Longrightarrow \hat{\tau}_\Omega\left(\hat{\tau}_\pi - 1\right) \geq \frac{8}{35}\hat{\tau}_\pi\hat{\eta}_\Omega. \tag{3.67}$$

It was shown in Ref. [50] that perturbations on a moving fluid lead to a linear stability condition identical to Eq. (3.67), with the additional constraint

$$\hat{\tau}_\pi + \hat{\tau}_\Omega > 1. \tag{3.68}$$

We note that this constraint reduces to the linear causality and stability conditions obtained for Israel-Stewart theory if $\tau_\Omega = 0$, see Eq. (1.95) [47, 48]. However, we further remark that the causality/stability condition given by Eq. (3.67) forbids this limit – a linearly causal and stable theory can only be obtained if τ_Ω is not zero.

3.3.3.2 Longitudinal modes

The longitudinal projection of Eq. (3.54) is obtained by contracting it with $\kappa_\mu\kappa_\nu$, thus leading to

$$\left(i\Omega\tau_\Omega + 1\right)\left(\frac{\kappa_\mu\kappa_\nu}{\kappa^2}\right)\kappa_\lambda\delta\tilde{\Omega}^{\mu\nu\lambda} = -\frac{9i}{35}\eta_\Omega\kappa^2\delta\tilde{\pi}_\parallel. \tag{3.69}$$

Inserting this result in the longitudinal projection of Eq. (3.52), we obtain

$$\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\eta}_\Omega\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega + 1} + 1\right)\frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0 + P_0} - \frac{4i}{3}\hat{\kappa}\delta\tilde{u}_\parallel = 0. \tag{3.70}$$

The above equation must be coupled with the longitudinal projection of the conservation of energy and momentum, Eqs. (1.68), which are reproduced below for convenience,

$$\Omega\frac{\delta\varepsilon}{\varepsilon_0 + P_0} - \kappa\delta\tilde{u}_\parallel = 0, \tag{3.71a}$$

$$\Omega\delta\tilde{u}_\parallel - \kappa\left(c_\mathrm{s}^2\frac{\delta\varepsilon}{\varepsilon_0 + P_0} + \frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0 + P_0}\right) = 0. \tag{3.71b}$$

These equations can be conveniently written a matrix form

$$\begin{pmatrix} i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\eta}_\Omega\hat{\tau}_\pi\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega+1} + 1 & -i\frac{4}{3}\hat{\kappa} & 0 \\ -\hat{\kappa} & \hat{\Omega} & -c_\mathrm{s}^2\hat{\kappa} \\ 0 & -\hat{\kappa} & \hat{\Omega} \end{pmatrix}\begin{pmatrix} \frac{\delta\tilde{\pi}_\parallel}{\varepsilon_0+P_0} \\ \delta\tilde{u}_\parallel \\ \frac{\delta\varepsilon}{\varepsilon_0+P_0} \end{pmatrix} = 0. \tag{3.72}$$

The dispersion related to the longitudinal degrees of freedom of the third-order formulation thus reads

$$\left(\hat{\Omega}^2 - c_s^2\hat{\kappa}^2\right)\left(i\hat{\tau}_\pi\hat{\Omega} + \frac{9}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\Omega\hat{\kappa}^2}{i\hat{\Omega}\hat{\tau}_\Omega + 1} + 1\right) - \frac{4i}{3}\hat{\Omega}\hat{\kappa}^2 = 0. \tag{3.73}$$

Considering perturbations on a static fluid, the dispersion relation associated with the longitudinal modes, Eq. (3.73), simply reads

$$\left(\hat{\omega}^2 - c_s^2\hat{k}^2\right)\left(i\hat{\tau}_\pi\hat{\omega} + \frac{9}{35}\frac{\hat{\tau}_\pi\hat{\eta}_\Omega\hat{k}^2}{i\hat{\omega}\hat{\tau}_\Omega + 1} + 1\right) - \frac{4i}{3}\hat{\omega}\hat{k}^2 = 0. \tag{3.74}$$

As before, we analyze the asymptotic form of the modes. In the small wave number limit, they read

$$\hat{\omega}_{\pm}^{\text{sound}} = \pm c_{\text{s}}\hat{k} + \frac{2i}{3}\hat{k}^2 \pm \frac{2\left(3\hat{\tau}_\pi c_{\text{s}}^2 - 1\right)}{9c_{\text{s}}}\hat{k}^3 + \mathcal{O}(\hat{k}^4), \tag{3.75}$$

$$\hat{\omega}_{L}^{\text{shear}} = \frac{i}{\hat{\tau}_\pi} + i\frac{\frac{27}{35}\hat{\tau}_\pi\hat{\eta}_\Omega + 4(\hat{\tau}_\Omega - \hat{\tau}_\pi)}{3(\hat{\tau}_\pi - \hat{\tau}_\Omega)}\hat{k}^2 + \mathcal{O}(\hat{k}^4), \tag{3.76}$$

$$\hat{\omega}_{L,\text{new}}^{\text{shear}} = \frac{i}{\hat{\tau}_\Omega} - i\frac{9\hat{\tau}_\pi\hat{\eta}_\Omega}{35(\hat{\tau}_\pi - \hat{\tau}_\Omega)}\hat{k}^2 + \mathcal{O}(\hat{k}^4). \tag{3.77}$$

We identify two hydrodynamic modes and two nonhydrodynamic modes. The hydrodynamic modes correspond to the usual sound modes and their small wave number limit remain identical to the corresponding results obtained in Israel-Stewart theory, see Eq. (1.105). In the small wave number limit, the nonhydrodynamic mode $\hat{\omega}_L^{\text{shear}}$ becomes identical to the nonhydrodynamic mode found in Israel-Stewart theory, with deviations only occurring at order $\mathcal{O}(\hat{k}^2)$, see Eq. (1.106). On the other hand, the nonhydrodynamic mode $\hat{\omega}_{L,\text{new}}^{\text{shear}}$ is intrinsically new and describes nonequilibrium modes that relax to equilibrium in times of order τ_Ω.

In the large wave number limit, all four longitudinal modes can be cast in the following form

$$\hat{\omega} = \pm\sqrt{\frac{\frac{27}{35}\hat{\tau}_\pi\hat{\eta}_\Omega + 3\hat{\tau}_\pi\hat{\tau}_\Omega c_{\text{s}}^2 + 4\hat{\tau}_\Omega \pm \sqrt{\left(\frac{27}{35}\hat{\tau}_\pi\hat{\eta}_\Omega + 3\hat{\tau}_\pi\hat{\tau}_\Omega c_{\text{s}}^2 + 4\hat{\tau}_\Omega\right)^2 - \frac{324}{35}\hat{\tau}_\pi^2\hat{\tau}_\Omega\hat{\eta}_\Omega c_{\text{s}}^2}}{6\hat{\tau}_\pi\hat{\tau}_\Omega}}\hat{k} + \mathcal{O}\left(1\right). \tag{3.78}$$

Since the hydrodynamic and nonhydrodynamic modes merge at finite values of wave number, it is not trivial to map these four solutions with the small wave number solutions displayed in Eqs. (3.75), (3.76), and (3.77). In order for these modes to be stable, it is essential that the term inside the outer square root is real and positive, otherwise leading to modes with a negative imaginary part, and thus unstable solutions. For this purpose, we must first impose that the term inside the inner square root is positive. If this is the case, it is straightforward to see that the term in the numerator is always positive. Therefore, in order to obtain purely real modes, it is sufficient to impose

$$\left(\frac{27}{35}\hat{\tau}_\pi\hat{\eta}_\Omega + 3\hat{\tau}_\pi\hat{\tau}_\Omega c_{\text{s}}^2 + 4\hat{\tau}_\Omega\right)^2 - \frac{324}{35}\hat{\tau}_\pi^2\hat{\tau}_\Omega\hat{\eta}_\Omega c_{\text{s}}^2 \geq 0. \tag{3.79}$$

In fact, this inequality is satisfied as long as the transport coefficients are positive definite quantities, i.e., $\hat{\tau}_\pi > 0$, $\hat{\tau}_\Omega > 0$, and $\hat{\eta}_\Omega > 0$. Therefore, the stability of the longitudinal modes perturbations on a static fluid is always fulfilled.

The linear causality of the theory can be verified by analyzing the asymptotic group velocity of the modes [89]. In order for these modes to propagate subluminally, the following condition must be satisfied

$$\lim_{\hat{k}\to\infty}\left|\frac{\partial\text{Re}(\hat{\omega})}{\partial\hat{k}}\right| \leq 1 \Longrightarrow \hat{\tau}_\Omega \geq \frac{27}{35}\hat{\eta}_\Omega\hat{\tau}_\pi\frac{1 - c_{\text{s}}^2}{3\hat{\tau}_\pi\left(1 - c_{\text{s}}^2\right) - 4}. \tag{3.80}$$

In order to obtain this relation, it is necessary to impose a first constraint to the shear relaxation time $\hat{\tau}_\pi$, which is given by

$$\hat{\tau}_\pi \geq \frac{4}{3(1-c_{\rm s}^2)}, \tag{3.81}$$

which, in the ultrarelativistic limit, $c_{\rm s}^2 = 1/3$, reduces to $\hat{\tau}_\pi \geq 2$. This is exactly the linear stability condition for the shear relaxation time in Israel-Stewart theory [47, 48], see Eq. (1.110).

In Ref. [50], perturbations on a moving fluid were also considered, and it was shown that the stability conditions obtained for such case exactly match Eqs. (3.80) and (3.81). Similarly to what was observed for the Israel-Stewart theory, the constraints obtained for the longitudinal modes supersede those coming from the analysis of the transverse modes. In particular, assuming a classical massless gas, the causality/stability conditions simplify, respectively, to

$$\eta_\Omega < \frac{49}{3} T\, \tau_\pi, \tag{3.82}$$

$$\tau_\pi \geq \frac{2\eta}{\varepsilon_0 + P_0}, \tag{3.83}$$

where we have used that $\gamma_{-1}^\Omega = 1/(7\,T)$ in the classical and massless limits. Our results for the transport coefficients, $\tau_\pi = 5\eta/(\varepsilon_0 + P_0)$ and $\eta_\Omega = 6\,T\,\tau_\pi$, listed in Appendix B, are thus consistent with the fundamental conditions listed above. Naturally, it is necessary to verify if these conditions are still satisfied when the full collision term is considered.

3.3.4 Bjorken flow

We are interested in analyzing the agreement between the solutions of the third-order equations of motion with solutions of the relativistic Boltzmann equation. For this purpose, we consider a relativistic gas of massless and classical particles undergoing Bjorken flow [103, 124]. This flow configuration is an idealized description of a heavy-ion collision that assumes that the matter produced shortly after the collision is homogeneous and azimuthally symmetric in the transverse plane, as well as invariant under Lorentz boosts along the longitudinal axis. Furthermore, in Bjorken flow, the otherwise convoluted partial differential equations that govern the evolution of relativistic fluids reduce to simpler ordinary differential equations that admit analytical solutions. Therefore, Bjorken flow often serves as a starting point in studies of relativistic kinetic theory [83, 125–130] and fluid dynamics [72, 74, 127, 131–134].

The spacetime in Bjorken flow is more conveniently described using hyperbolic coordinates, with a geometry defined by the following metric tensor

$$g_{\mu\nu} = \mathrm{diag}\,(g_{\tau\tau}, g_{xx}, g_{yy}, g_{\eta_s\eta_s}) = \mathrm{diag}\,(1, -1, -1, -\tau^2), \tag{3.84}$$

where τ is the proper time and η_s is the spacetime rapidity. These coordinates are related to the usual Cartesian coordinates through

$$\tau = \sqrt{t^2 - z^2}, \quad \eta_s = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right). \tag{3.85}$$

In this coordinate system, the only non-zero Christoffel symbols are

$$\Gamma^{\tau}_{\eta_s\eta_s} = \tau, \quad \Gamma^{\eta_s}_{\tau\eta_s} = \Gamma^{\eta_s}_{\eta_s\tau} = \frac{1}{\tau}. \tag{3.86}$$

Therefore, all usual derivatives must be replaced by covariant derivatives in the equations of motion for the irreducible moments. In addition, we make the following set of assumptions:

1. The system is symmetric under reflections with the respect to the η_s axis, i.e., $\eta_s \to -\eta_s$.
2. The system is homogeneous, i.e., invariant under translations, along the η_s axis. This implies the system is boost-invariant and thus all fluid-dynamical quantities depend solely on the proper time τ.
3. The system is homogeneous and isotropic (invariant under translations and rotations) in the transverse xy-plane.

These assumptions combined lead to a static fluid velocity in hyperbolic coordinates, $u^\mu = (u^\tau, u^x, u^y, u^{\eta_s}) = (1, 0, 0, 0)$. Naturally, the fluid itself is not static, as its expansion is embedded in the metric tensor and thus manifests itself via the covariant derivatives of the 4-velocity [58]. In particular, the expansion rate and shear tensor are given by

$$\theta = D_\mu u^\mu = \frac{1}{\tau}, \qquad \sigma_{\mu\nu} = D_{\langle\mu} u_{\nu\rangle} = \mathrm{diag}\left(0, \frac{1}{3\tau}, \frac{1}{3\tau}, -\frac{2\tau}{3}\right). \tag{3.87}$$

Furthermore, without loss of generality, the shear-stress tensor can be decomposed in an analogous form,

$$\pi^{\mu\nu} = \mathrm{diag}\left(0, \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{\tau^2}\right). \tag{3.88}$$

The conservation of energy, Eq. (1.34a), reduces to

$$\frac{d\varepsilon}{d\tau} = -\frac{1}{\tau}\left(\varepsilon + P - \pi\right). \tag{3.89}$$

The momentum conservation equation, Eq. (1.34b), is trivially satisfied. In order to compare our results to those of Ref. [1], we neglect any contribution of the particle density, setting it to zero, and Eq. (1.34c) becomes trivially satisfied as well.

Since we consider a classical gas of massless particles, the energy density and thermodynamic pressure are related through $\varepsilon = 3P$. On top of that, the energy density is a quartic function of the temperature, $\varepsilon \sim T^4$. It is then convenient to rewrite Eq. (3.89) as a differential equation for the temperature,

$$\frac{dT}{d\tau} = \frac{T}{3\tau}(\hat{\pi} - 1), \tag{3.90}$$

with $\hat{\pi} \equiv \pi/(\varepsilon_0 + P_0)$.

The next step is to obtain the equations of motion for the dissipative currents that couple with the conservation of energy and momentum in Bjorken flow. First, we note that

the bulk viscous pressure is zero, since we are considering a system of massless particles. As already stated, the particle diffusion 4-current is identically zero in Bjorken flow, since it is orthogonal to the 4-velocity. Furthermore, all irreducible moments of odd rank also vanish in this framework [58]. Wherefore, the relevant fluid-dynamical equations reduce to

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} - \tau_\pi\gamma^\Theta_{-2}\Theta^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \tag{3.91a}$$

$$\tau_\Theta \dot{\Theta}^{\langle\mu\nu\alpha\beta\rangle} + \Theta^{\mu\nu\alpha\beta} = \delta_{\Theta\Theta}\Theta^{\mu\nu\alpha\beta}\theta + \tau_{\Theta\Theta}\sigma_\lambda^{\langle\mu}\Theta^{\nu\alpha\beta\rangle\lambda} + \ell_{\Theta\pi}\sigma^{\langle\mu\nu}\pi^{\alpha\beta\rangle}. \tag{3.91b}$$

In the massless limit, these transport coefficients are

$$\tau_\pi = \frac{5\eta}{Ts},\ \frac{\delta_{\pi\pi}}{\tau_\pi} = \frac{4}{3},\ \frac{\tau_{\pi\pi}}{\tau_\pi} = \frac{10}{7},\ \gamma^\Theta_{-2} = \frac{1}{72T^2},\ \frac{\delta_{\Theta\Theta}}{\tau_\Theta} = -2,\ \frac{\tau_{\Theta\Theta}}{\tau_\Theta} = -\frac{36}{11},\ \ell_{\Theta\pi} = 64T^2, \tag{3.92}$$

where η is the shear viscosity coefficient and s is the entropy density. We remark that the first three transport coefficients were first calculated in Ref. [2], while the last four were calculated in Ref. [72]. Once again, general expressions for the latter can be found in Appendix B.

It is convenient to define a unitary 4-vector, $z_\mu = (0, 0, 0, -\tau)$, and project Eqs. (3.91a) and (3.91b) with $z_\mu z_\nu$ and $z_\mu z_\nu z_\alpha z_\beta$, respectively, in order to obtain a closed equation of motion for the longitudinal components of $\pi^{\mu\nu}$ and $\Theta^{\mu\nu\alpha\beta}$. These equations then become

$$\frac{d\hat{\pi}}{d\tau} = -\frac{\hat{\pi}}{\tau_\pi} + \frac{4}{15\tau} - \frac{10}{21}\frac{\hat{\pi}}{\tau} - \frac{4}{3}\frac{\hat{\pi}^2}{\tau} - \frac{1}{72}\frac{\hat{\varphi}}{\tau}, \tag{3.93a}$$

$$\frac{d\hat{\varphi}}{d\tau} = -\frac{\hat{\varphi}}{\tau_\Theta} + \frac{768}{35}\frac{\hat{\pi}}{\tau} - \frac{60}{77}\frac{\hat{\varphi}}{\tau} - 2\frac{\hat{\varphi}\hat{\pi}}{\tau}, \tag{3.93b}$$

where we have used Eq. (3.90) to obtain an equation of motion for the dimensionless variable $\hat{\varphi} \equiv \Theta^{\eta_s\eta_s}_{\eta_s\eta_s}/[(\varepsilon + P)T^2]$ and employed the transport coefficients given in Ref. [2].

In Fig. 1, we compare the results for the pressure anisotropy in Bjorken flow, defined as $P_L/P_T = (1 - 4\hat{\pi})/(1 + 2\hat{\pi})$, calculated within the third-order formalism developed in Ref. [1] (blue dashed lines), the one proposed in this chapter (red solid lines), Israel-Stewart theory (green solid lines) – recovered by setting $\hat{\varphi} = 0$ in Eqs. (3.93) –, and solutions of the Boltzmann equation (black dots). In the left panel, we compare to solutions of the Boltzmann equation calculated assuming the relaxation time approximation, with initial time and temperature calibrated to describe the matter produced in heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) [122, 135]. In the right panel, we compare to solutions of the Boltzmann equation calculated using the Boltzmann Approach To Multi-Parton Scatterings (BAMPS) [119]. The initial time and temperature were calibrated to describe the matter produced in heavy-ion collisions at the Large Hadron Collider (LHC) [135]. In both scenarios, we have assumed an initially isotropic pressure configuration, $P_L/P_T = 1$. We see that solutions of both third-order fluid-dynamical theories are in good agreement with solutions of the microscopic theory, providing a more accurate description when compared to Israel-Stewart theory. In particular, the formulation derived in this chapter displays a slightly better description than the one proposed in Ref. [1]. We remark that BAMPS solves the full Boltzmann equation without relying on

the relaxation time approximation for the collision term. Thus, the agreement with the fluid-dynamical calculations may suggest that this approximation is reasonable, at least for the purposes of describing the time evolution of the shear-stress tensor.

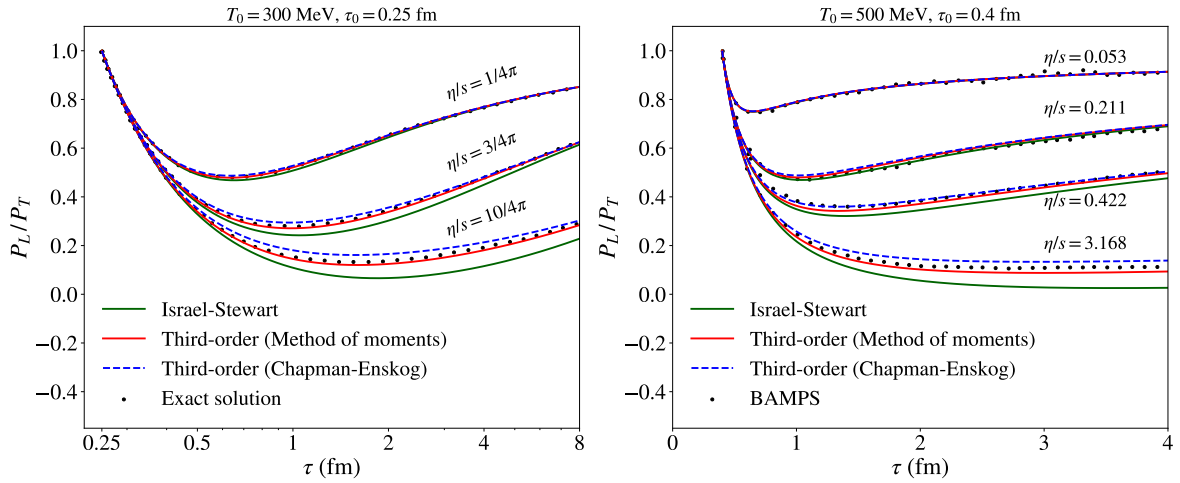

Figure 1 – Pressure anisotropy in Bjorken flow for RHIC (left panel) and LHC (right panel) energies, as calculated from the Chapman-Enskog method [1], method of moments and solutions of the Boltzmann equation from BAMPS for several values of η/s, considering $\tau_\Theta = \tau_\pi = \frac{5\eta}{\varepsilon+P}$ [2].

In Fig. 2, we display the pressure anisotropy assuming nonequilibrium initial conditions. In particular, we consider two different scenarios, in which the pressure anisotropy is either initially positive [$\hat{\pi}(0) = 3/14$] or negative [$\hat{\pi}(0) = 1/2$]. In both cases, we consider $\tau_0 = 0.4$ fm, $T_0 = 500$ MeV and $\hat{\varphi}(0) = 0.5$. It can be readily seen that the agreement between solutions for the third-order theories obtained from the Chapman-Enskog method and the method of moments is stronger at sufficiently late times, especially when η/s is small. We remark that different values of $\hat{\varphi}(0)$ do not qualitatively change this behavior.

For the sake of completeness, in Figs. 3 and 4, we display $\hat{\pi}$ and $\hat{\varphi}$, respectively, as function of τ/τ_π, for a wide set of initial values of $\hat{\pi}$ (black solid lines) and $\hat{\varphi}$ (red dashed lines), considering both the RHIC and LHC scenarios described above. In both cases, we observe that these quantities approach the same universal values at large proper times, regardless of which set of initial conditions is being used. This universal behavior displayed by the fluid-dynamical variables at late times in spite of the initial conditions is called the *hydrodynamic attractor* and was first investigated in Ref. [125]. Here, we see that the novel field $\hat{\varphi}$ also displays this attractor behavior.

Last, in Fig. 5, we compare a solution of Eq. (3.93b) to two of its asymptotic solutions: (i) its lowest contribution in a gradient expansion, $\hat{\varphi}_{\rm grad} = 768\hat{\pi}/(35\tau)$ and (ii) its zeroth order slow-roll solution [125, 131, 136, 137], obtained by setting $\dot{\hat{\varphi}} = 0$ in Eq. (3.93b), i.e.,

$$\hat{\varphi}_{\rm slow-roll} = \frac{768\hat{\pi}}{35\left(\frac{\tau}{\tau_\Theta} + \frac{60}{77} + 2\hat{\pi}\right)}. \tag{3.94}$$

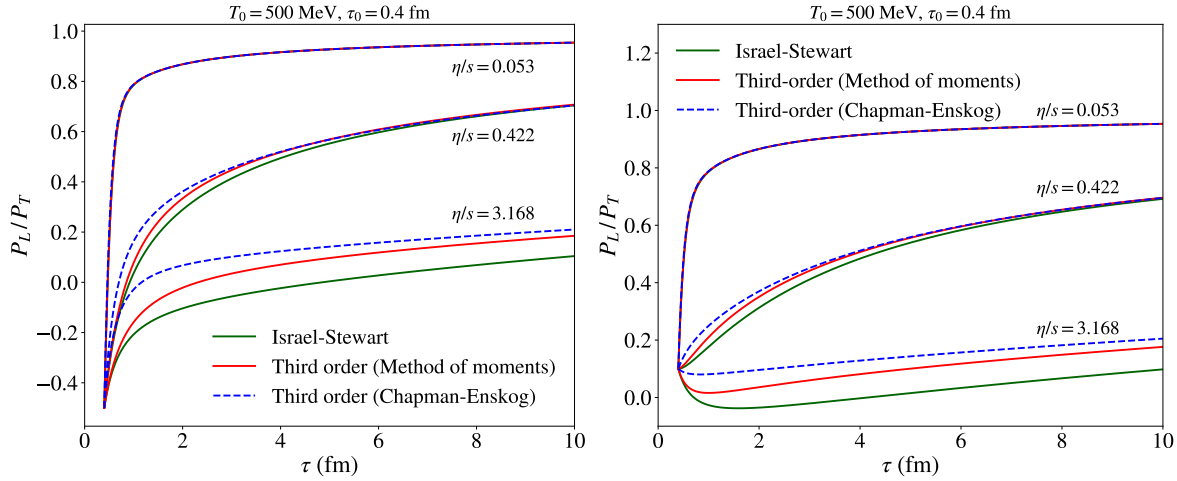

Figure 2 – Pressure anisotropy in Bjorken flow for LHC energies, as calculated from the Chapman-Enskog method [1] and method of moments for several values of η/s, considering $\tau_\Theta = \tau_\pi = \frac{5\eta}{\varepsilon+P}$ [2], as well as $\tau_0 = 0.4$ fm, $T_0 = 500$ MeV and $\hat{\varphi}(0) = 0.5$.

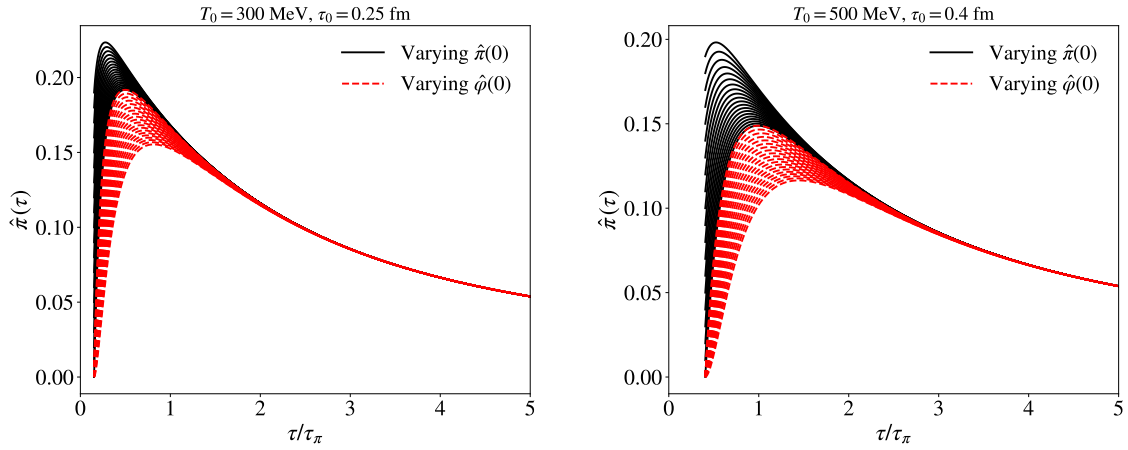

Figure 3 – $\hat{\pi}$ as a function of τ/τ_π for RHIC (left panel) and LHC (right panel) energies, considering several initial conditions for $\hat{\pi}$ and $\hat{\varphi}$, assuming $\tau_\Theta = \tau_\pi = \frac{5\eta}{\varepsilon+P}$ [2].

We consider LHC and RHIC energies and systems that are initially in local equilibrium. In both cases, we observe that the lowest order gradient expansion value of $\hat{\varphi}$ can surpass its third-order solution by a factor of ~ 4, while the zeroth slow-roll solution provides a considerably better agreement with the actual solution at early times. On the other hand, the gradient expansion leading solution converges to the hydrodynamic attractor faster than the slow-roll solution. In the end, neither of these asymptotic solutions provide a good description for the time evolution of $\hat{\varphi}$.

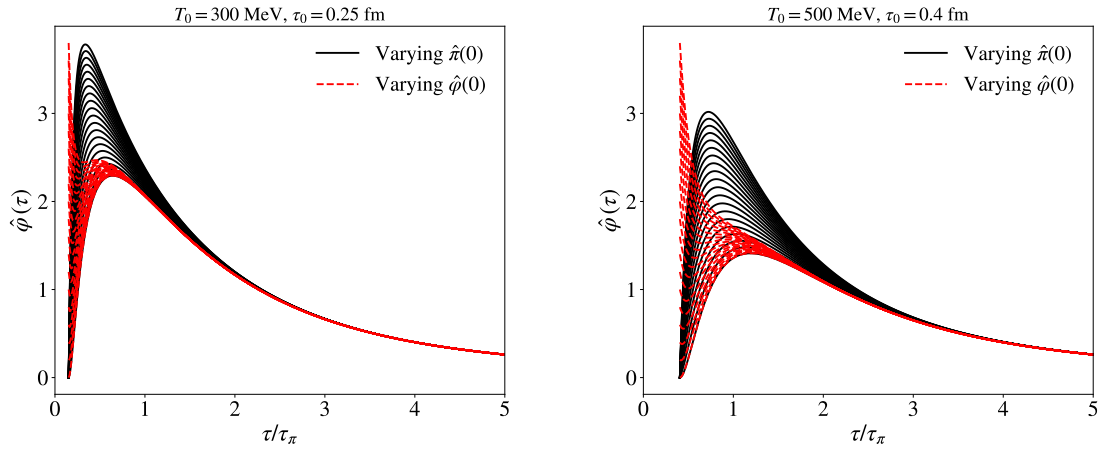

Figure 4 – $\hat{\varphi}$ as a function of τ/τ_π for RHIC (left panel) and LHC (right panel) energies, considering several initial conditions for $\hat{\pi}$ and $\hat{\varphi}$, assuming $\tau_\Theta = \tau_\pi = \frac{5\eta}{\varepsilon+P}$ [2].

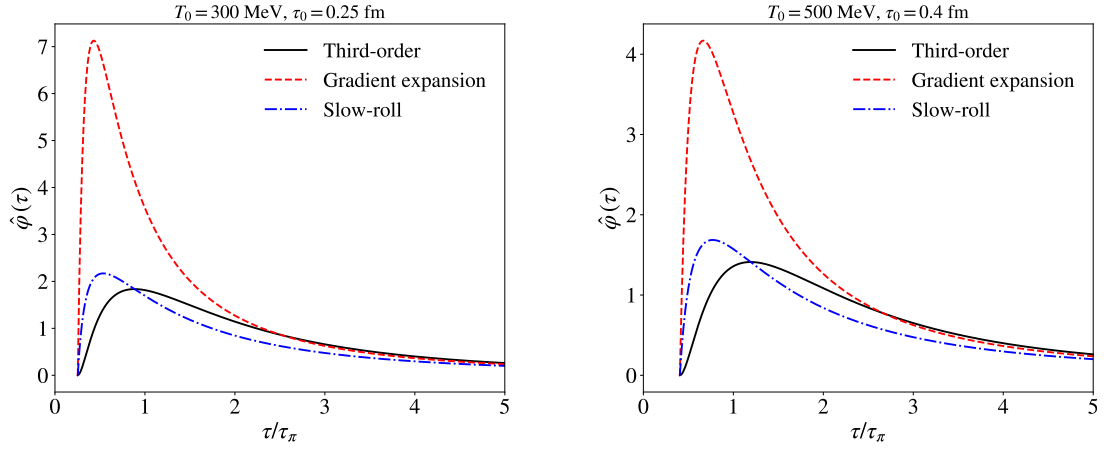

Figure 5 – (Color online) Solution of $\hat{\varphi}$, starting from equilibrium, as a function of τ/τ_π and for $\eta/s = 0.5$, compared to its respective Navier-Stokes limit and zeroth-order slow-roll solution. Left panel shows the comparison for RHIC energies while the right panel shows the comparison for LHC energies.

3.4 Discussion

In this chapter, we have formally derived a linearly causal and stable third-order fluid-dynamical theory from the Boltzmann equation using the method of moments. We demonstrated that equations of motion that include all terms that are asymptotically of third order in a gradient expansion can only be obtained including novel degrees of freedom, corresponding to irreducible tensors of rank 3 and 4. This is in contrast to the fluid-dynamical theories developed so far, the so-called second-order theories, that only require the inclusion of irreducible tensors of rank 0, 1, and 2 – which are usually matched to the traditional fluid-dynamical variables appearing in the conserved currents. We generalized the minimal truncation scheme derived by Israel and Stewart [40], so that these novel degrees of freedom are taken into account in

the derivation procedure. We derived all the equations of motion of this theory and calculated its corresponding transport coefficients. Furthermore, we demonstrated that such transport coefficients are consistent with the linear causality and stability conditions derived in Ref. [50].

Last, we analyzed the derived third-order fluid-dynamical equations within the highly symmetric framework of Bjorken flow. We observed that the currents $\pi^{\mu\nu}$ and $\Theta^{\mu\nu\alpha\beta}$ are the only ones that provide nonvanishing contributions in this flow profile, since we considered massless particles in this comparison. We observed that third-order fluid dynamics derived from the method of moments provides results that are slightly different than a derivation from the Chapman-Enskog method [1], but are still in good agreement with solutions of the relativistic Boltzmann equation both for LHC and RHIC energies. Nevertheless, the formalism developed throughout this work satisfies causality and stability in the linear regime [50] and, thus, may be solved in more general flow configurations.

The derivation of the third-order fluid-dynamical formulation developed in this chapter is an extension of the 14-moment approximation to include contributions of third-order in Knudsen number. It relies on a minimal truncation scheme. That is, the moment expansion of the single-particle distribution function is truncated such that each term in the expansion matches a degree of freedom. In this sense, this approach represents a truncation purely in degrees of freedom that does not rely on a small parameter, and therefore does not possess a clear domain of validity in terms of the gradients of the hydrodynamic variables. In order to address this issue, instead of directly truncating the moment expansion, one should truncate the moment equation employing a systematic power-counting scheme [2, 58], see Appendix C. In fact, such a derivation will provide more accurate expressions for the transport coefficients of our theory.

4 Method of moments II: Convergence of solutions

In the previous Chapter, we discussed the method of moments and the derivation of the dynamical equations for the irreducible moments of the single particle distribution function. Until this thesis, equations of motion for the irreducible moments have always been calculated separately (rank by rank), and, to this day, have only been obtained for a handful of moments [2, 72].[1,2] This task becomes progressively more exhausting as moments of higher ranks are included in the expansion given by Eq. (3.13). For this reason, the convergence of the moment equations has never been fully explored in the relativistic regime. The equations of motion for the moments are highly coupled, with the dynamics of moments that are of a lower order in the moment expansion coupling to those that are of a higher order. How this hierarchy of equations can be properly truncated and how this truncation quantitatively affects the solutions for each moment is not well known.

In this chapter, we bridge this gap by deriving the equations of motion for all the *irreducible* moments of the distribution function. Furthermore, we show how these equations simplify in the highly symmetric configuration of Bjorken flow [103]. In this case, we recover the results first obtained in Ref. [58], where the method of moments was developed employing the aforementioned symmetries from the start. We then investigate the convergence properties of the solutions for this hierarchy of equations for a gas of classical massless particles in Bjorken flow within the relaxation time approximation [104].

4.1 General equations of motion

The main goal of this chapter is to derive a general equation of motion for an irreducible moment of arbitrary rank ℓ, reproducing the results originally derived in Ref. [83]. Then, the dynamics of *any* irreducible moment can be straightforwardly recovered from this general equation. First, it is convenient to define the irreducible moments of a *generic* single-particle distribution function, $f_{\mathbf{k}}$,

$$\varrho_r^{\mu_1\cdots\mu_\ell} = \int dK\, E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} f_{\mathbf{k}}. \tag{4.1}$$

We emphasize that the irreducible moments $\varrho_r^{\mu_1\cdots\mu_\ell}$ are defined slightly differently than $\rho_r^{\mu_1\cdots\mu_\ell}$. They are integrals of the single-particle distribution function, $f_{\mathbf{k}}$, rather than integrals of its non-equilibrium component, $\delta f_{\mathbf{k}}$, cf. Eq. (3.12). The latter prescription can be straightforwardly

[1] In Ref. [138], general equations of motion for the *reducible* moments of the single-particle distribution function were thoroughly investigated for a system of electrically charged particles, in the context of the Boltzmann-Vlasov-Maxwell equation.

[2] During the writing of this manuscript, general moment equations were also derived in Ref. [139].

recovered by simply factorizing $f_{\mathbf{k}}$ as in Eq. (3.1). We remark that the equilibrium distribution function has only scalar moments[3],

$$\int dK E_{\mathbf{k}}^{r} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} f_{0\mathbf{k}} \equiv \varrho_r^{\mathrm{eq}}\delta_{\ell 0}, \tag{4.2}$$

where we defined ϱ_r^{eq} as the equilibrium value of the corresponding scalar irreducible moment.

The Boltzmann equation can be expressed as

$$\frac{d}{d\tau}f_{\mathbf{k}} = \frac{C[f]}{E_{\mathbf{k}}} - \frac{1}{E_{\mathbf{k}}}k_{\langle\mu\rangle}\nabla^{\mu}f_{\mathbf{k}}, \tag{4.3}$$

where $d/d\tau = u^\mu\partial_\mu$ is the comoving time derivative and $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ is the 4-gradient operator. Using this result, it is possible to show that the irreducible moments of *arbitrary* rank, $\varrho_r^{\mu_1\cdots\mu_\ell}$, satisfy the following equations of motion

$$\begin{aligned}
\dot{\varrho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} &= C_{r-1}^{\mu_1\cdots\mu_\ell} + r\varrho_{r-1}^{\mu_1\cdots\mu_{\ell+1}}\dot{u}_{\mu_{\ell+1}} - \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\varrho_{r-1}^{\nu_1\cdots\nu_{\ell+1}} + (r-1)\varrho_{r-2}^{\mu_1\cdots\mu_{\ell+2}}\sigma_{\mu_{\ell+1}\mu_{\ell+2}} \\
&+ \ell\varrho_r^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}\omega^{\mu_\ell\rangle}{}_\alpha + \frac{\ell}{2\ell+1}\left[rm^2\varrho_{r-1}^{\langle\mu_1\cdots\mu_{\ell-1}} - (r+2\ell+1)\varrho_{r+1}^{\langle\mu_1\cdots\mu_{\ell-1}}\right]\dot{u}^{\mu_\ell\rangle} \\
&+ \frac{1}{3}\left[(r-1)m^2\varrho_{r-2}^{\mu_1\cdots\mu_\ell} - (r+\ell+2)\varrho_r^{\mu_1\cdots\mu_\ell}\right]\theta - \frac{\ell}{2\ell+1}\nabla^{\langle\mu_1}\left(m^2\varrho_{r-1}^{\mu_2\cdots\mu_\ell\rangle} - \varrho_{r+1}^{\mu_2\cdots\mu_\ell\rangle}\right) \\
&+ \frac{\ell}{2\ell+3}\left[(2r-2)m^2\varrho_{r-2}^{\alpha\langle\mu_1\cdots\mu_{\ell-1}} - (2r+2\ell+1)\varrho_r^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}\right]\sigma_\alpha^{\mu_\ell\rangle} \\
&+ \frac{\ell(\ell-1)}{4\ell^2-1}\left[(r-1)m^4\varrho_{r-2}^{\langle\mu_1\cdots\mu_{\ell-2}} - (2r+2\ell-1)m^2\varrho_r^{\langle\mu_1\cdots\mu_{\ell-2}} + (r+2\ell)\varrho_{r+2}^{\langle\mu_1\cdots\mu_{\ell-2}}\right]\sigma^{\mu_{\ell-1}\mu_\ell\rangle},
\end{aligned} \tag{4.4}$$

where we remark that $C_r^{\mu_1\cdots\mu_\ell}$ is the generalized collision term, defined according to Eq. (3.25). Furthermore, in the derivation of the above equation, we have used the following identities,

$$\begin{aligned}
k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} &= k^{\langle\mu_1\rangle}\cdots k^{\langle\mu_\ell\rangle} \\
&+ \sum_{q=1}^{[\ell/2]}\left(\Delta_{\alpha\beta}k^\alpha k^\beta\right)^q\frac{\ell!}{2^q q!}\frac{C(\ell,q)}{\mathcal{N}_{\ell,q}}\sum_{\mathcal{P}_\mu^\ell}\Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2q-1}\mu_{2q}}k^{\langle\mu_{2q+1}\rangle}\cdots k^{\langle\mu_\ell\rangle},
\end{aligned} \tag{4.5a}$$

$$k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle} = k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle} + \frac{\ell}{2\ell+1}\left(\Delta_{\lambda\beta}k^\lambda k^\beta\right)\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle}. \tag{4.5b}$$

The calculations to obtain Eqs. (4.4) and (4.5) are reported in detail in Appendices D and E. In particular, we remark that Eq. (4.5a) is the generalization of Eqs. (3.21).

The result obtained in Eq. (4.4) is consistent with previous calculations for $\ell = 0, 1, 2$ [2], as well as for $\ell = 3, 4$ [72], originally derived in the context of second- and third-order theories of relativistic fluid dynamics, respectively. In particular, we remark that this result is the relativistic

[3] This happens because, unless $\ell = 0$, it is not possible to construct an ℓ-th rank irreducible tensor solely in terms of T, μ, u^μ, and $g^{\mu\nu}$.

generalization of Eq. (8) from Ref. [140].[4] Furthermore, we note that Eqs. (4.4) do not clearly display the traditional Navier-Stokes terms, i.e., terms that are of first-order in gradients of thermal potential and 4-velocity. This happens because such terms stem solely from derivatives of the local equilibrium distribution function, while the irreducible moments $\varrho_r^{\mu_1\cdots\mu_\ell}$ were constructed in terms of the single-particle distribution function, without factorizing its equilibrium component. In order to identify the Navier-Stokes-like terms in the equations, the irreducible moments must be separated into their equilibrium and non-equilibrium parts. In particular, we remark that only the equations of motion for the irreducible moments with $\ell \leq 2$ have non-zero Navier-Stokes-like terms – these terms vanish for all moments of rank $\ell \geq 3$, as a consequence of Eq. (4.2). Nevertheless, we remark that these terms were explicitly calculated in Ref. [2].

In order to obtain an expression for $f_{\mathbf{k}}$, it is necessary to compute the dynamics of the irreducible moments, which in turn satisfy a hierarchy of coupled differential equations. In particular, as previously discussed, the *exact* single-particle distribution function is obtained when the sums in Eq. (3.13) are taken to infinity, i.e., when the dynamics of *all* its irreducible moments is taken into account. In practice, however, we are required to truncate the expansion in Eq. (3.13), including the number of moments required for the series to converge up to a given momentum. Nevertheless, the goal of this chapter is different. We will not evaluate the distribution function itself but will instead verify how one can effectively truncate the moment equations derived above and how well this truncation procedure converges. The reconstruction of the distribution function will be discussed in detail in the next Chapter in the context of boost-invariant expanding systems, where the moment equations (4.4) can be solved to a very high order.

Once again, we shall define the local equilibrium state using Landau matching conditions [37]. In this case, the values of temperature and chemical potential are defined in such a way that the particle and energy densities are fixed to their corresponding equilibrium values, which implies that $\varrho_1 \equiv \varrho_1^{\text{eq}}$ and $\varrho_2 \equiv \varrho_2^{\text{eq}}$, respectively. Furthermore, the 4-velocity is defined so that the energy diffusion 4-current vanishes, $\varrho_1^{\mu} = 0$.

4.2 A gas of massless particles in Bjorken flow

We consider a gas of massless particles undergoing an intense longitudinal expansion, described by the Bjorken flow, introduced in Sec. 3.3.4. In such flow profile, the Boltzmann equations reduces simply to,

$$\partial_\tau f_{\mathbf{k}} = -\frac{1}{\tau_R}\left(f_{\mathbf{k}} - f_{0\mathbf{k}}\right). \tag{4.6}$$

where we have once again employed the relaxation-time approximation for the collision term [104, 117], with the parameter τ_R dictating the time scale over which the single-particle distribu-

[4] We remark that an extension of this result considering an anisotropic expansion was recently obtained in Ref. [141].

tion function relaxes to its equilibrium value. The relaxation time can be expressed in terms of the shear viscosity, η, as [2],

$$\tau_R = \frac{5\eta}{\varepsilon + P}\,, \tag{4.7}$$

where, following the same convention adopted in the previous chapters, ε is the energy density and P is the thermodynamic pressure. In the following, we assume that the shear viscosity over entropy density, η/s, is constant.

Furthermore, the spacetime dependence of $f_{\mathbf{k}}$ is restricted to the time coordinate τ. On top of that, the assumption of isotropy in the xy-plane implies that the momentum dependence of $f_{\mathbf{k}}$ can be fully determined by the transverse momentum, $k_\perp \equiv \sqrt{k_x^2 + k_y^2}$, and the longitudinal component, k_{η_s}. In particular, given the on-shell condition, $k_\mu k^\mu = 0$, $k_\perp$ can be expressed solely in terms of k_0 and k_{η_s}, via, $k_0 = \sqrt{k_\perp^2 + k_{\eta_s}^2/\tau^2}$. Then, in Bjorken flow, the single-particle distribution function can be written as,

$$f_{\mathbf{k}} = f(\tau, k_0, k_{\eta_s}). \tag{4.8}$$

At this point, it is convenient to define a normalized space-like 4-vector $z_\mu = (0, 0, 0, -\tau)$, such that $z_\mu z^\mu = -1$, that is orthogonal to the fluid 4-velocity, $u_\mu z^\mu = 0$, so that Eq. (4.8) can be expressed in a covariant form,

$$f_{\mathbf{k}} = f(\tau, u^\mu k_\mu, z^\mu k_\mu). \tag{4.9}$$

The irreducible moments $\varrho_r^{\mu_1\cdots\mu_\ell}$ are defined as integrals of $f_{\mathbf{k}}$ over momentum space. Thus, in Bjorken flow, they can only depend on τ, u^μ, and z^μ, see Eqs. (4.1) and (4.9). In particular, their tensor structure must be constructed solely from combinations of u^μ, z^μ, and the metric tensor, $g^{\mu\nu}$. The only combination of these tensors that form an irreducible tensor of rank ℓ is $z^{\langle\mu_1}\cdots z^{\mu_\ell\rangle}$ [58]. Thus, the irreducible moments must have the following general form in Bjorken flow,

$$\varrho_n^{\mu_1\cdots\mu_\ell} = \mathcal{F}(\tau) z^{\langle\mu_1}\cdots z^{\mu_\ell\rangle}, \tag{4.10}$$

where we identify,

$$\mathcal{F}(\tau) = (-1)^\ell \frac{(2\ell-1)!!}{\ell!} z_{\langle\mu_1}\cdots z_{\mu_\ell\rangle}\varrho_n^{\mu_1\cdots\mu_\ell}, \tag{4.11}$$

making use of the identity [58]

$$z_{\langle\mu_1}\cdots z_{\mu_\ell\rangle} z^{\langle\mu_1}\cdots z^{\mu_\ell\rangle} = (-1)^\ell \frac{\ell!}{(2\ell-1)!!}. \tag{4.12}$$

The next step it to obtain the explicit form of $\mathcal{F}(\tau)$. First, we note that [58]

$$z_{\langle\mu_1}\cdots z_{\mu_\ell\rangle} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} = \sum_{q=0}^{[\ell/2]} C(\ell,q)(-E_{\mathbf{k}}^2)^q\,(-1)^q \left(\frac{k_{\eta_s}}{\tau}\right)^{\ell-2q} = E_{\mathbf{k}}^\ell \sum_{q=0}^{[\ell/2]} C(\ell,q)(\cos\Theta)^{\ell-2q}, \tag{4.13}$$

where we have defined $\cos\Theta \equiv k_{\eta_s}/(\tau k_0)$, and the coefficients $C(\ell,q)$ were defined in Eq. (3.4) and are reproduced below for convenience,

$$C(\ell,q) = (-1)^q \frac{(\ell!)^2}{(2\ell)!}\frac{(2\ell-2q)!}{q!(\ell-q)!(\ell-2q)!}. \tag{4.14}$$

The Legendre polynomials can be expressed as [142]

$$P_n(x) = \frac{(2n-1)!!}{n!} \sum_{q=0}^{[n/2]} C(n,q)\, x^{n-2q}, \tag{4.15}$$

thus leading to

$$z_{\langle\mu_1}\cdots z_{\mu_\ell\rangle} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} = \frac{\ell!}{(2\ell-1)!!}\, E_{\mathbf{k}}^{\ell}\, P_\ell(\cos\Theta). \tag{4.16}$$

Wherefore, it follows that

$$z_{\langle\mu_1}\cdots z_{\mu_\ell\rangle}\varrho_n^{\mu_1\cdots\mu_\ell} = \frac{\ell!}{(2\ell-1)!!}\int dK k_0^{n+\ell} P_\ell(\cos\Theta) f_{\mathbf{k}}. \tag{4.17}$$

The system's invariance under reflections around the η_s-axis further implies that

$$f(\tau, k_0, k_{\eta_s}) = f(\tau, k_0, -k_{\eta_s}). \tag{4.18}$$

Moreover, the Legendre polynomials, $P_\ell(\cos\Theta)$, are even (odd) functions of k_{η_s}, as long as ℓ is equally even (odd). Therefore, irreducible moments of odd rank are identically zero in Bjorken flow [58]. Taking $\ell \to 2\ell$, we have

$$z_{\langle\mu_1}\cdots z_{\mu_{2\ell}\rangle}\varrho_n^{\mu_1\cdots\mu_{2\ell}} \equiv \frac{(2\ell)!}{(4\ell-1)!!}\varrho_{n+2\ell,\ell}, \tag{4.19}$$

where we have defined the new fields[5]

$$\varrho_{n,\ell} = \int dK k_0^n P_{2\ell}(\cos\Theta) f_{\mathbf{k}}. \tag{4.20}$$

In particular, taking $f_{\mathbf{k}} = f_{0\mathbf{k}}$, we obtain the equilibrium value of the irreducible moments, which vanishes unless $\ell = 0$, as a consequence of the orthogonality of the Legendre polynomials [142],

$$\varrho_{n,\ell}^{\text{eq}} = e^{\alpha}\frac{(n+1)!}{2\pi^2} T^{n+2}\delta_{\ell 0}. \tag{4.21}$$

In summary, the irreducible moments of a generic single-particle distribution function in Bjorken flow can be written as

$$\varrho_n^{\mu_1\cdots\mu_{2\ell}} = \varrho_{n+2\ell,\ell} z^{\langle\mu_1}\cdots z^{\mu_{2\ell}\rangle}. \tag{4.22}$$

In order to obtain the equation of motion satisfied by $\varrho_{n,\ell}$, it is still necessary to provide an expression for the collision term, $C[f]$. In general, obtaining a closed expression for $C[f]$ is the most challenging part of solving the Boltzmann equation with the method of moments. It typically requires the computation of several integrals involving the single-particle distribution function and the transition rate that determines the scattering processes, which can be rather cumbersome even in the linear regime [143, 144]. As was done in the previous chapter, we once

[5] We remark that the moments $\mathcal{L}_\ell$, investigated in Refs. [75–79], are a subset of the irreducible moments $\varrho_{n,\ell}$ for $n = 2$. Furthermore, what these references refer to as mode coupling theory is equivalent to the traditional method of moments discussed here.

again adopt the relaxation time approximation [104] for the collision term. In this approach, the single-particle distribution function is assumed to relax to its equilibrium value within a timescale τ_R, see Eq. (4.23), and the collision term becomes simply

$$C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}\left(f_{\mathbf{k}} - f_{0\mathbf{k}}\right) \Longrightarrow C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\frac{1}{\tau_R}\left(\varrho_r^{\mu_1\cdots\mu_\ell} - \varrho_{r,\,\mathrm{eq}}^{\mu_1\cdots\mu_\ell}\right). \tag{4.23}$$

Hence, all irreducible moments also evolve towards their equilibrium values within the same timescale. As previously mentioned, in order to be consistent with the conservation of energy and momentum, the relaxation time approximation requires the imposition of Landau matching conditions, in which the values of the temperature and chemical potential out of equilibrium are defined so that the particle and energy densities are fixed to their equilibrium values

$$n \equiv n_0 \Longrightarrow \varrho_{1,0} = \varrho_{1,0}^{\mathrm{eq}}\,, \quad \varepsilon \equiv \varepsilon_0 \Longrightarrow \varrho_{2,0} = \varrho_{2,0}^{\mathrm{eq}}. \tag{4.24}$$

We are now finally in position to obtain a set of coupled equations of motion for the irreducible moments in the framework of Bjorken flow. Replacing Eqs. (4.22) and Eq. (4.23) into Eq. (4.4) and contracting it with $z_{\langle\mu_1}\cdots z_{\mu_{2\ell}\rangle}$, we obtain

$$\frac{d\varrho_{n,\ell}}{d\tau} = -\frac{1}{\tau_R}\left(\varrho_{n,\ell} - \varrho_{n,\ell}^{\mathrm{eq}}\right) - \mathcal{P}_{n,\ell}\frac{\varrho_{n,\ell-1}}{\tau} - \mathcal{Q}_{n,\ell}\frac{\varrho_{n,\ell}}{\tau} - \mathcal{R}_{n,\ell}\frac{\varrho_{n,\ell+1}}{\tau}, \tag{4.25}$$

where we have introduced the following coefficients

$$\mathcal{P}_{n,\ell} = 2\ell\frac{(n+2\ell)(2\ell-1)}{(4\ell+1)(4\ell-1)}, \tag{4.26a}$$

$$\mathcal{Q}_{n,\ell} = \frac{2\ell(2\ell+1) + n(24\ell^2+12\ell-3)}{3(4\ell-1)(4\ell+3)} + \frac{2}{3}, \tag{4.26b}$$

$$\mathcal{R}_{n,\ell} = (n-2\ell-1)\frac{(2\ell+1)(2\ell+2)}{(4\ell+1)(4\ell+3)}. \tag{4.26c}$$

We recovered the set of differential equations previously obtained in Ref. [58], where the method of moments was directly constructed assuming the symmetries of Bjorken flow within the relaxation time approximation. In this chapter, however, we constructed the method of moments for a general flow configuration and only then employed the assumptions of Bjorken flow and the relaxation time approximation. In particular, we remark that moments of a given rank are always coupled with those of higher and lower ranks, given by $\varrho_{n,\ell+1}$ and $\varrho_{n,\ell-1}$, respectively. Meanwhile, moments with different powers of energy do not couple with each other, a particular feature of the relaxation-time approximation for massless particles [58].

In order to work with dimensionless variables, it is convenient to obtain a hierarchy of differential equations for the irreducible moments normalized by the appropriate equilibrium value. For that purpose, we define the *rescaled* moments

$$\chi_{n,\ell} \equiv \frac{\varrho_{n,\ell}}{\varrho_{n,0}^{\mathrm{eq}}}. \tag{4.27}$$

Furthermore, we also define the *normalized* proper time, $\hat{\tau} \equiv \tau/\tau_R$. We remark that with this choice of variables, the dynamics of the (rescaled) fields no longer explicitly depends on the viscosity and more dissipative systems can only be studied by considering smaller values of the initial rescaled time. The rescaled moments then satisfy the following equations of motion

$$\frac{d\chi_{n,\ell}}{d\hat{\tau}} = \frac{3}{2}\left[1 - \frac{\chi_{2,1}(1-\alpha)}{4-\alpha}\right]^{-1} \times \left[-\chi_{n,\ell} + \delta_{\ell 0} - \mathcal{P}_{n,\ell}\frac{\chi_{n,\ell-1}}{\hat{\tau}} - \mathcal{S}_{n,\ell}\frac{\chi_{n,\ell}}{\hat{\tau}} - \mathcal{R}_{n,\ell}\frac{\chi_{n,\ell+1}}{\hat{\tau}} + \frac{2(n-1)}{3\hat{\tau}}\chi_{2,1}\chi_{n,\ell}\right], \quad (4.28)$$

where $\mathcal{P}_{n,\ell}$ and $\mathcal{R}_{n,\ell}$ are the ones defined in Eq. (4.26) and we have introduced

$$\mathcal{S}_{n,\ell} = \frac{2\ell(2\ell+1)(2n+1)}{3(4\ell-1)(4\ell+3)}. \quad (4.29)$$

In terms of the rescaled moments, Landau matching conditions imply that $\chi_{1,0} = \chi_{2,0} = 1$. Therefore, these moments are non-dynamic and thus the thermal potential and temperature must be obtained from the conservation laws, which can be expressed as

$$\frac{d\alpha}{d\hat{\tau}} - \frac{3}{2}\left[1 - \frac{\chi_{2,1}(1-\alpha)}{4-\alpha}\right]^{-1}\left(\frac{2\chi_{2,1}}{\hat{\tau}}\right) = 0, \quad (4.30)$$

$$\frac{dT}{d\hat{\tau}} + \frac{3}{2}\left[1 - \frac{\chi_{2,1}(1-\alpha)}{4-\alpha}\right]^{-1}\frac{T}{3\hat{\tau}}(1+2\chi_{2,1}) = 0. \quad (4.31)$$

Once the hierarchy of moment equations (4.28) is solved coupled with the equations of motion for thermal potential and temperature – Eqs. (4.30) and (4.31), respectively – it is possible to reconstruct the single-particle distribution function. In principle, an *exact* solution for the Boltzmann equation can only be achieved by the inclusion of an infinite number of moments in the moment expansion. However, this is surely impossible, in practice and one must truncate the moment expansion including a sufficiently large number of moments. In particular, we expect that the inclusion of more terms in the moment expansion (3.13) will lead to a more accurate solution for the single-particle distribution function. The reconstruction of the single-particle distribution function from its moments will be thoroughly investigated in the next chapter.

4.3 Convergence of the solutions

We solve Eqs. (4.25) by imposing a truncation of the moment expansion. In practice, this means that irreducible moments of a given rank (or higher) will be set to zero in the calculations. In the highly symmetric flow configuration considered in the previous section, this is implemented by taking $\varrho_{n,\ell} = 0, \forall\ \ell > \ell_{\max}$, with the parameter $\ell_{\max}$ quantitatively specifying our truncation. Once this is done, Eqs. (4.25) can be solved using numerical algorithms. Unless stated otherwise, we consider the system to be in equilibrium at an initial time τ_0, with a temperature $T(\tau_0) = 1$ GeV and a vanishing chemical potential. Also, we consider three distinct values of initial rescaled time, namely $\hat{\tau}_0 = 0.01$, 0.1 and 1.

We first look at the convergence of solutions for the temperature and thermal potential as a function of the rescaled time, τ/τ_R, for $\ell_{\rm max} = 2$, 4 and 8. These quantities are obtained by solving Eqs. (4.25) and then using Landau matching conditions to extract T and α from the particle number and energy densities. In Fig. 6, we display solutions for the temperature (upper panels) and thermal potential (lower panels) as functions of the rescaled time. As already mentioned, we note that all the dependence on η/s is embedded in the relaxation time, τ_R, and, by plotting our results as a function of the rescaled time, the magnitude of the dissipative effects is solely determined by the initial value of τ/τ_R – that is, more dissipative systems can be probed by considering smaller values for the initial rescaled time. We observe that, as the initial rescaled time is smaller, i.e. as the system becomes more dissipative, more moments have to be included in the hierarchy of differential equations (4.25) in order for the solutions to converge.

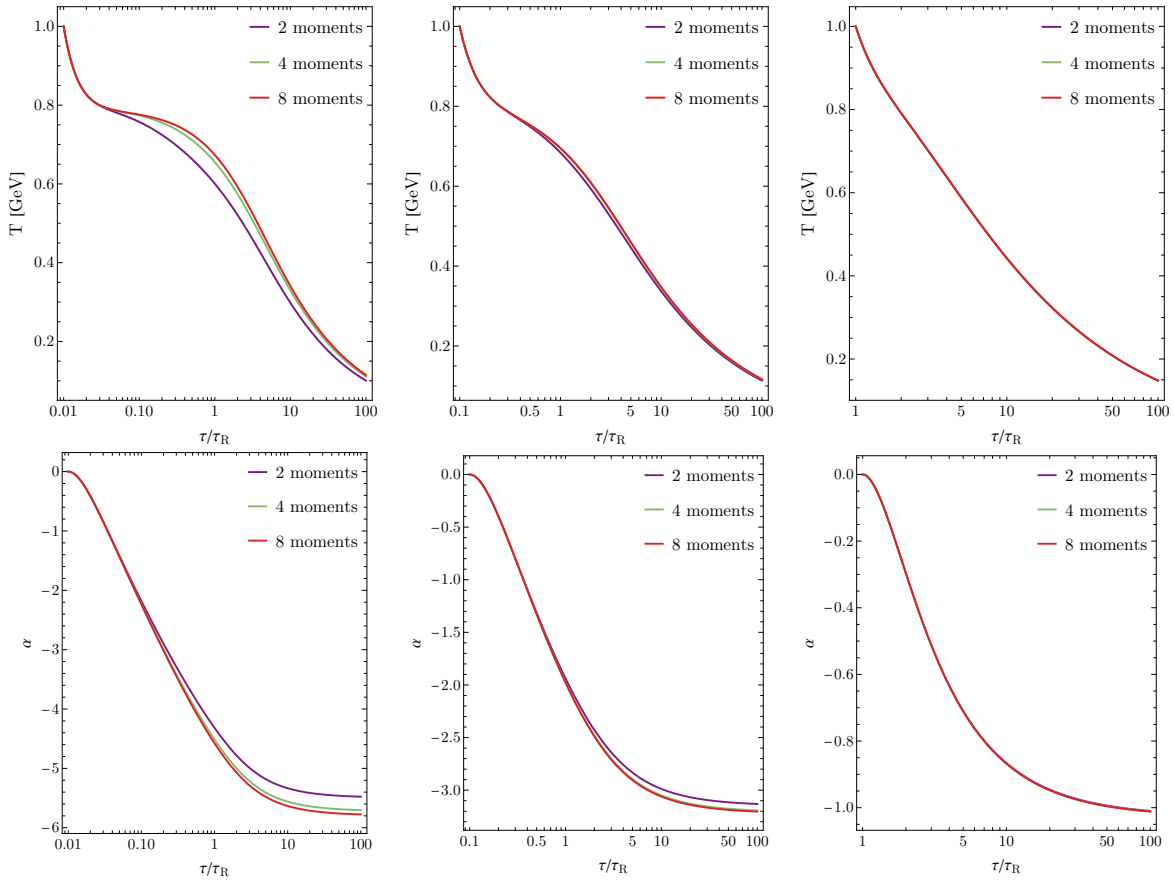

Figure 6 – Solutions for the temperature (upper panels) and thermal potential (lower panels) considering different truncations and values for the initial rescaled time and assuming $T(\tau_0) = 1$ GeV and $n(\tau_0) = T(\tau_0)^3/\pi^2$.

Next we look at the convergence of certain irreducible moments. In Fig. 7, we display $\varrho_{2,1}/\varrho^{\rm eq}_{2,0}$ (which is related to the shear-stress tensor as $\pi^{\eta_s\eta_s} = -2\varrho_{2,1}/3$ [58]), $\varrho_{2,5}/\varrho^{\rm eq}_{2,0}$ and $\varrho_{2,10}/\varrho^{\rm eq}_{2,0}$ as functions of the rescaled time τ/τ_R, for several values of $\ell_{\rm max}$. It can readily be seen that, as more moments are included in the hierarchy of differential equations, the solutions for the irreducible moments gradually converge to a unique curve. Moreover, as the initial rescaled time becomes smaller, the irreducible moments become larger in magnitude and more moments

have to be included in order to observe convergence, i.e., a larger value of $\ell_{\max}$ is required in order for the solutions to converge. We remark that this behavior is also observed for different values of n, which defines the power of energy in Eq. (4.20), but we do not display these cases for the sake of simplicity.

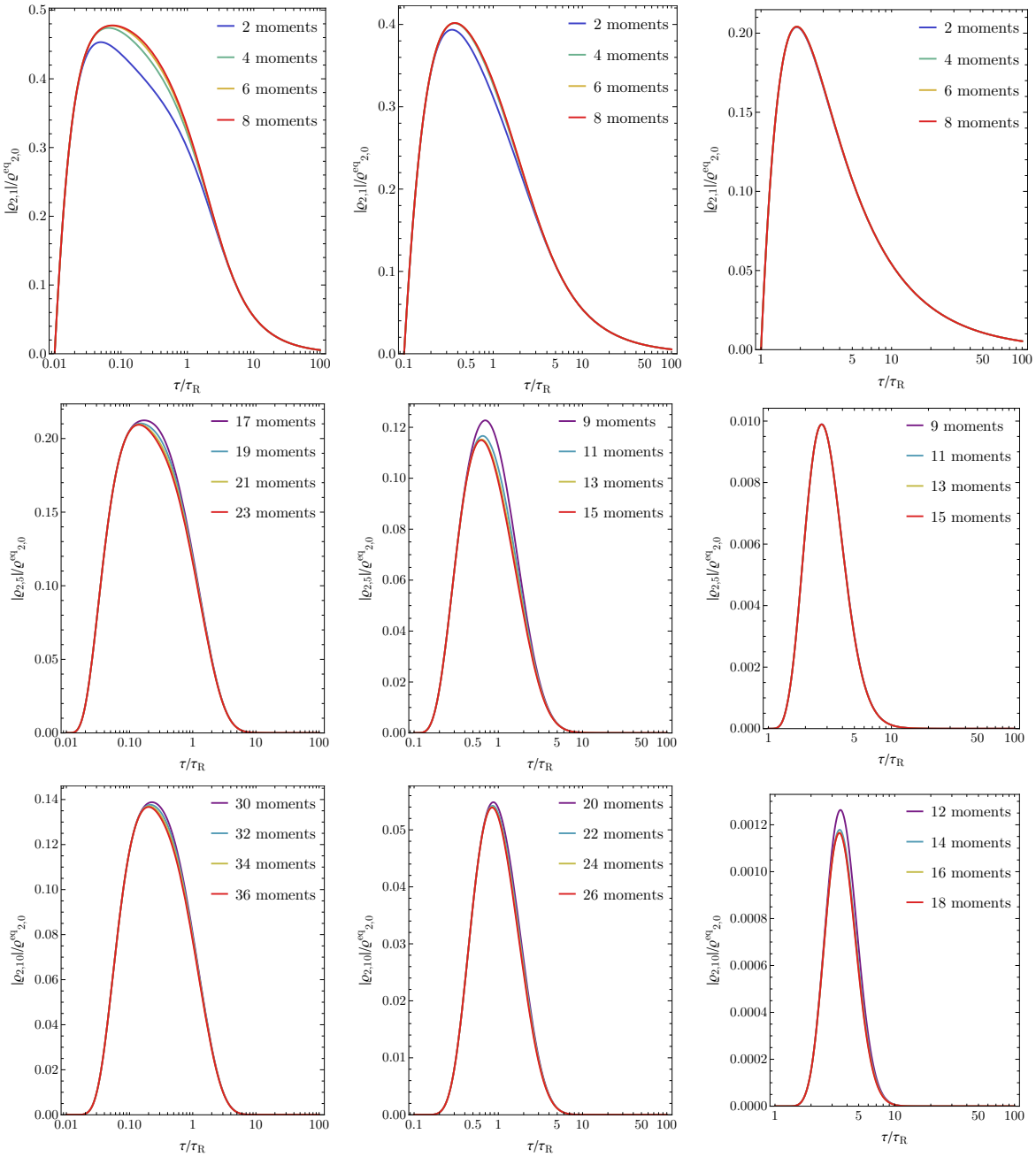

Figure 7 – Solutions for $\varrho_{2,1}/\varrho^{\rm eq}_{2,0}$ (upper panels), $\varrho_{2,5}/\varrho^{\rm eq}_{2,0}$ (middle panels) and $\varrho_{2,10}/\varrho^{\rm eq}_{2,0}$ (lower panels) considering different truncations and values for the initial rescaled time and assuming $T(\tau_0) = 1$ GeV and $n(\tau_0) = T(\tau_0)^3/\pi^2$.

We now look at the irreducible moments, $\varrho_{n\ell}$, fixing the value of n and varying the parameter ℓ. We solve Eqs. (4.25) for $\ell_{\max} = 100$ and portray, in Fig. 8, the absolute value of the irreducible moments normalized by their equilibrium value, $|\rho_{n,\ell}|/\rho^{\rm eq}_{n,0}$, for $n = 1$ and $\ell = 1$–10 as a function of τ/τ_R. Each panel of Fig. 8 displays solutions obtained for a different choice of

initial time. We observe that, for a fixed n, the normalized irreducible moments become smaller (in magnitude) as the value of ℓ is increased. This is consistent with the apparent convergence observed for the solutions of the moment equations obtained with our truncation scheme – moments become smaller as the value of ℓ is increased and, for a sufficiently large value of ℓ, it becomes a good approximation to simply set them to zero. Naturally, the moments with $\ell \ll \ell_{\rm max}$ are well approximated in this scheme, whereas those with $\ell \sim \ell_{\rm max}$ are usually not. This is in agreement with the behavior observed in Figs. 6 and 7.

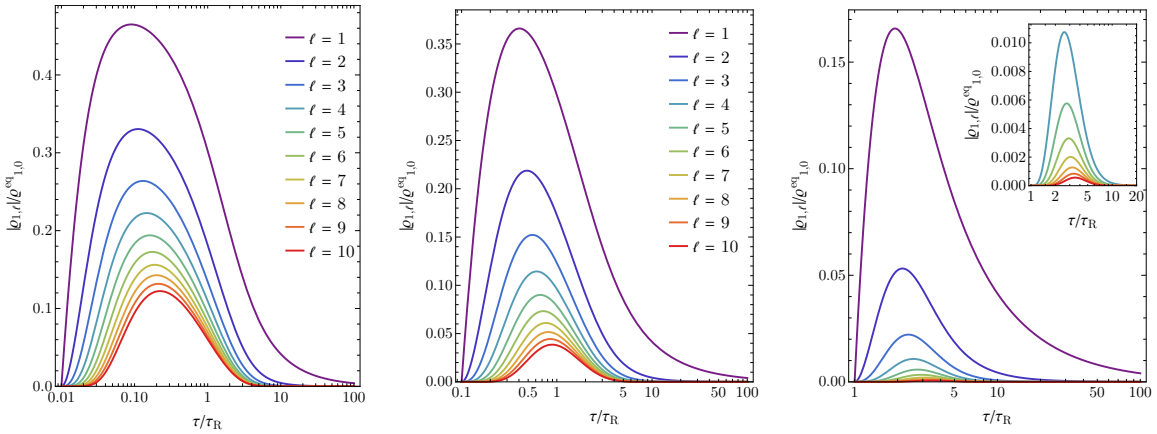

Figure 8 – Normalized irreducible moments for $n = 1$ and considering different values for ℓ and initial rescaled time and assuming $T(\tau_0) = 1$ GeV and $n(\tau_0) = T(\tau_0)^3/\pi^2$.

Next, we analyze the behavior of the irreducible moments, $\varrho_{n\ell}$, for $\ell = 1$ and several values of n. We solve Eqs. (4.25) with $\ell_{\rm max} = 10$ (which is sufficiently large to ensure the convergence of the solutions) and considering $n = 1$–10. In Fig. 9, we display the absolute value of the normalized irreducible moments $|\varrho_{n1}|/\varrho^{\rm eq}_{n0}$ as a function of the rescaled time τ/τ_R. We observe that as the value of n is increased, the magnitude of the normalized moments is also increased, and this behavior becomes even more manifested as the initial rescaled time is smaller. We remark that this behavior is also observed for different choices of ℓ, but we not display these results here for the sake of simplicity. This behavior of the irreducible moments as one increases the parameter n is significant, since it renders the task of determining the single-particle distribution function complicated. In the method of moments, the single-particle distribution function is expressed in terms of a sum of irreducible moments, cf. Eq. (3.13), and it is generally assumed that this series converges rapidly, at least for relatively small values of momentum. However, here we see that $\varrho_{n\ell}$ grows significantly with n at intermediate (rescaled) times, making it challenging to calculate the moment expansion to a sufficiently high order and establishing its convergence. This will be thoroughly investigated in the next chapter.

We now test if the solutions converge to the correct value by comparing to semi-analytical and numerical results. In particular, we compute the pressure anisotropy, defined as the ratio of the longitudinal pressure to the transverse pressure, $P_L/P_T = (1 + 2\varrho_{2,1})/(1 - \varrho_{2,1})$. We assume different values for the shear viscosity over entropy density, η/s, and initial temperatures calibrated to emulate the hot and dense matter created at the Relativistic Heavy-Ion Collider

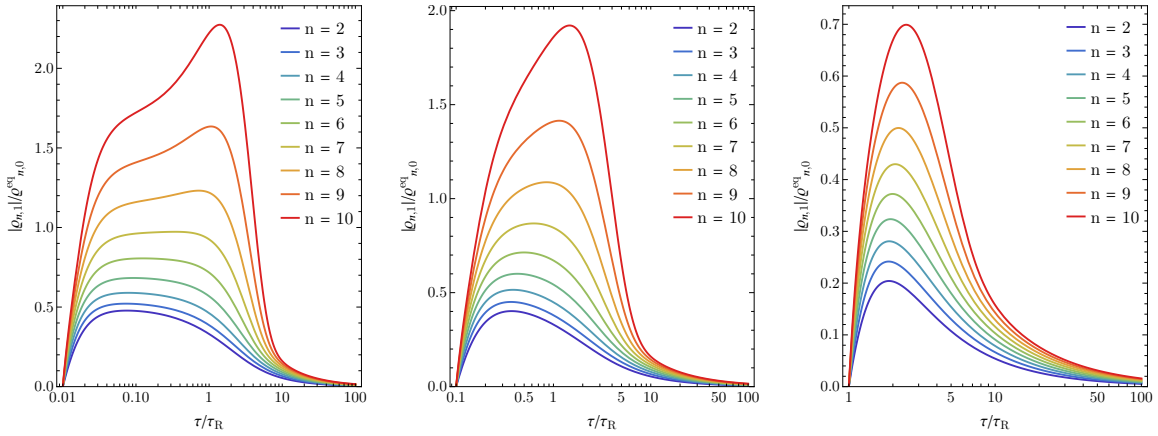

Figure 9 – Normalized irreducible moments for $\ell_{\rm max} = 10$ considering different values for the initial rescaled time and assuming $T(\tau_0) = 1$ GeV and $n(\tau_0) = T(\tau_0)^3/\pi^2$.

(RHIC) and the Large Hadron Collider (LHC). In the left panel of Fig. 10, we display the pressure anisotropy as a function of the proper time, calculated from Eqs. (4.25) with $\ell_{\rm max} = 6$, and compare them to semi-analytical solutions for this quantity derived in Ref. [98]. Three values of shear viscosity over entropy density are considered: $\eta/s = 1/(4\pi)$, $3/(4\pi)$, and $10/(4\pi)$ – here, since we do not display the solution as a function of the rescaled time, the curves will depend on the magnitude of the shear viscosity coefficient. For the sake of comparison, we consider the initial conditions proposed in Ref. [98], with the system being initially at equilibrium at an initial time $\tau_0 = 0.25$ fm, with an initial temperature of $T(\tau_0) = 300$ MeV. For this type of initial condition, we observe that including 6 moments in the moment equations is sufficient to obtain a convergent solution. In the right panel of Fig. 10, we compare our solutions to numerical solutions of the Boltzmann equation obtained using the BAMPS simulation code [145, 146], in Ref. [119]. We now use the initial conditions employed in Ref. [119], i.e., a system in equilibrium at $\tau = 0.4$ fm with an initial temperature of $T = 500$ MeV. We note that the solutions calculated with the method of moments are in surprisingly good agreement with the numerical solutions from BAMPS for practically all values of shear viscosity employed. This did not have to be the case, since in our solutions the collision term was simplified significantly by imposing the relaxation time approximation. This indicates that the relaxation time approximation, even though extremely rudimentary, can still provide a reasonable approximation for the pressure anisotropy. Nevertheless, this may be a feature of the highly symmetric flow configuration considered here.

4.3.1 Anisotropic initial conditions

So far, we have restricted our analysis to systems that are initially in equilibrium. For the sake of completeness, we now solve the moment equations considering an anisotropic initial distribution function. In Ref. [147], an anisotropic distribution function is constructed by changing the 3-momentum of an isotropic distribution as follows

$$|\mathbf{k}| \to \sqrt{\mathbf{k}^2 + \xi(\mathbf{k}\cdot\hat{\mathbf{n}})^2}, \tag{4.32}$$

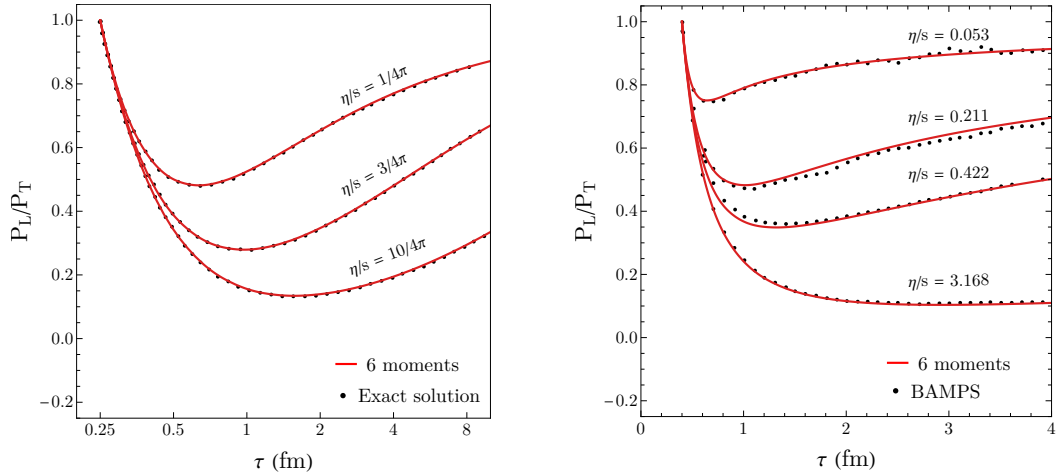

Figure 10 – Pressure anisotropy for $T(\tau_0) = 300$ MeV and $\tau_0 = 0.25$ fm (left panel) and $T(\tau_0) = 500$ MeV and $\tau_0 = 0.4$ fm (right panel).

where $\xi > -1$ is the anisotropy coefficient and $\hat{\mathbf{n}}$ is an unitary vector pointing in the direction of the anisotropy. Naturally, if $\xi = 0$, one recovers the original isotropic distribution. In Bjorken flow, the z-axis corresponds to the anisotropic direction, and thus $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. It follows that $\mathbf{k} \cdot \hat{\mathbf{n}} = k_\mu z^\mu = k_{\eta_s}/\tau$, and therefore

$$\sqrt{\mathbf{k}^2 + \xi(\mathbf{k} \cdot \hat{\mathbf{n}})^2} = k_0 \sqrt{1 + \xi \cos^2 \Theta}. \tag{4.33}$$

Here we apply this procedure over the equilibrium Maxwell-Boltzmann distribution function, thus leading to

$$f_{\text{aniso}} = \exp\left(\alpha^\star - \beta^\star E_{\mathbf{k}} \sqrt{1 + \xi \cos^2 \Theta}\right). \tag{4.34}$$

We shall use the distribution above to compute the initial value of the irreducible moments. In particular, we remark that $\alpha^\star$ and $\beta^\star$ do not necessarily correspond to the usual thermal potential and inverse temperature, respectively. They are simply free parameters that define the anisotropic distribution function and, without loss of generality, we take $\alpha^\star = 1$ and $\beta^\star = 1$. The physical temperature and chemical potential are then extracted by imposing Landau matching conditions and used to compute the corresponding equilibrium values for the irreducible moments.

We consider $\tau_0/\tau_R = 0.1$ and assume three different values for the anisotropy coefficient, namely $\xi = 0.01, 0.1, 1$. In Fig. 11, we display the resulting rescaled irreducible moments. We observe that the irreducible moments slightly increase in magnitude as we probe more anisotropic initial conditions. Nevertheless, we note that the system does not display a strong sensitivity on the magnitude of the anisotropy coefficient. A similar behavior is observed when we consider smaller initial rescaled times, but we do not display these cases here. Overall, there are no qualitative differences to the case of an isotropic initial distribution displayed in Fig. 9.

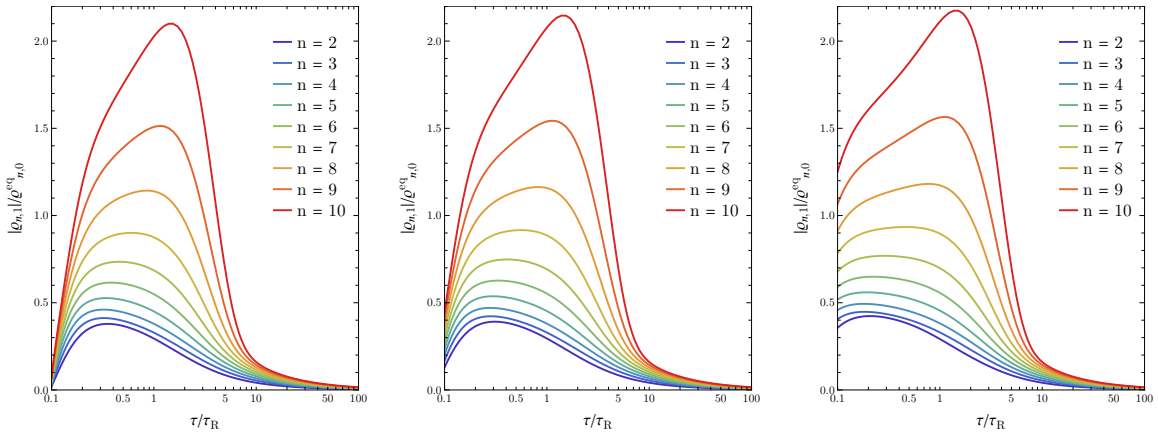

Figure 11 – Normalized irreducible moments assuming an initially anisotropic distribution function for $\ell_{\max} = 10$ considering different values for the anisotropy coefficient, $\xi = 0.1$, 1, 10. In all cases, we assume the initial rescaled time is $\tau_0/\tau_R = 0.1$, $T(\tau_0) = 1$ GeV and $n(\tau_0) = T(\tau_0)^3/\pi^2$.

4.4 Discussion

We have derived the equations of motion for all the *irreducible* moments of the single-particle distribution function. As expected from previous less general results [2,72], the equations of motion for the moments are coupled, with the particular challenging feature of the dynamics of lower-rank moments being always coupled to those of a higher-rank, leading to an endless tower of equations. In this Chapter we investigated how this hierarchy of equations can be properly truncated and solved. This is a fundamental step in making it possible to solve the Boltzmann equation using the method of moments.

We have specifically investigated the solutions for the irreducible moments considering a gas of classical massless particles undergoing a Bjorken expansion, assuming the relaxation time approximation for the collision term [104]. In this flow profile, the dynamics of the irreducible moments is entirely contained in the scalars $\varrho_{n\ell}$, which then satisfy a simplified set of coupled differential equations of motion. We observe that, as we increase the truncation of the hierarchy of differential equations for the irreducible moments, Eqs. (4.25), i.e., as the dynamics of more moments is taken into account, their solutions gradually converge. In particular, we observe that for more dissipative systems, the irreducible moments increase in absolute value, as expected, and even more moments must be included in order for the solutions to converge. In addition, we computed the pressure anisotropy taking $\ell_{\max} = 6$ and obtained results in rather good agreement with numerical solutions of the Boltzmann equation for initial conditions calibrated to reproduce the hot and dense matter created at RHIC and LHC considering a wide range of values for the shear viscosity over entropy density. This serves as an important test for the capability of the method of moments in providing accurate solutions for moments of the single-particle distribution function.

Finally, we observed that the magnitude of the moments $\rho_{n\ell}$ always increases with

increasing n. This makes it a challenge to calculate the single-particle distribution function using the method of moments, since higher-order terms in the moment expansion will not be necessarily small. Understanding the convergence of this series is of the utmost important and will be carefully investigated in Chapter 5.

5 Divergence and resummation of the moment expansion

So far, the moment expansion for the single-particle distribution function has been assumed to converge and its truncations are widely employed in the description of dilute plasmas; for instance the 14-moment approximation has been crucial in modeling the electromagnetic radiation [148, 149] and soft hadrons [150, 151] that are produced in heavy-ion collisions. However, the convergence of this series has never been thoroughly investigated.[1] In this chapter, we investigate the convergence of the moment expansion considering a classical gas of massless particles in a simplified description of a heavy-ion collision referred to as Bjorken flow [103]. In contrast to common belief, we demonstrate that this expansion actually diverges due to the factorial growth of the moments of the distribution function. We then resum this series using the Borel-Padé method [153, 154] and finally determine the solution of the Boltzmann equation from the method of moments. These solutions demonstrate that truncations of the moment expansion often employed in theoretical descriptions of heavy-ion collisions might not be reliable approximations.

5.1 Multipole expansion of the distribution function

Following the method of moments, we now expand the single-particle distribution using a complete basis in momentum space – in our case, we employ Legendre, P_n, and associated Laguerre, $L_n^{(m)}$, polynomials. The result is [58, 83],

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} c_{n,\ell} (\beta E_{\mathbf{k}})^{2\ell} P_{2\ell}(\cos\Theta) L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \,, \tag{5.1}$$

where we defined the angle Θ in momentum space via $\cos\Theta \equiv k_{\eta_s}/(\tau E_{\mathbf{k}})$, with k_{η_s} being the longitudinal component of the 4-momentum. The expansion coefficients $c_{n,\ell}$ are determined using the orthogonality relations satisfied by the Legendre and associated Laguerre polynomials, leading to

$$c_{n,\ell} = (4\ell+1) n! \sum_{m=0}^{n} \frac{(-1)^m \, (m+2\ell+1)!}{(n-m)!(m+4\ell+1)!m!} \, \chi_{m+2\ell,\ell} \,, \tag{5.2}$$

where $\chi_{n,\ell}$ are the normalized irreducible moments, introduced in Eq. (4.27). Once the moment equations are solved, it is then possible to reconstruct the single-particle distribution function using Eq. (5.1). In principle, an *exact* solution for the distribution function would only be obtained with the inclusion of an infinite number of moments in the moment expansion (5.1). *A*

[1] The convergence of the moment expansion has been only investigated for homogeneous and isotropic fluids, in the context of cosmology [152].

priori, we expect that the inclusion of more moments in the moment expansion (5.1) will lead to a more accurate solution for the single-particle distribution function.

The main goal of this chapter is to compute the single-particle distribution function of a gas via the moment expansion. We first re-express Eq. (5.1) as,

$$f_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \frac{4\ell+1}{2} P_{2\ell}(\cos\Theta)\mathcal{F}_{\mathbf{k}}^{(\ell)}, \tag{5.3}$$

where we introduced the ℓ-th multipole coefficient

$$\mathcal{F}_{\mathbf{k}}^{(\ell)} \equiv \int_{-1}^{1} \mathrm{d}\cos\Theta P_{2\ell}(\cos\Theta)\ f_{\mathbf{k}} = \frac{2}{4\ell+1}(\beta E_{\mathbf{k}})^{2\ell} f_{0\mathbf{k}} \sum_{n=0}^{\infty} c_{n,\ell} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}})\ . \tag{5.4}$$

In order to reconstruct the single-particle distribution function from its moments, we first must solve the set of coupled equations for the rescaled irreducible moments, Eq. (4.28). The moment equations are truncated considering $\ell_{\max} = 40$ and $n_{\max} = 30$, which implies setting $\chi_{N,L} = 0$ for all $N > n_{\max}$ and/or $L > \ell_{\max}$. In all calculations investigated in this Chapter, we always assumed equilibrium initial conditions, i.e., $\chi_{n,\ell}(\hat{\tau}_0) = \delta_{\ell,0}$. Then, we compute the expansion coefficients $c_{n,\ell}$ using Eq.. (5.2), and finally compute the multipoles of the single-particle distribution according to Eq. (5.4).

The divergence of the series can be observed using Fig. 12, where we display the expansion coefficients $c_{n,1}$ (used to reconstruct the multipole $\mathcal{F}_{\mathbf{k}}^{(1)}$) as a function of the rescaled time. At both early and late times, we see that the expansion coefficients decrease with n. However, for intermediate values of the rescaled proper time, the expansion coefficients display a qualitatively different behavior and start to increase with n, with this behavior becoming more pronounced for smaller values of the *initial* rescaled time. We remark that, in order for the expansion (5.4) in associated Laguerre polynomials to converge, the expansion coefficients $c_{n,1}$ must not increase with n and, thus, the aforementioned behavior guarantees that Eq. (5.4) diverges at least for these intermediate values of the rescaled time. A qualitatively similar behavior is also observed for other values of ℓ.

We explicitly verify the non-convergence of the moment expansion in Fig. 13, where we compute a snapshot ($\hat{\tau} = 10$) of the first three multipoles of $f_{\mathbf{k}}$, for $\hat{\tau}_0 = 0.5$, as a function of $\beta E_{\mathbf{k}}$ and considering three different truncations of the series – the parameter $n_{\max}$ specifies the number of terms included in the series (5.4). We observe that the expansion for the three multipoles portrayed does not converge, exhibiting an unphysical oscillatory pattern as more terms are included in the moment expansion. Hence, this implies that the expansion of the single-particle distribution itself is also divergent. We note that this divergence is not expected and that truncations of the moment expansion are widely used in the derivation of hydrodynamics and in the phenomenology of heavy ion collisions. Furthermore, the moment expansion is crucial in deriving the equations of motion for the moments for general collision terms. In the following we obtain a resummed expression for the moment expansion and discuss the reasons behind this

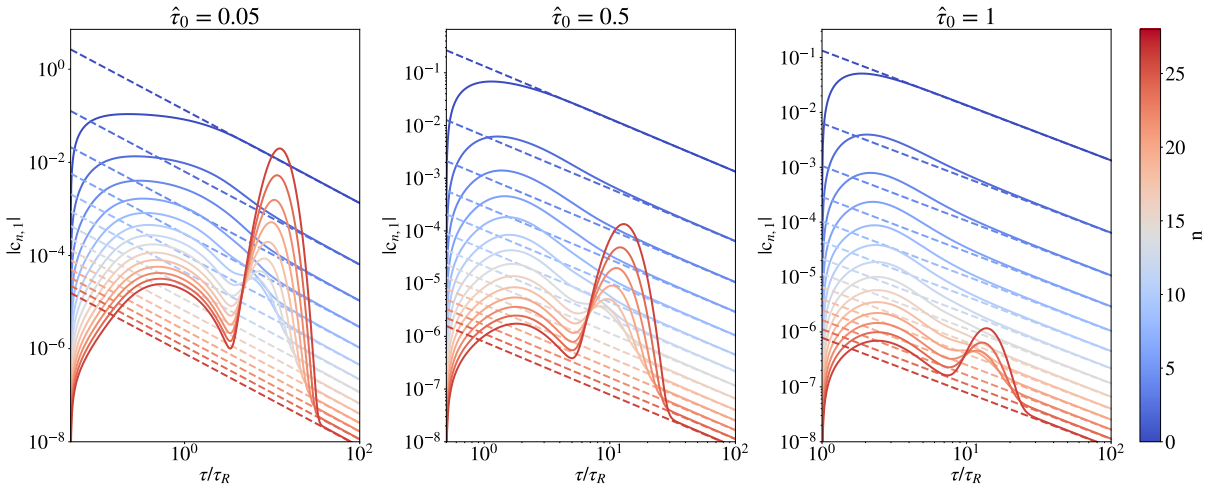

Figure 12 – Exact solutions (solid lines) and Navier-Stokes limit (dashed lines) for the expansion coefficients as a function of the rescaled proper time considering $\hat{\tau}_0 = 0.05$ (left panel), 0.5 (middle panel), and 1 (right panel). In all three cases, we assume equilibrium initial conditions, $\chi_{n\ell} = \delta_{\ell 0}$, $T(\hat{\tau}_0) = 10$ GeV, and $n(\hat{\tau}_0) = T(\hat{\tau}_0)^3/\pi^2$.

divergence. In particular, in Appendix F, we analytically demonstrate that the moment expansion for the distribution function diverges for rescaled times of $\hat{\tau}/\hat{\tau}_0 > 8$ in the case of a constant relaxation time.

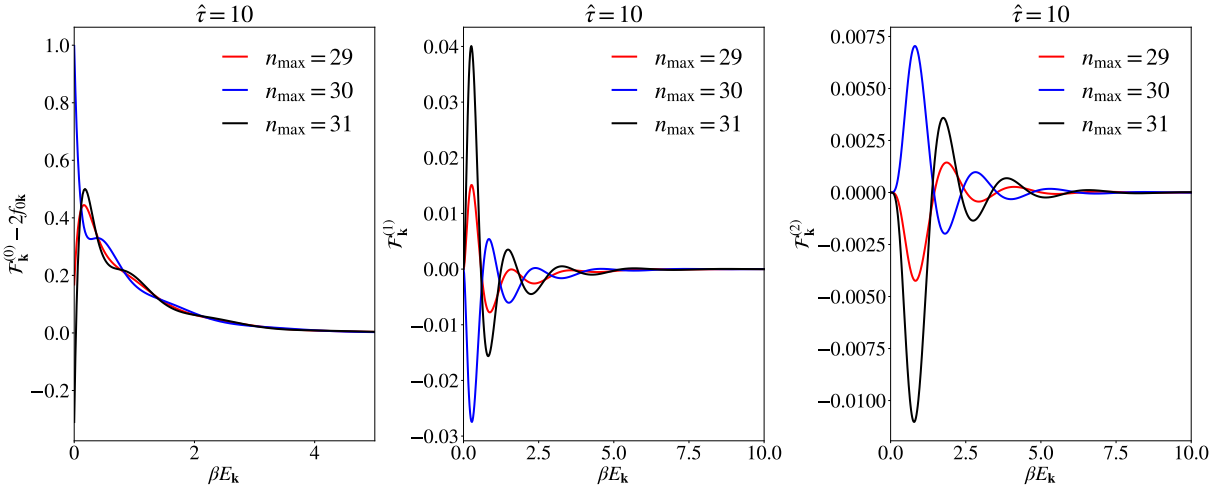

Figure 13 – Multipoles of the single-particle distribution function as functions of $\beta E_{\mathbf{k}}$, for different truncations, considering $\hat{\tau}_0 = 0.5$ and $\hat{\tau} = 10$.

5.2 Borel-Padé resummation algorithm

We have shown that the expansion of the distribution function (5.4) diverges for a wide range of values of the rescaled proper time. Therefore, in order to assess physically meaningful results from the moment expansion in Bjorken flow, it is of the utmost importance to employ resummation techniques that will be carefully discussed in this section.

5.2.1 Borel transform

In Appendix F, we *analytically* show that the moment expansion (5.1) diverges factorially assuming a simplified case for the moment equations in which all equilibrium moments are identically set to zero. Such pathological behavior suggests that a convergent series might be achievable by employing a Borel summation [153]. First, we define the Borel transform of an arbitrary power series as

$$\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\text{Original series}} \xrightarrow{\text{Borel transform}} \underbrace{\sum_{n=0}^{\infty} c_n \frac{x^n}{n!}}_{\text{Borel-transformed series}} \tag{5.5}$$

Then, if the Borel transform of the original series is such that the following integral is defined, the Borel sum of such a series is

$$\int dx\, e^{-x} \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}. \tag{5.6}$$

Finally, if this integral converges, the series is referred to as Borel summable, and thus we have

$$\sum_{n=0}^{\infty} c_n x^n \equiv \int dx\, e^{-y} \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}. \tag{5.7}$$

We then compute the Borel sum of the multipoles of the single-particle distribution function, Eq. (5.4), leading to

$$\mathcal{F}_{\mathbf{k}}^{(\ell)} = \frac{2}{4\ell+1} (\beta E_{\mathbf{k}})^{2\ell} f_{0\mathbf{k}} \int_0^{\infty} dx\, e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} c_{n,\ell} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \,. \tag{5.8}$$

In particular, let us analyze the integrand of the equation above,

$$\mathcal{I}_{\ell}(\beta E_{\mathbf{k}}, x) \equiv \frac{2}{4\ell+1} (\beta E_{\mathbf{k}})^{2\ell} f_{0\mathbf{k}}\, e^{-x} \sum_{n=0}^{N} \frac{x^n}{n!} c_{n,\ell} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \,. \tag{5.9}$$

In Fig. 14, we portray $\mathcal{I}_{\ell}(\beta E_{\mathbf{k}} = 1, y)$ as a function of y for a wide range of different truncations of the sum in Eq. (5.9). From this plot, it can be straightforwardly seen that there exists an optimal truncation N for the Borel sum of the multipoles, beyond which the results for $\mathcal{I}_{\ell}$ start to diverge in the range $20 \lesssim y \lesssim 40$. This is a clear signature of a divergent series, and thus serves as another striking evidence of such pathological behavior of the moment expansion in Bjorken flow. We thus conclude that the moment expansion is not Borel summable. Therefore, in order to obtain a convergent series, it is necessary to employ further resummation techniques.

5.2.2 Padé approximants

A Padé approximant is a rational representation of a function whose Maclaurin expansion recovers the power series representation of the original function up to a given order [155]. Considering a function which can be expressed as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i, \tag{5.10}$$

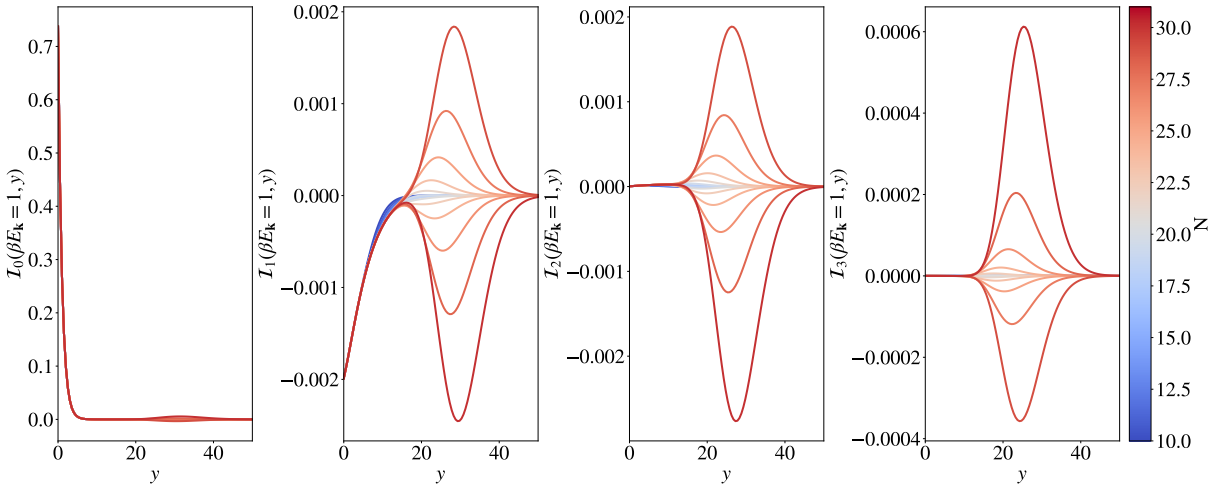

Figure 14 – Integrand of the Borel-transformed series of different multipoles for a wide range of truncations N, considering $\beta E_{\mathbf{k}} = 1$, $\hat{\tau}_0 = 0.5$ and $\hat{\tau} = 10$.

its Padé approximant of order (m, n) is defined as

$$f^{(m,n)}(x) = \frac{\sum_{i=0}^{m} b_i x^i}{\sum_{i=0}^{n} c_i x^i}. \tag{5.11}$$

Without loss of generality, we take $c_0 = 1$, which effectively reduces the number of unknown coefficients to $m + n + 1$. This implies that the Padé approximants capture the behavior of the original function up to $\mathcal{O}(x^{m+n})$, that is,

$$\sum_{i=0}^{\infty} a_i x^i = \frac{\sum_{i=0}^{m} b_i x^i}{\sum_{i=0}^{n} c_i x^i} + \mathcal{O}(x^{m+n+1}). \tag{5.12}$$

The so-called numerator and denominator coefficients – b_i and c_i, respectively – can thus be obtained iteratively. We have

$$(a_0 + a_1 x + \cdots)(1 + c_1 x + \cdots + c_n x^n) - (b_0 + b_1 x + \cdots + b_m x^m) = \mathcal{O}(x^{m+n+1}). \tag{5.13}$$

Then, the coefficients b_i and c_i can be obtained by successively equating Eq. (5.13) to zero, order by order in x. Up to $\mathcal{O}(x^m)$, we have

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= a_1 + a_0 c_1, \\ b_2 &= a_2 + a_1 c_1 + a_0 c_2, \\ &\vdots \\ b_m &= a_m + \sum_{i=1}^{\min(m,n)} c_i a_{m-i}. \end{aligned} \tag{5.14}$$

Similarly, equating the coefficients of the terms of order $\mathcal{O}(x^{m+1})$ up to $\mathcal{O}(x^{m+n})$, we obtain

$$\begin{aligned} a_{m+1} + a_m c_1 + \cdots + a_{m-n+2} c_{n-1} + a_{m-n+1} c_n &= 0, \\ a_{m+2} + a_{m+1} c_1 + \cdots + a_{m-n+3} c_{n-1} + a_{m-n+2} c_n &= 0, \\ &\vdots \\ a_{m+n} + a_{m+n-1} c_1 + \cdots + a_{m+1} c_{n-1} + a_m c_n &= 0. \end{aligned} \tag{5.15}$$

Since we set $c_0 = 1$, we obtain a set of n linear equations for the denominator coefficients, which can be expressed as

$$\begin{pmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ a_{m+n} \end{pmatrix} + \begin{pmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0. \tag{5.16}$$

Therefore, the denominator coefficients are obtained by inverting the matrix containing the expansion coefficients of the original function, thus leading to

$$\vec{c} = \mathcal{A}^{-1} \vec{a}. \tag{5.17}$$

Equations (5.15) and (5.16) [or, conversely, (5.17)] are referred to as Padé equations. Overall, in order to obtain the Padé approximant of a function $f(x)$, it is necessary to:

1. Expand $f(x)$ in power series of x and compute the expansion coefficients a_i;
2. Construct the matrix $\mathcal{A}$ as in Eq. (5.16) and compute its inverse. Then, from Eq. (5.17), compute the denominator coefficients c_i;
3. Calculate the numerator coefficients using Eq. (5.15).

Finally, using the results obtained in steps 2 and 3, we are able to construct a rational representation of a function.

Application to the moment expansion

In order to obtain a convergent result, the Borel sum of the multipoles, Eq. (5.8), shall be computed using Padé approximants. First, we re-express the series $f(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} c_{n,\ell} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}})$ using Padé approximants [154] and only then compute the integral in Eq. (5.8) numerically.

We then compute $\mathcal{I}_\ell(\beta E_{\mathbf{k}}, y)$ using Padé approximants with a quadratic polynomial in the denominator. In Fig. 15, we portray $\mathcal{I}_\ell(\beta E_{\mathbf{k}} = 1, y)$ as a function of y for a wide range of different truncations of the sum in Eq. (5.9), computed using Padé approximants. In this case, it can be straightforwardly seen that the Padé approximants (black dashed lines) seem to capture the optimal truncation of the series.

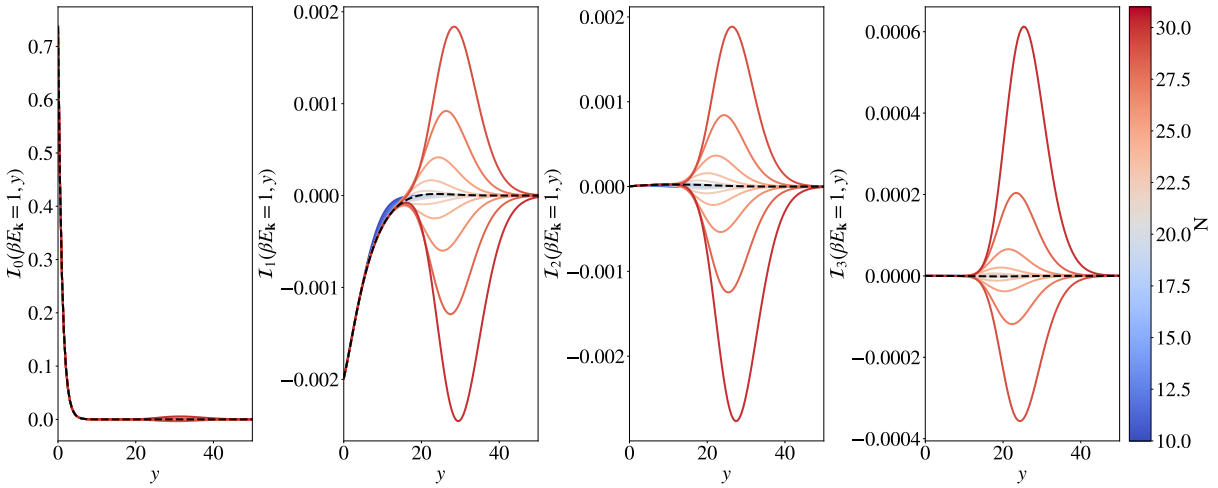

Figure 15 – Integrand of the Borel-transformed series of different multipoles for a wide range of truncations N, considering $\beta E_{\mathbf{k}} = 1$, $\hat{\tau}_0 = 0.5$ and $\hat{\tau} = 10$. The black dashed lines correspond to the Borel-transformed series computed using Padé approximants for the maximal truncation considered here.

The next step is to apply the Borel-Padé resummation algorithm in order to obtain physically meaningful results from the moment expansion. In Fig. 16, we show the first three multipoles of the distribution function as a function of $\beta E_{\mathbf{k}}$ for $\hat{\tau}_0 = 0.5$, computed using a Borel-Padé resummation scheme, considering the same truncations as in Fig. 13. We display results for three different times, namely $\hat{\tau} = 1$ (solid lines), $\hat{\tau} = 5$ (dashed lines), and $\hat{\tau} = 10$ (dotted lines), for which it can be readily seen that all multipoles depicted display convergent behavior. We remark that in computing the zeroth ($\ell = 0$) and second ($\ell = 2$) multipoles, we have fixed the order of the polynomial in the denominator of the Padé approximant to be $s = 2$, whereas for the first ($\ell = 1$), we take $s = 4$, with the order of the corresponding numerator being computed accordingly in each case. This choice is made to avoid the occurrence of spurious poles – pseudo-singularities that do not exist in the original function – in the real plane, which ultimately lead to numerical errors.

Overall, we conclude that even though a Borel transform is not sufficient to render the moment expansion of the single-particle distribution function in Bjorken flow well-behaved, a convergent result can be achieved when such technique is further complemented by Padé approximants.

5.2.3 Reconstructing the moments from the resummation

In the previous subsection, the Borel-Padé resummation scheme was demonstrated to be appropriate to render the otherwise divergent moment expansion in Bjorken flow a convergent series. In this subsection, we investigate the consistency of this procedure and provide additional evidence that the aforementioned resummation scheme was correctly implemented.

From Eqs. (4.20) and (5.4), we can write the irreducible moments in terms of the

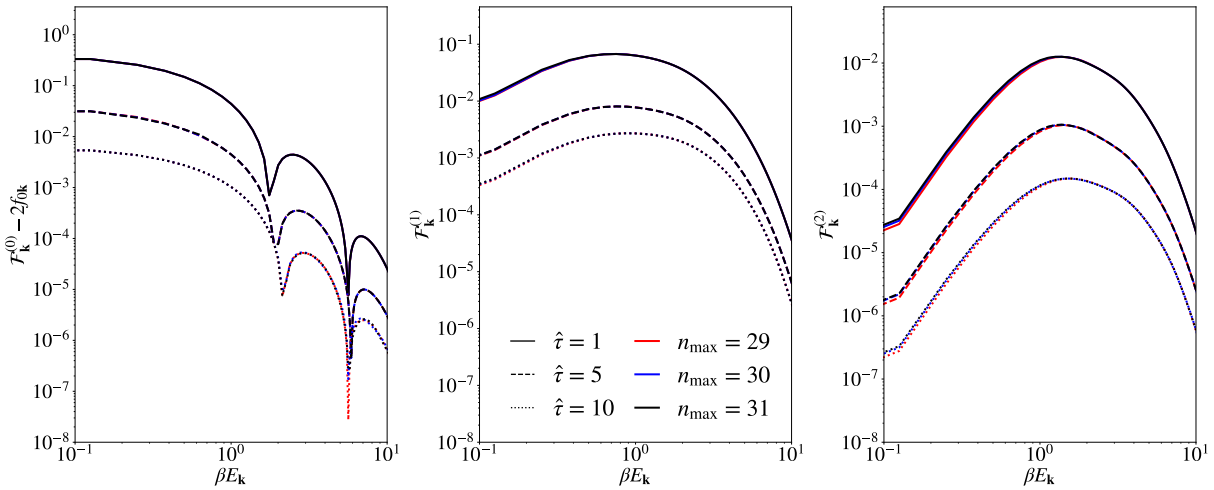

Figure 16 – Multipoles of the single-particle distribution function as functions of $\beta E_{\mathbf{k}}$ for different truncations, computed with a Borel-Padé resummation scheme, for $\hat{\tau} = 1, 5, 10$ considering $\hat{\tau}_0 = 0.5$.

multipoles of the single-particle distribution function as

$$\varrho_{n,\ell} = \frac{1}{(2\pi)^2} \int_0^\infty dk_0 \, k_0^{n+1} \mathcal{F}^{(\ell)}_{\mathbf{k}}. \tag{5.18}$$

Thus, the *rescaled* moments (4.27) are given by

$$\chi_{n,\ell} = \frac{1}{2(n+1)!\, e^\alpha} \int_0^\infty d(\beta k_0) \, (\beta k_0)^{n+1} \mathcal{F}^{(\ell)}_{\mathbf{k}}. \tag{5.19}$$

In Chapter 4, we showed that the (rescaled) irreducible moments $\chi_{n,\ell}$ satisfy a set of differential equations (4.28), which couple the moments of different ranks, encoded in the parameter ℓ, but not different values of n – a consequence of the assumption of massless particles combined with the relaxation time approximation. In particular, provided a truncation in rank, i.e., a value of $\ell_{\max}$ for which $\chi_{n,\ell} = 0$ for $\ell > \ell_{\max}$, we can systematically calculate the dynamics of *any* irreducible moment using the appropriate numerical scheme, even though the single-particle distribution function is completely unknown. We further showed that these solutions converge as the chosen truncation is increased. In practice, we choose the truncation by including enough equations in the system until the solutions stop displaying appreciable changes.

In the present chapter, we showed how the irreducible moments can be employed in the reconstruction of the single-particle distribution function. However, even though the solutions for the moments themselves converge, the moment expansion is divergent and must be consistently resummed. In Fig. 16, the Borel-Padé resummation procedure was shown to yield results for the nonequilibrium multipoles of the single-particle distribution function that are marginally affected by increasing the truncation, thus corresponding to a well-behaved, convergent result. The goal of this subsection is to provide further scrutiny to this affirmation.

In Fig. 17, we compare the rescaled irreducible moments $\chi_{n,0}$, $\chi_{n,1}$ and $\chi_{n,2}$ as function of the rescaled proper time, for several values of n considering $\hat{\tau}_0 = 0.5$, calculated from

Eqs. (4.28) (colored lines) and from the direct integration of the multipoles resummed using a Borel-Padé algorithm, as given by Eq. (5.19) (circles) for $\hat{\tau} = 1, 2, 5, 10$. We remark that this provides a reliable consistency check for the resummation techniques implemented here since it confronts these results with those where the solutions have been extensively studied and are well in control [75–79]. It can readily be seen that the integration of the resummed multipoles yields results that are in fairly good agreement with the exact solutions obtained from the differential equations.

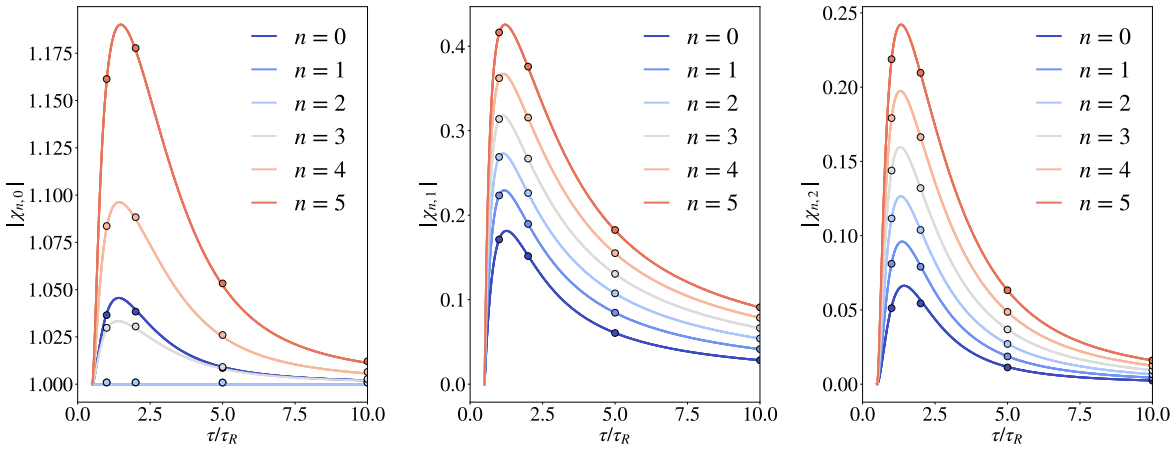

Figure 17 – Reconstruction of the rescaled moments from the resummation of the multipoles of the single-particle distribution function for $\hat{\tau} = 0.5$.

5.3 Nonequilibrium effects in conformal systems

So far, we have shown that, in order to obtain convergent results for the moment expansion of the single-particle distribution function for massless particles in Bjorken flow is necessary to employ a Borel-Padé resummation algorithm. However, in order to actually solve the Boltzmann equation, it is still necessary to reconstruct the single-particle distribution function from its *resummed* multipoles, cf. Eq. (5.3). This procedure involves summing multipoles at all ranks up to a given truncation. Before proceeding to this computation, it is crucial to understand how exactly different multipoles contribute to the moment expansion.

In this context, we first compare the exact solutions for the multipoles with traditional approximations that are expected to be valid in the fluid-dynamical limit: the first-order Chapman-Enskog solution [59] and the 14-moment approximation [42]. In particular, fluid dynamics implies a coarse-graining of the, in principle, infinite degrees of freedom contained in the moment expansion. In general, such solutions imply in the following expressions for the single-

particle distribution function

$$f_{\mathbf{k}}^{\text{14 moment}} = f_{0\mathbf{k}}\left[1 + \frac{1}{2T^2(\varepsilon_0 + P_0)}\pi^{\mu\nu}k_{\langle\mu}k_{\nu\rangle}\right], \tag{5.20}$$

$$f_{\mathbf{k}}^{\text{CE}} = f_{0\mathbf{k}}\left[1 + \tau_R\frac{\beta}{k_0}\sigma^{\mu\nu}k_{\langle\mu}k_{\nu\rangle}\right]. \tag{5.21}$$

In Bjorken flow, these truncations can be expressed as

$$f_{\mathbf{k}}^{\text{14 moment}} = f_{0\mathbf{k}}\left[1 + \frac{1}{4T^2}k_0^2P_2(\cos\Theta)\chi_{2,1}\right], \tag{5.22}$$

$$f_{\mathbf{k}}^{\text{CE}} = f_{0\mathbf{k}}\left[1 + \frac{2\tau_R}{3T\tau}k_0P_2(\cos\Theta)\right]. \tag{5.23}$$

Since both of these hydrodynamic solutions only contain the $\ell = 1$ multipole of the single-particle distribution function, we initially restrict our comparison to this case. In Fig. 18, we plot $\mathcal{F}_{\mathbf{k}}^{(1)}$ as a function of $\beta E_{\mathbf{k}}$ considering $\hat{\tau}_0 = 0.5$. We observe that the 14-moment approximation, traditionally employed in fluid-dynamical models of heavy-ion collisions, is not in good agreement with the exact solution for $\mathcal{F}_{\mathbf{k}}^{(1)}$. On the other hand, the first-order Chapman-Enskog solution is in better agreement with the exact solution, in particular at late times. In Fig. 16 we can also see that, at early times, the $\ell = 0$ and $\ell = 2$ multipole components, which should vanish in the fluid-dynamical limit, are comparable in magnitude to the $\ell = 1$ component. This further indicates a significant deviation of the solution of the Boltzmann equation from the fluid-dynamical limit. We have explicitly shown that the 14-moment approximation does not

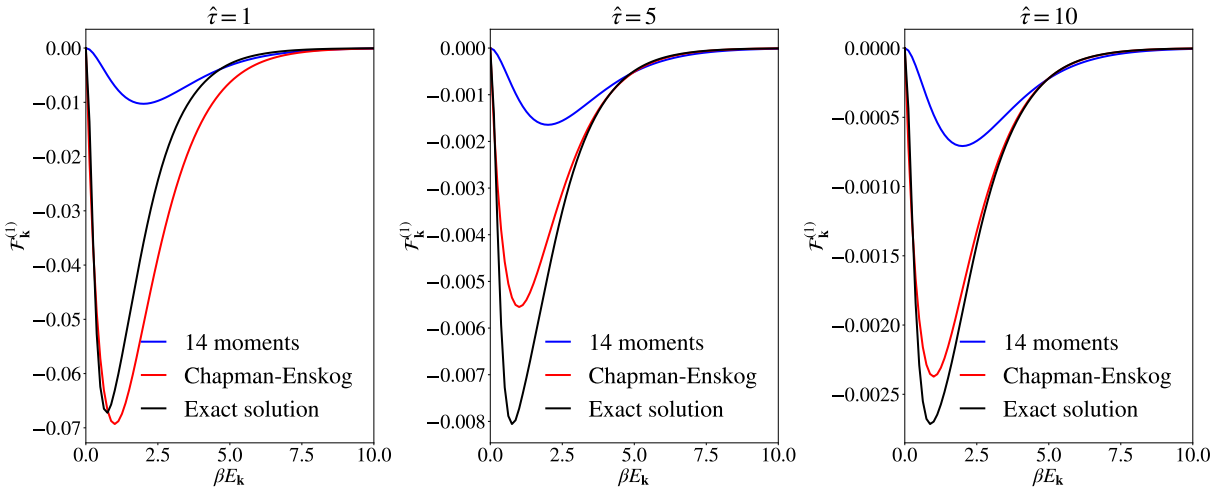

Figure 18 – Comparison of the exact and hydrodynamic solutions with the Chapman-Enskog limit of $\mathcal{F}_{\mathbf{k}}^{(1)}$ as functions of $\beta E_{\mathbf{k}}$ for $\hat{\tau} = 1, 5, 10$ considering $\hat{\tau}_0 = 0.5$.

provide a quantitatively accurate description of the exact solution for the first multipole of the single-particle distribution function.

For the sake of illustration, in Fig. 19 we compare the nonequilibrium terms of the first three (resummed) multipoles of the single-particle distribution function for different times, namely $\hat{\tau} = 0.05, 0.5, 1$, and considering $\hat{\tau}_0 = 0.5$. We observe that the scalar multipole

dominates at small values of $\beta E_{\mathbf{k}}$ and later becomes subleading as compared to the others. Furthermore, the first multipole dominates at $\beta E_{\mathbf{k}} \sim 1$ and remains being the dominant term of the expansion throughout the entire evolution in this range of energy values. In particular, this pattern remains considerably unchanged for the different times considered here.

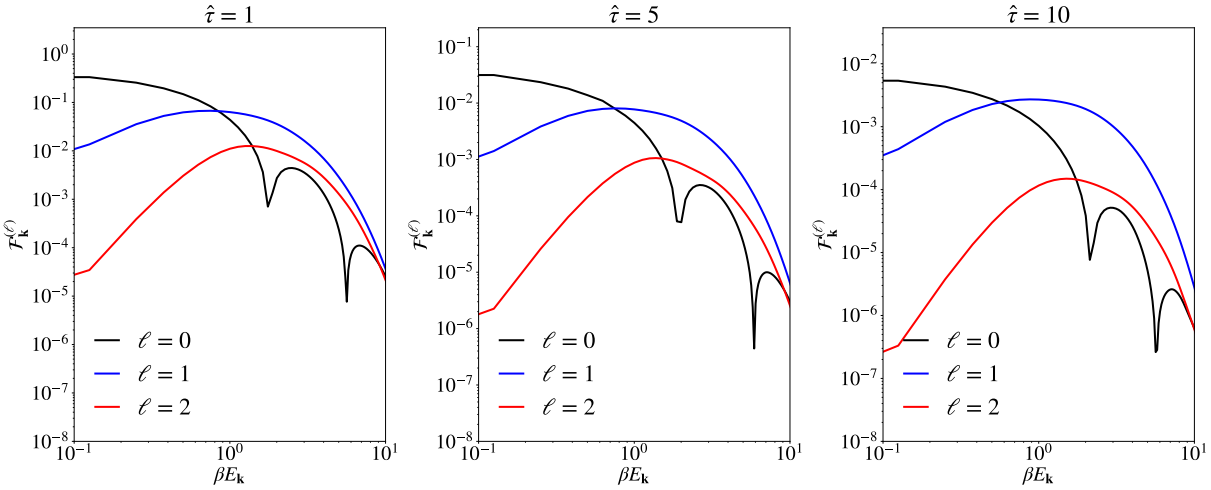

Figure 19 – Dominance of the first three resummed multipoles of $f_{\mathbf{k}}$ for $\hat{\tau} = 0.05, 0.5, 1$, considering $\hat{\tau}_0 = 0.5$.

A longitudinal expansion drives conformal systems into a nonequilibrium state whose evolution cannot be accurately captured by a fluid-dynamical description. In particular, we remark that, for a gas of massless particles in which the bulk viscosity is zero, the conserved currents N^{μ} and $T^{\mu\nu}$ do not depend on the scalar multipole $\mathcal{F}_{\mathbf{k}}^{(0)}$, and thus this term is referred to as a nonhydrodynamic current. We have shown that the scalar multipole dominates over all others at small values of energy over temperature, and shall provide non-negligible contributions to the reconstruction of the single-particle distribution function. Moreover, these nonhydrodynamic, nonequilibrium effects might have relevant implications in the calculations of different observables in heavy-ion collisions.

The energy-momentum tensor of a massless gas in Bjorken flow is given by $T^{\mu\nu} = \mathrm{diag}(\varepsilon, P + \pi/2, P + \pi/2, P - \pi)$, where the energy density is determined by Landau matching conditions. That is, the fluid-dynamical degrees of freedom reduce to the energy density ε and the scalar π associated with the shear-stress tensor, which are defined in terms of the irreducible moments respectively as $\varepsilon = \varrho_{2,0}$ and $\pi = -3\varrho_{2,1}/2$. In Fig. 20, we compare the fluid-dynamical (blue lines) and exact (red lines) solutions for the rescaled shear-stress scalar as function of the rescaled proper time, considering $\hat{\tau}_0 = 0.05, 0.5, 1$. The hydrodynamic solution is obtained by truncating the moment equations (4.28) at $\ell_{\mathrm{max}} = 1$, thus disregarding the dynamics of any degrees of freedom that do not appear in Eq. (5.20). In order to obtain the so-called exact solution, on the other hand, we impose a sufficiently large number of degrees of freedom in the hierarchy of moment equations, in this case taking $\ell_{\mathrm{max}} = 20$. In particular, further increasing this truncation barely affects the solution for π, such that this solution can be in fact regarded as

exact. We observe that the two solutions are in good agreement, especially for larger values of the initial rescaled proper time, that is, when considering less dissipative systems. Hence, the energy-momentum tensor of a longitudinally expanding conformal gas is fairly well described by fluid dynamics, even though such a system is not in an equilibrium state, as shown previously.

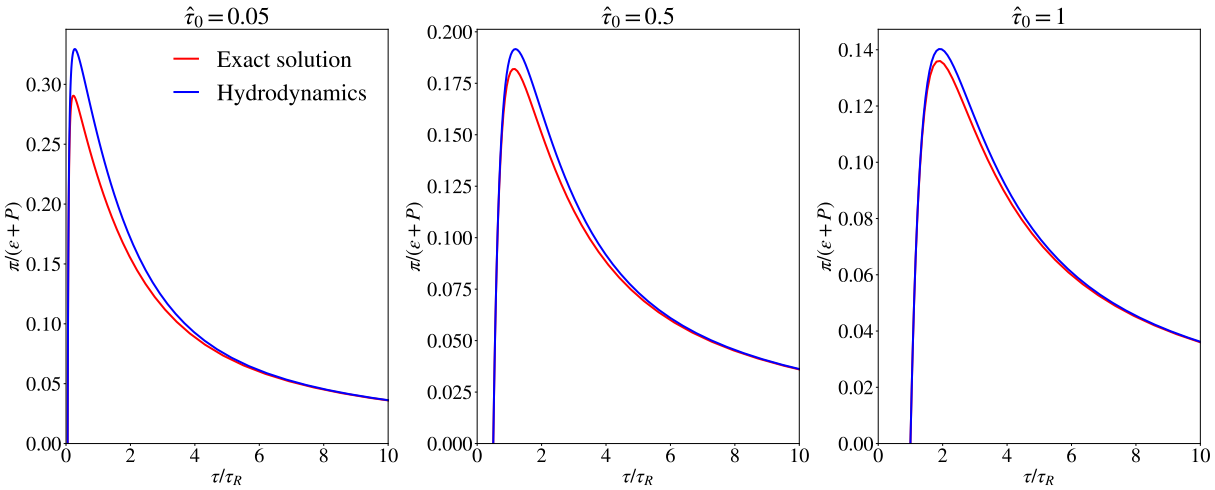

Figure 20 – Exact and hydrodynamic solutions for the rescaled shear-stress $\pi/(\varepsilon_0 + P_0)$ as function of the proper time for $\hat{\tau}_0 = 0.05, 0.5, 1$.

5.4 Hybrid solution of the Boltzmann equation

So far, this chapter has been dedicated to solve the Boltzmann equation in Bjorken flow by reconstructing the single-particle distribution function from its moments. We have shown that the moment expansion diverges and thus must be appropriately resummed, leading to physically meaningful results. Nevertheless, we remark that, in Bjorken flow, the Boltzmann equation assumes a rather simple form, as shown in Eq. (4.6), which we shall reproduce below for convenience

$$\partial_\tau f_{\mathbf{k}} + \frac{1}{\tau_R} f_{\mathbf{k}} = \frac{1}{\tau_R} f_{0\mathbf{k}}. \tag{5.24}$$

This ordinary differential equation admits *exact* analytical solutions, given by

$$f_{\mathbf{k}}(\tau) = f_{\mathbf{k}}(\tau_0) \exp\left[-\int_{\tau_0}^{\tau} \frac{1}{\tau_R(\tau')} d\tau'\right] + \int_{\tau_0}^{\tau} \exp\left[-\int_{\tau'}^{\tau} \frac{1}{\tau_R(\tau'')} d\tau''\right] \frac{1}{\tau_R(\tau')} f_{0\mathbf{k}}(\tau') d\tau', \tag{5.25}$$

with τ_0 being the initial time. In the above expression, the first term on the right-hand side corresponds to the homogeneous solution of Eq. (5.24), whereas the second term is the particular solution. In addition, we remark that

$$f_{0\mathbf{k}}(\tau') = \exp\left[\alpha(\tau') - \frac{k_0(\tau')}{T(\tau')}\right]. \tag{5.26}$$

We note that the energy of the particle at time τ' can be expressed in terms of the energy at a time τ in the following way,

$$k_0(\tau') = k_0(\tau)\sqrt{1+\cos^2\Theta\left(\frac{\tau^2}{\tau'^2}-1\right)}. \tag{5.27}$$

Therefore, we may express the single-particle distribution function as a function of $k_0(\tau)$ and $\cos\Theta(\tau)$,

$$f_{\mathbf{k}}(\tau,k_0,\cos\Theta) = f_{\mathbf{k}}(\tau_0)\exp\left[-\int_{\tau_0}^{\tau}\frac{1}{\tau_R(\tau')}d\tau'\right] + \int_{\tau_0}^{\tau}d\tau'\frac{1}{\tau_R(\tau')}\exp\left[-\int_{\tau'}^{\tau}\frac{1}{\tau_R(\tau'')}d\tau''\right] \times \exp\left[\alpha(\tau') - \frac{k_0}{T(\tau')}\sqrt{1+\cos^2\Theta\left(\frac{\tau^2}{\tau'^2}-1\right)}\right]. \tag{5.28}$$

In particular, for a constant relaxation time and assuming equilibrium initial conditions, this expression assumes a rather simpler form

$$f_{\mathbf{k}}(\tau,k_0(\tau),\cos\Theta) = \exp\left[\alpha(\tau_0) - \frac{k_0(\tau)}{T(\tau_0)}\sqrt{1+\cos^2\Theta\left(\frac{\tau^2}{\tau_0^2}-1\right)}\right]\exp\left[-\frac{\tau-\tau_0}{\tau_R}\right] + \frac{1}{\tau_R}\int_{\tau_0}^{\tau}d\tau'\exp\left[-\frac{\tau-\tau'}{\tau_R}\right]\exp\left[\alpha(\tau') - \frac{k_0(\tau)}{T(\tau')}\sqrt{1+\cos^2\Theta\left(\frac{\tau^2}{\tau'^2}-1\right)}\right]. \tag{5.29}$$

In order to obtain an exact solution for the single-particle distribution function, it is still necessary to input expressions for the thermal potential and temperature. Here, these quantities are determined from the hierarchy of moment equations (4.28) coupled with the dynamics of the aforementioned thermodynamical variables, Eqs. (4.30) and (4.31) – we shall refer to this as a hybrid solution. Assuming a constant relaxation time, this system of equations reduce simply to

$$\frac{d\chi_{n,\ell}}{d\tau} + \chi_{n,\ell} - \delta_{\ell 0} + \mathcal{P}_{n,\ell}\frac{\chi_{n,\ell-1}}{\tau} + \mathcal{Q}_{n,\ell}\frac{\chi_{n,\ell}}{\tau} + \mathcal{R}_{n,\ell}\frac{\chi_{n,\ell+1}}{\tau} - \frac{2(n-1)}{3\tau}\chi_{2,1}\chi_{n,\ell} = 0, \tag{5.30}$$

$$\frac{d\alpha}{d\tau} - \frac{2\chi_{2,1}}{\tau} = 0, \tag{5.31}$$

$$\frac{dT}{d\tau} + \frac{T}{3\tau}(1+2\chi_{2,1}) = 0, \tag{5.32}$$

where the coefficients $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ have been introduced in Eqs. (4.26).

These results are then plugged in Eq. (5.29), thus leading to a semi-analytical solution for the Boltzmann equation. In particular, in this approach, the single-particle distribution function is computed directly from the Boltzmann equation and does not rely on a reconstruction using its moments, thus not requiring any resummation techniques. We remark that the scope of such approach is strictly limited to the relaxation approximation in Bjorken flow, where the Boltzmann equation reduces to an ordinary differential equation that admits these rather simple solutions. Nevertheless, these solutions will be used as a validation for the results obtained from the resummation schemes thoroughly discussed in previous sections.

In Fig. 21, we compare the first three multipoles of the single-particle distribution function computed from the moment expansion with the Borel-Padé resummation algorithm and via the hybrid solution, Eq. (5.29), considering $\hat{\tau}_0 = 0.5$ and $\hat{\tau} = 2$, for a constant relaxation time, $\tau_R = 1$. We observe that both solutions display a significant agreement, especially for larger values of energy. This leads us to conclude that the resummation techniques applied for the moment expansion in this chapter are indeed correct and consistent with the semi-analytical solution of the Boltzmann equation.

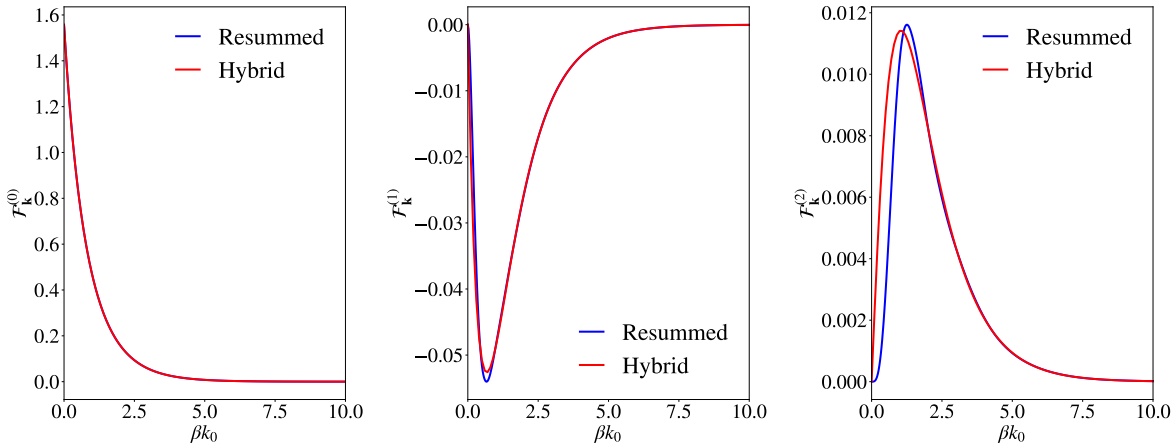

Figure 21 – Multipoles of the single-particle distribution function calculated from the hybrid solution (red lines) and Borel-Padé resummation scheme (blue lines), considering $\hat{\tau}_0 = 0.5$ and $\hat{\tau} = 2$, for a constant relaxation time, $\tau_R = 1$.

5.5 Discussion

In this Chapter we have shown that, in contrast to the usual assumption made in the field, the moment expansion diverges. We then demonstrated how it can be resummed, via a Borel-Padé scheme. We later compared our resummed expressions with hybrid solutions of the Boltzmann equation and verified that the resummation scheme worked well. Once the momentum distribution functions were reconstructed, we managed to demonstrate that, even though the energy-momentum tensor is well described by second-order hydrodynamics, the system is not close to local equilibrium. The traditional 14-momentum approximation fails dramatically in describing the momentum distribution of an ultrarelativistic gas undergoing Bjorken flow. This may have repercussions in heavy ion collisions, where the 14-moment approximation is often used to approximate the momentum distribution of partons at the early stages of the collision, in order to calculate the energy loss of heavy quarks [156] and thermal photon production [148,157].

6 Conclusions and outlook

In this thesis, we have thoroughly investigated the derivation of several different formulations of relativistic fluid dynamics. We started considering phenomenological derivations, in which a non-equilibrium entropy 4-current is constructed including all possible combinations involving the dissipative currents up to a given order. The fluid-dynamical equations are then obtained from the requirement of non-negativity of the entropy production. Eckart [36] and Landau [37], independently, were the first to develop relativistic theories of dissipative fluid dynamics. In their works, the non-equilibrium entropy 4-current involves terms up to first order in the dissipative currents and yield the so-called relativistic Navier-Stokes theory, thus being referred to as a first-order theory. In particular, the Navier-Stokes equations describe the dissipative currents as gradients of the traditional fluid-dynamical variables, namely 4-velocity, temperature and chemical potential.

However, even though its derivation is grounded entirely on conservation laws and on the second law of thermodynamics, Navier-Stokes theory suffers from mathematical pathologies that forbid it to be employed for practical purposes. As a matter of fact, even the smallest gradients of hydrodynamic variables yield an instant response from the system in the form of dissipative currents – that is, signals travel with infinite speed, thus violating the causality principle. In particular, the acausal behavior of the Navier-Stokes equations, which can be traced to its parabolic character, leads to instabilities in the system [38]. Therefore, Navier-Stokes theory is unsuitable to describe realistic fluids and one must resort to alternative descriptions.

In order to obtain a fluid-dynamical theory that does not display the aforementioned unphysical features of Navier-Stokes theory, Israel and Stewart proposed an extension of Eckart and Landau's work [40]. For this purpose, they considered a non-equilibrium entropy 4-current that also includes all possible terms of second order in the dissipative currents, thus leading to a second-order theory. Analogously, the Israel-Stewart equations are then obtained requiring the entropy production is non-negative. Unlike in Navier-Stokes theory, Israel-Stewart equations describe the dissipative currents as independent degrees of freedom, satisfying their own relaxation equations, which are coupled to the conservation laws. In fact, the inclusion of second-order terms in the entropy 4-current naturally lead to the occurrence of relaxation times, which are essential for the theory's causality. In this case, gradients of hydrodynamic variables do not generate an instant response from the system, but it rather takes a finite time – given by the aforementioned relaxation times – for the dissipative to appear. In particular, when these timescales are set to zero, one immediately recovers Navier-Stokes theory.

There are two fundamental requirements a fluid-dynamical theory must satisfy in order to be applicable to describe realistic systems, namely linear causality and stability. The first

dictates that perturbations in a system must propagate subluminally, whereas the second imposes that such perturbations must be exponentially damped with time. It has been extensively shown that these properties are intrinsically connected, i.e., one cannot exist without the other. As a matter of fact, linear causality implies linear stability [158], as well as causality is guaranteed by thermodynamic stability [159, 160]. Nevertheless, these properties are not necessarily satisfied and it is therefore crucial to verify whether this is the case. In order to investigate the linear causality and stability of a theory, we consider a fluid initially in a global equilibrium state and then perform *small* perturbations around such a state. The fluid-dynamical equations are then linearized, with terms of second order (or higher) in perturbations being identically disregarded. A linear causality and stability analysis then consists in computing the dispersion relations and constraining the transport coefficients of the theory in order for the modes to satisfy the aforementioned properties. We have performed this task for different formulations, namely Navier-Stokes, Israel-Stewart and third-order fluid dynamics. We have shown that Navier-Stokes theory is linearly acausal and unstable for perturbations on a moving fluid, whereas the other formulations can be linearly causal and stable as long as the transport coefficients satisfy certain constraints.

Phenomenological approaches from the second law of thermodynamics are by no means the unique form to derive fluid-dynamical theories. In fact, we also focused in understanding how fluid-dynamical formulations emerge from an underlying microscopic theory, in particular in the context of the Boltzmann equation. The Boltzmann equation describes the dynamics of the momentum distribution in a dilute gas (encoded in the single-particle distribution function) as a consequence of binary collisions between the particles. We have carried a careful and thorough derivation of the Boltzmann equation from collisional arguments. In particular, the so-called collision term contains rather complicated momentum integrals of the cross sections involved in the interactions. In fact, analytical computations of the collision term are only known for a handful of cases [143]. Therefore, obtaining exact solutions for the Boltzmann equation is often a fairly convoluted task and thus it is often necessary to resort to approximate methods.

The most widespread approach to solve the Boltzmann equation is the Chapman-Enskog method [61]. It consists in expanding the single-particle distribution function in powers of a book-keeping parameter, ϵ, that corresponds to powers of gradients of fluid-dynamical fields. The Boltzmann equation is then solved order by order in such parameter, which is later set to unity and one obtains a solution for the original problem. Therefore, in this approach, the Boltzmann equation is replaced by a perturbative problem. A truncation of the Chapman-Enskog expansion in zeroth order in ϵ leads to the usual Euler equations of ideal fluid dynamics. On the other hand, a first order truncation yields the Navier-Stokes equations. Furthermore, truncations in second order and beyond lead to Burnett and super-Burnett equations, which are unstable also in the nonrelativistic regime. Therefore, we conclude that the Chapman-Enskog method leads to fluid-dynamical theories that cannot be employed for practical applications, and thus one must resort to different approaches to pursue a microscopic derivation of fluid dynamics from the

Boltzmann equation.

An alternative approach to solve the Boltzmann equation was introduced by Grad [63] in the context of nonrelativistic gases. The so-called method of moments consists in expanding the single-particle distribution function in terms of its moments using a basis of generalized Hermite polynomials [41]. The problem of solving an integro-differential equation for the distribution function is then converted into solving a set of coupled partial differential equations for its moments. The first relativistic generalization of Grad's work was put forward by Israel and Stewart [42], introducing an expansion of the distribution function in a basis of 4-momenta, $1, k^\mu, k^\mu k^\nu, \cdots$. Since this is not an orthogonal basis, obtaining the expansion coefficients is no trivial task. In this context, in Ref. [2], the authors proposed an expansion using a basis of *irreducible* momenta $1, k^{\langle\mu\rangle}, k^{\langle\mu}k^{\nu\rangle}, \cdots$. Unlike the basis used by Israel and Stewart, the irreducible momenta form a complete and orthogonal basis [60], and thus the expansion coefficients can be exactly obtained. Then, the single-particle distribution function can be reconstructed from its moments.

The irreducible moments of the single-particle distribution function satisfy an infinite hierarchy of coupled partial differential equations, with moments of a given rank always being coupled with those of higher and lower ranks. In order to obtain a fluid-dynamical picture of the system, it is then necessary to truncate the moment expansion of the distribution function, hence describing the system's dynamics using a reduced – and, in particular, *finite* – number of degrees of freedom. In traditional formulations of transient fluid dynamics, such a coarse-graining procedure is performed by expressing the distribution function solely in terms of the 14 independent fields that appear in the conserved currents. This implies in including in the hierarchy of moment equations only moments of rank 0, 1 and 2. This leads to fluid-dynamical equations that are up to second order in gradients, thus being regarded as second-order theories. Overall, the method of moments is a truncation in degrees of freedom rather than in a small parameter, as is the case of the Chapman-Enskog expansion.

Notwithstanding, one can go further and also include additional non-hydrodynamic degrees of freedom – i.e., that do not feature the conserved currents – in the moment expansion. We have shown that, in order to obtain a third-order theory of fluid dynamics, it is necessary to account for the dynamics of moments of rank 3 and 4 [72], which leads to a total of 30 independent degrees of freedom. In particular, the third-order terms appear in the equations of motion as corrections to the traditional Israel-Stewart-like theories – setting these contributions to zero, one immediately recovers the aforementioned traditional theories. The causality and stability of this theory have been carefully analyzed, and we conclude that the third-order currents must satisfy relaxation equations themselves. We remark that the linear causality and stability conditions derived for this formulation encompass those obtained for Israel-Stewart theory [50].

Obtaining analytical solutions the hierarchy of coupled partial differential equations for the irreducible moments is, in general, not possible. Numerical solutions involve complex

computations and thus are also rather convoluted. We thus investigate the solutions for the third-order theory within a highly symmetric flow configuration, called Bjorken flow [103]. In this flow profile, the system is assumed to be invariant under boost along the longitudinal direction, as well as homogeneous and isotropic in the transverse plane. This dramatically simplifies the fluid-dynamical equations, since all quantities depend solely on the proper time. Furthermore, all dynamics of tensors are now contained in scalar variables, so the otherwise convoluted differential equations for tensor variables reduce to simple ordinary differential equations for scalars, which admit semi-analytical solutions. We then analyzed the novel third-order theory under these simplifications, and observed that it provides a good agreement with exact and numerical solutions for the Boltzmann equation.

In order to obtain *exact* solutions for the Boltzmann equation from the method of moments, one must include an infinite number of moments in the expansion. Naturally, this is unfeasible in practice and the moment expansion must be necessarily truncated at some point. In particular, this requires the computation of the equations of motion for moments lying far beyond the hydrodynamic truncation. However, until recently, these equations were known only up to rank 4 [2, 72]. In this context, we have derived a general differential equation of motion for irreducible moments of *arbitrary* rank [83], from which the previously known results can be straightforwardly recovered. These equations were then solved assuming Bjorken flow, and we observe that, as we include more moments in the hierarchy of equations, the solutions converge. That is, as the dimension of the set of coupled equations is increased, solutions become less sensitive to the inclusion of additional degrees of freedom. In particular, we observe that we obtain a fairly good agreement with numerical solutions of the Boltzmann equation even with a considerably small number of moments in the moment expansion.

The final step was to combine the dynamics of the irreducible moments to reconstruct the single-particle distribution function, thus actually solving the Boltzmann equation. So far, the moment expansion has always been assumed to converge, and, in fact, never investigated further beyond the hydrodynamic truncations. We have shown, considering Bjorken flow, that the moment expansion is, in fact, *divergent*, and this can only be seen when a sufficiently large number of terms in included in the moment expansion [84]. This divergence was shown both numerically and analytically, cf. Appendix F. In order to obtain physically meaningful results, it is thus necessary to employ resummation schemes. We then computed the moment expansion implementing a Borel-Padé resummation algorithm and obtained a well-behaved, convergent series. In particular, when comparing the exact resummed solutions for the multipoles with traditional fluid-dynamical approximations, we concluded that the 14-moment approximation does not provide a quantitatively accurate description of the exact solution for the first multipole of the single-particle distribution function. That is, the nonequilibrium dynamics of longitudinally expanding massless gases in Bjorken flow is not quantitatively well described by fluid dynamics. Furthermore, we observed that nonhydrodynamic effects, that arise from contributions that do not feature the conserved currents, are dominant for small values of energy and might impact the

computation of different observables in heavy-ion collisions, e.g. photon emission spectra [148] and transport of heavy quarks [156].

In Bjorken flow, the Boltzmann equation takes a rather simple form and admits exact analytical solutions. The single-particle distribution can be computed without relying on a reconstruction from its moments and thus does not rely on any resummation techniques. We then compute the temperature and thermal potential by solving the moment equations coupled with differential equations for the aforementioned thermodynamic variables, which are then used to obtain the distribution function – this approach is referred to as the hybrid solution. For the sake of simplicity, we consider a case of constant relaxation time. We observed that the results are in good agreement with those obtained from the moment expansion using a Borel-Padé resummation algorithm, thus serving as convincing validation of the method developed in this thesis.

The failure of the moment expansion as an approximation for the single-particle distribution may have severe consequences for the method of moments. These consequences were not explored in this thesis, but will be certainly developed in future works. The main challenge is how to re-derive the moment equations for an arbitrary interaction (not the relaxation time approximation used in this thesis) without resorting to a moment expansion. Or even how to insert the Borel-Padé resummation scheme developed here into the moment equations derived to solve the Boltzmann equation. This may have consequences to the theoretical description of certain observables in heavy ion collisions, which depend significantly on the 14-moment approximation. For instance the production of thermal photons [148] or the energy loss of heavy quarks [156].

APPENDIX A – Properties of the linearized collision operator

In this section, we investigate a few properties of the linearized collision operator, defined as

$$f_{0\mathbf{k}}\hat{L}[\phi_{\mathbf{k}}] = \frac{1}{2}\int dK'dPdP'W_{kk'\leftrightarrow pp'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\left(\phi_{\mathbf{p}}+\phi_{\mathbf{p}'}-\phi_{\mathbf{k}}-\phi_{\mathbf{k}'}\right). \tag{A.1}$$

We first show this is a self-adjoint operator, that is,

$$\int dK f_{0\mathbf{k}}\varphi_{\mathbf{k}}\hat{L}[\phi_{\mathbf{k}}] = \int dK f_{0\mathbf{k}}\phi_{\mathbf{k}}\hat{L}[\varphi_{\mathbf{k}}]. \tag{A.2}$$

Then,

$$\int dK f_{0\mathbf{k}}\varphi_{\mathbf{k}}\hat{L}[\phi_{\mathbf{k}}] = \frac{1}{2}\int dKdK'dPdP'W_{kk'\leftrightarrow pp'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\varphi_{\mathbf{k}}\left(\phi_{\mathbf{p}}+\phi_{\mathbf{p}'}-\phi_{\mathbf{k}}-\phi_{\mathbf{k}'}\right). \tag{A.3}$$

For the sake of illustration, let us work the first term inside parenthesis on the right-hand side of the above equation. Exchanging the incoming and outgoing momenta $k \leftrightarrow p$ and $k' \leftrightarrow p'$, we obtain a different form of the same integral,

$$\begin{aligned}
&\frac{1}{2}\int dKdK'dPdP'W_{kk'\leftrightarrow pp'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\varphi_{\mathbf{k}}\phi_{\mathbf{p}}\\
&=\frac{1}{2}\int dKdK'dPdP'W_{pp'\leftrightarrow kk'}f_{0\mathbf{p}}f_{0\mathbf{p}'}\tilde{f}_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}'}\varphi_{\mathbf{p}}\phi_{\mathbf{k}}\\
&=\frac{1}{2}\int dKdK'dPdP'W_{kk'\leftrightarrow pp'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\varphi_{\mathbf{p}}\phi_{\mathbf{k}},
\end{aligned} \tag{A.4}$$

where we have employed the detailed balance relation, Eq. (2.69), as well as the parity and time-reversal symmetry of the transition rate, $W_{kk'\leftrightarrow pp'} = W_{pp'\leftrightarrow kk'}$. An analogous reasoning can be employed to work the remaining terms, further leading to

$$\begin{aligned}
\int dK f_{0\mathbf{k}}\varphi_{\mathbf{k}}\hat{L}[\phi_{\mathbf{k}}] &= \frac{1}{2}\int dKdK'dPdP'W_{kk'\leftrightarrow pp'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}\phi_{\mathbf{k}}\left(\varphi_{\mathbf{p}}+\varphi_{\mathbf{p}'}-\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'}\right)\\
&= \int dK f_{0\mathbf{k}}\phi_{\mathbf{k}}\hat{L}[\varphi_{\mathbf{k}}],
\end{aligned} \tag{A.5}$$

thus completing the proof of the self-adjointness of the linearized collision operator.

APPENDIX B – Third-order transport coefficients

In the derivation of the equations of motion of the theory, in Sec. 3.3.2, we have introduced several transport coefficients.

First, in Eq. (3.45), we have defined

$$\lambda_{\pi\Omega} = -\left(\frac{\partial\gamma^{\Omega}_{-1}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma^{\Omega}_{-1}}{\partial\beta_0}\right), \tag{B.1}$$

$$\tau_{\pi\Omega} = \beta_0\frac{\partial\gamma^{\Omega}_{-1}}{\partial\beta_0}, \tag{B.2}$$

where $h_0 = (\varepsilon_0 + P_0)/n_0$. Then, in Eq. (3.46) we introduced

$$\tau_{\Omega} = t_R, \tag{B.3}$$

$$\delta_{\Omega\Omega} = -\frac{1}{3}\tau_{\Omega}\left(m^2\gamma^{\Omega}_{-2} + 5\right), \tag{B.4}$$

$$\ell_{\Omega n} = -\frac{6}{35}\tau_{\Omega}\left(m^4\gamma^{n}_{-2} + 5m^2 - 6\gamma^{n}_{2}\right), \tag{B.5}$$

$$\ell_{\Omega\Omega} = -\frac{1}{3}\tau_{\Omega}\left(2m^2\gamma^{\Omega}_{-2} + 7\right), \tag{B.6}$$

$$\eta_{\Omega} = -\tau_{\Omega}\left(m^2\gamma^{\pi}_{-1} - \gamma^{\pi}_{1}\right), \tag{B.7}$$

$$\lambda_{\Omega\pi} = -\frac{3}{7}\tau_{\Omega}\left[m^2\left(\frac{\partial\gamma^{\pi}_{-1}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma^{\pi}_{-1}}{\partial\beta_0}\right) - \left(\frac{\partial\gamma^{\pi}_{1}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma^{\pi}_{1}}{\partial\beta_0}\right)\right], \tag{B.8}$$

$$\lambda_{\Omega\Theta} = -\tau_{\Omega}\left(\frac{\partial\gamma^{\Theta}_{-1}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma^{\Theta}_{-1}}{\partial\beta_0}\right), \tag{B.9}$$

$$\tau_{\Omega\pi} = -\frac{3}{7}\tau_{\Omega}\frac{\beta_0}{\varepsilon_0 + P_0}\left(m^2\frac{\partial\gamma^{\pi}_{-1}}{\partial\beta_0} - \frac{\partial\gamma^{\pi}_{1}}{\partial\beta_0}\right), \tag{B.10}$$

$$\tau_{\Omega\Theta} = \tau_{\Omega}\beta_0\frac{\partial\gamma^{\Theta}_{-1}}{\partial\beta_0}, \tag{B.11}$$

Finally, in Eq. (3.47), we have

$$\tau_\Theta = t_R, \tag{B.12}$$

$$\delta_{\Theta\Theta} = -\frac{1}{3}\tau_\Theta\left(\gamma^\Theta_{-2}m^2+6\right), \tag{B.13}$$

$$\tau_{\Theta\Theta} = -\frac{4}{11}\tau_\Theta\left(2\gamma^\Theta_{-2}m^2+9\right), \tag{B.14}$$

$$\ell_{\Theta\Omega} = -\frac{4}{9}\tau_\Theta\left(m^2\gamma^\Omega_{-1}-\gamma^\Omega_1\right), \tag{B.15}$$

$$\ell_{\Theta\pi} = -\frac{4}{21}\tau_\Theta\left(\gamma^\pi_{-2}m^4+7m^2-8\gamma^\pi_2\right), \tag{B.16}$$

$$\lambda_{\Theta\Omega} = -\frac{4}{9}\tau_\Theta\left[m^2\left(\frac{\partial\gamma^\Omega_{-1}}{\partial\alpha_0}+h_0^{-1}\frac{\partial\gamma^\Omega_{-1}}{\partial\beta_0}\right)-\left(\frac{\partial\gamma^\Omega_1}{\partial\alpha_0}+h_0^{-1}\frac{\partial\gamma^\Omega_1}{\partial\beta_0}\right)\right], \tag{B.17}$$

$$\tau_{\Theta\Omega} = -4\tau_\Theta\left[\gamma^\Omega_1-\frac{\beta_0}{9}\left(m^2\frac{\partial\gamma^\Omega_{-1}}{\partial\alpha_0}-\frac{\partial\gamma^\Omega_1}{\partial\beta_0}\right)\right]. \tag{B.18}$$

The remaining transport coefficients that were not explicitly defined here can be found in Ref. [2].

In the massless and classical limits, the transport coefficients listed above reduce to

$$\begin{aligned}
&\lambda_{\pi\Omega} = -\frac{\beta_0}{28},\ \tau_{\pi\Omega} = \frac{\beta_0}{7},\ \delta_{\Omega\Omega} = -\frac{5}{3}\tau_\Omega,\ \ell_{\Omega n} = \frac{144\tau_\Omega}{7\beta_0^2},\ \tau_{\Omega\Omega} = -\frac{7}{3}\tau_\Omega,\ \eta_\Omega = \frac{6}{\beta_0}\tau_\Omega,\\
&\lambda_{\Omega\pi} = -\frac{9}{14}\frac{\tau_\Omega}{\beta_0}, \lambda_{\Omega\Theta} = -\frac{\beta_0\tau_\Omega}{36},\ \tau_{\Omega\pi} = -\frac{9\pi^2}{14}\tau_\Omega n_0,\ \tau_{\Omega\Theta} = \frac{\beta_0\tau_\Omega}{9},\ \delta_{\Theta\Theta} = -2\tau_\Theta,\\
&\ell_{\Theta\Theta} = -\frac{36}{11}\tau_\Theta,\ \ell_{\Theta\Omega} = \frac{32\tau_\Theta}{9\beta_0}, \ell_{\Theta\pi} = \frac{64\tau_\Theta}{\beta_0^2},\ \lambda_{\Theta\Omega} = -\frac{8}{9}\frac{\tau_\Theta}{\beta_0},\ \tau_{\Theta\Omega} = -\frac{256\tau_\Theta}{9\beta_0},
\end{aligned} \tag{B.19}$$

where we have used that $h_0 = 4/\beta_0$.

APPENDIX C – Denicol-Niemi-Molnár-Rischke (DNMR) approach

In the fluid-dynamical limit, the system can be described solely by the conserved currents N^μ and $T^{\mu\nu}$. In the context of kinetic theory, the applicability of fluid dynamics relies on the separation between microscopic and macroscopic scales. This is quantified by the Knudsen number,

$$\mathrm{Kn} = \frac{\lambda}{L} \ll 1, \tag{C.1}$$

where λ denotes a microscopic length scale (for gases, it is usually taken to the mean free path [60]), whereas L corresponds to a macroscopic scale – a typical scale over which the primary fluid-dynamical variables vary.

Furthermore, fluid dynamics is expected to be valid whenever the system is sufficiently close to a local equilibrium state, i.e., $\delta f_{\mathbf{k}} \ll f_{0\mathbf{k}}$. In this context, it is useful to introduce the inverse Reynolds number, defined as ratios between a given dissipative current and the associated equilibrium quantity with same dimension,

$$\mathrm{Re}_\Pi^{-1} = \frac{|\Pi|}{P_0}, \quad \mathrm{Re}_n^{-1} = \frac{|n^\mu|}{n_0}, \quad \mathrm{Re}_\pi^{-1} = \frac{|\pi^{\mu\nu}|}{P_0}. \tag{C.2}$$

Therefore, the Knudsen and inverse Reynolds numbers quantify the proximity of the system to the fluid-dynamical limit [58]. In this subsection, we shall outline the DNMR procedure to reduce the hierarchy of moment equations to a set of macroscopic equations neglecting terms of third order (or higher) in Knudsen and inverse Reynolds numbers.

The starting point is to rewrite the linearized generalized collision term from Eqs. (2.74) and (3.25), and use the moment expansion for $\phi_{\mathbf{k}}$, given in Eq. (3.13), leading to

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_m} + \text{higher order terms}, \tag{C.3}$$

where $\mathcal{A}_{rn}^{(\ell)}$ is the (rn) element of a $(N_\ell+1)\times(N_\ell+1)$ matrix $\mathcal{A}^{(\ell)}$ which contains all the information of the underlying microscopic theory [2], defined as (see Appendix A of Ref. [2] for details)

$$\begin{aligned}\mathcal{A}_{rn}^{(\ell)} = \frac{1}{2(2\ell+1)} \int dK dK' dP dP'\, W_{kk'\leftrightarrow pp'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} \\ \times \left(\mathcal{H}_{n\mathbf{k}}^{(\ell)} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} + \mathcal{H}_{n\mathbf{k}'}^{(\ell)} k'_{\langle\mu_1}\cdots k'_{\mu_\ell\rangle} - \mathcal{H}_{n\mathbf{p}}^{(\ell)} p_{\langle\mu_1}\cdots p_{\mu_\ell\rangle} - \mathcal{H}_{n\mathbf{p}'}^{(\ell)} p'_{\langle\mu_1}\cdots p'_{\mu_\ell\rangle}\right).\end{aligned} \tag{C.4}$$

The DNMR approach consists in diagonalizing the matrix $\mathcal{A}^{(\ell)}$ in order to find the eigenvectors that correspond to the smallest eigenvalues. These eigenvalues are referred to as the slowest eigenmodes. It is necessary to take the smallest eigenvalue of $\mathcal{A}^{(\ell)}$ since the relaxation times τ appearing in the moment equations are related to the inverse of $\mathcal{A}^{(\ell)}$. Therefore, the smallest eigenvalue yields the longest relaxation time [161]. The fluid-dynamical behavior is then expected to emerge when the microscopic degrees of freedom are integrated out and the system can be described entirely by the conserved currents. For the aforementioned diagonalization procedure, following Ref. [2], we introduce the matrix $\Omega^{(\ell)}$, which diagonalizes $\mathcal{A}^{(\ell)}$,

$$\left(\Omega^{-1}\right)^{(\ell)} \mathcal{A}^{(\ell)} \Omega^{(\ell)} = \text{diag}\left(\chi_0^{(\ell)}, \ldots, \chi_j^{(\ell)}, \ldots\right), \tag{C.5}$$

where $\chi_i^{(\ell)}$ are the eigenvalues of $\mathcal{A}^{(\ell)}$. Without loss of generality, we order these eigenvalues such that

$$\chi_i^{(\ell)} < \chi_{i+1}^{(\ell)}, \quad \text{for } i = 0, 1, 2, \cdots. \tag{C.6}$$

Moreover, we introduce the tensors

$$X_i^{\mu_1 \cdots \mu_\ell} = \sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} \rho_j^{\mu_1 \cdots \mu_\ell}, \tag{C.7}$$

which can be identified as the eigenmodes of the linearized Boltzmann equation.

Multiplying the linearized irreducible collision term (C.3) by $(\Omega^{-1})_{ij}^{(\ell)}$,

$$\sum_{j=0}^{N_\ell} \left(\Omega^{-1}\right)_{ij}^{(\ell)} C_{r-1}^{\langle \mu_1 \cdots \mu_\ell \rangle} = -\chi_i^{(\ell)} X_i^{\mu_1 \cdots \mu_\ell} + \text{higher order terms}, \tag{C.8}$$

where the repeated indices on the right-hand side of the above equations do not imply a sum.

The next step is to multiply the equations for the irreducible moments of ranks 0, 1 and 2 [Eqs. (3.16)–(3.18)] by $(\Omega^{-1})_{ir}^{(\ell)}$ and sum over r. Then, using Eqs. (C.7) and (C.8), the dominant terms can be identified as

$$\dot{X}_i + \chi_i^{(0)} X_i = \beta_i^{(0)} \theta + \cdots, \tag{C.9a}$$

$$\dot{X}_i^{\langle \mu \rangle} + \chi_i^{(1)} X_i^{\mu} = \beta_i^{(1)} \nabla^\mu \alpha + \cdots, \tag{C.9b}$$

$$\dot{X}_i^{\langle \mu\nu \rangle} + \chi_i^{(2)} X_i^{\mu\nu} = \beta_i^{(2)} \sigma^{\mu\nu} + \cdots, \tag{C.9c}$$

where we have defined

$$\beta_i^{(0)} = \sum_{\substack{j=0 \\ (\neq 1,2)}}^{N_0} \left(\Omega^{-1}\right)_{ij}^{(0)} \alpha_j^{(0)}, \tag{C.10a}$$

$$\beta_i^{(1)} = \sum_{\substack{j=0 \\ (\neq 1)}}^{N_1} \left(\Omega^{-1}\right)_{ij}^{(1)} \alpha_j^{(1)}, \tag{C.10b}$$

$$\beta_i^{(0)} = 2 \sum_{j=0}^{N_2} \left(\Omega^{-1}\right)_{ij}^{(2)} \alpha_j^{(2)}. \tag{C.10c}$$

The ellipsis in Eq. (C.9) denote nonlinear terms in $\delta f_{\mathbf{k}}$ stemming from Eq. (C.8), as well as to higher order terms appearing in the right-hand side of the moment equations (3.16)–(3.18), i.e., eigenmodes multiplied by gradients or gradients of eigenmodes. Furthermore, at sufficiently late times, the nonlinear terms in Eq. (C.9) are expected to be negligible, in which case the tensors $X_i^{\mu_1\cdots\mu_\ell}$ relax exponentially to the values on the right-hand side of Eqs. (C.9) divided by the corresponding $\chi_i^{(\ell)}$. These asymptotic solutions are hereon referred to as Navier-Stokes values.

In the limit of vanishing relaxation time, i.e., $\chi_i^{(\ell)} \to \infty$, while the ratio $\beta_i^{(\ell)}/\chi_i^{(\ell)}$ is kept fixed, all irreducible moments $\rho_n^{\mu_1\cdots\mu_\ell}$ become proportional to gradients of α_0, β_0 and u^μ. This corresponds to the Navier-Stokes equations,

$$X_{r\geq 0} \simeq \frac{\beta_r^{(0)}}{\chi_r^{(0)}}\theta + \cdots\,, \tag{C.11a}$$

$$X^\mu_{r\geq 0} \simeq \frac{\beta_r^{(1)}}{\chi_r^{(1)}}\nabla^\mu\alpha + \cdots\,, \tag{C.11b}$$

$$X^{\mu\nu}_{r\geq 0} \simeq \frac{\beta_r^{(2)}}{\chi_r^{(2)}}\sigma^{\mu\nu} + \cdots\,. \tag{C.11c}$$

We note that these terms are of first order in Knudsen number since θ, $\nabla^\mu\alpha$ and $\sigma^{\mu\nu}$ are all of order $\sim 1/L$, whereas $\chi_r^{(i)} \sim \lambda$.

At this point, we assume that only the slowest modes (those corresponding to the smallest eigenvalues and hence relaxing within a larger timescale) of each rank up to 2[1] remain in the transient and satisfy the following

$$\dot{X}_0 + \chi_0^{(0)} X_0 = \beta_0^{(0)}\theta + \cdots\,, \tag{C.12a}$$

$$\dot{X}_0^{\langle\mu\rangle} + \chi_0^{(1)} X_0^\mu = \beta_0^{(1)}\nabla^\mu\alpha + \cdots\,, \tag{C.12b}$$

$$\dot{X}_0^{\langle\mu\nu\rangle} + \chi_0^{(2)} X_0^{\mu\nu} = \beta_0^{(2)}\sigma^{\mu\nu} + \cdots\,, \tag{C.12c}$$

while all other modes X_r, X_r^μ and $X_r^{\mu\nu}$ for $r > 0$ are approximated by their asymptotic values (C.11).

In the moment equations (3.16)-(3.20), the dynamics of the usual dissipative currents, namely Π, n^μ and $\pi^{\mu\nu}$, are coupled to moments that do not appear in the conservation laws. Nonetheless, Eqs. (C.11) enable us to approximate such moments in terms of the dissipative currents up to a first order in Knudsen number. In this case, we obtain a closed set of coupled differential equations in terms solely of the traditional fluid-dynamical variables. It is convenient

[1] Moments of rank higher than 2 are at least of second order in Knudsen and inverse Reynolds numbers. In particular, since these terms appear in the moment equations either being differentiated or contracted with other moments, they provide contributions of

to rewrite Eqs. (C.11) as, for $r, n > 0$,

$$X_r \simeq \frac{\chi_n^{(0)}}{\chi_r^{(0)}} \frac{\beta_r^{(0)}}{\beta_n^{(0)}} X_n + \cdots , \tag{C.13a}$$

$$X_r^\mu \simeq \frac{\chi_n^{(1)}}{\chi_r^{(1)}} \frac{\beta_r^{(1)}}{\beta_n^{(1)}} X_n^\mu + \cdots , \tag{C.13b}$$

$$X_r^{\mu\nu} \simeq \frac{\chi_n^{(2)}}{\chi_r^{(2)}} \frac{\beta_r^{(2)}}{\beta_n^{(2)}} X_n^{\mu\nu} + \cdots . \tag{C.13c}$$

We then take $n = 3, 2, 1$ for the scalar, vector and tensor moments, respectively. This choice implies the inclusion of 3 scalar moments, 2 vector moments and 1 tensor moment. In this case, we obtain equations that involve solely the 14 quantities appearing in the conservation laws. Hence,

$$X_r \simeq \frac{\chi_3^{(0)}}{\chi_r^{(0)}} \frac{\beta_r^{(0)}}{\beta_3^{(0)}} X_3 + \cdots , \tag{C.14a}$$

$$X_r^\mu \simeq \frac{\chi_2^{(1)}}{\chi_r^{(1)}} \frac{\beta_r^{(1)}}{\beta_2^{(1)}} X_2^\mu + \cdots , \tag{C.14b}$$

$$X_r^{\mu\nu} \simeq \frac{\chi_1^{(2)}}{\chi_r^{(2)}} \frac{\beta_r^{(2)}}{\beta_1^{(2)}} X_1^{\mu\nu} + \cdots . \tag{C.14c}$$

Inverting Eq. (C.7),

$$\rho_n^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}. \tag{C.15}$$

Then, from Eq. (C.14), we obtain

$$\rho_i = \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta + \cdots = \Omega_{i0}^{(0)} X_0 + \mathcal{O}(\mathrm{Kn}), \tag{C.16a}$$

$$\rho_i^\mu = \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha + \cdots = \Omega_{i0}^{(1)} X_0^\mu + \mathcal{O}(\mathrm{Kn}), \tag{C.16b}$$

$$\rho_i^{\mu\nu} = \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} + \cdots = \Omega_{i0}^{(2)} X_0^{\mu\nu} + \mathcal{O}(\mathrm{Kn}). \tag{C.16c}$$

The primary fluid-dynamical currents are identified as moments of the single-particle distribution function, as depicted in Eq. (2.43). In particular, the dissipative currents are defined as

$$\rho_0 = -\frac{3}{m^2}\Pi, \quad \rho_0^\mu = n^\mu, \quad \rho_0^{\mu\nu} = \pi^{\mu\nu}. \tag{C.17}$$

Then, taking $r = 0$ in Eqs. (C.16) and $\Omega_{00}^{(\ell)} = 1$,

$$X_0 = -\frac{3}{m^2}\Pi - \frac{\chi_3^{(0)}}{\beta_3^{(0)}} \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} X_3, \tag{C.18a}$$

$$X_0^{\mu} = n^{\mu} - \frac{\chi_2^{(1)}}{\beta_2^{(1)}} \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} X_2^{\mu}, \tag{C.18b}$$

$$X_0^{\mu\nu} = \pi^{\mu\nu} - \frac{\chi_1^{(2)}}{\beta_1^{(2)}} \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} X_1^{\mu\nu}. \tag{C.18c}$$

Substituting Eqs. (C.18) into (C.16), we have

$$\frac{m^2}{3}\rho_i = -\Omega_{i0}^{(0)}\Pi + \frac{\chi_3^{(0)}}{\beta_3^{(0)}} \left(\frac{3}{m^2}\zeta_i - \Omega_{i0}^{(0)}\zeta_0 \right) X_3 = -\Omega_{i0}^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}), \tag{C.19a}$$

$$\rho_i^{\mu} = \Omega_{i0}^{(1)} n^{\mu} + \frac{\chi_2^{(1)}}{\beta_2^{(1)}} \left(\kappa_i - \Omega_{i0}^{(1)}\kappa_0 \right) X_2^{\mu} = \Omega_{i0}^{(1)} n^{\mu} + \mathcal{O}(\mathrm{Kn}), \tag{C.19b}$$

$$\rho_i^{\mu\nu} = \Omega_{i0}^{(2)} \pi^{\mu\nu} + \frac{\chi_1^{(2)}}{\beta_1^{(2)}} \left(\eta_i - \Omega_{i0}^{(2)}\eta_0 \right) X_1^{\mu\nu} = \Omega_{i0}^{(2)} \pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}). \tag{C.19c}$$

In deriving these equations, we have introduced the following transport coefficients

$$\zeta_i = \frac{m^2}{3} \sum_{\substack{j=0 \\ (\neq 1,2)}}^{N_0} \tau_{ij}^{(0)} \alpha_j^{(0)}, \quad \kappa_i = \sum_{\substack{j=0 \\ (\neq 1)}}^{N_1} \tau_{ij}^{(1)} \alpha_j^{(1)}, \quad \eta_i = \sum_{j=0}^{N_2} \tau_{ij}^{(2)} \alpha_j^{(2)}, \tag{C.20}$$

where

$$\tau_{ij}^{(\ell)} = \sum_{k=0}^{N_\ell} \Omega_{ik}^{(\ell)} \frac{1}{\chi_k^{(\ell)}} \left(\Omega^{-1} \right)_{kj}^{(\ell)}. \tag{C.21}$$

We remark that for $\ell = 0$, the terms $k = 1, 2$ must be excluded from the sum, whereas for $\ell = 1$, one must exclude the term $k = 1$.

Therefore, we have shown that, up to first order in Knudsen number, the irreducible moments of the nonequilibrium distribution function can be expressed in terms of the dissipative currents Π, n^{μ} and $\pi^{\mu\nu}$. This could only be achieved by diagonalizing the matrix $\mathcal{A}^{(\ell)}$ and then taking the slowest eigenmodes i.e., considering the moments that relax to their corresponding Navier-Stokes values within the largest timescales. This procedure enables us to reduce the number of variables of the hydrodynamic moment equations to the 14 quantities appearing in the conserved quantities. In the traditional 14-moment approximation [40], the hydrodynamic equations are obtained by truncating the moment expansion in such a way that only the primary fluid-dynamical fields contribute to the expansion of the single-particle distribution function. In the context of the DNMR approach, on the other hand, the moment expansion of the distribution function is truncated in orders of the Knudsen and inverse Reynolds number. This procedure requires a power counting scheme that shall be introduced in the following.

We note that Eqs. (C.19) for the irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$ are valid only for positive values of r. Nevertheless, we remark that moments with negative values of r appear in the moment equations, even though such terms do not appear in the moment expansion. Hence, it is also necessary to express those in terms of the dissipative currents, which can be performed using the completeness of the basis of irreducible momenta, in particular from Eqs. (3.6) and (3.13). Thus,

$$\rho_{-r}^{\mu_1\cdots\mu_\ell} = \sum_{n=0}^{N_\ell} \mathcal{F}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell}, \tag{C.22}$$

where we have defined the thermodynamic integrals

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{n\mathbf{k}}^{(\ell)} \left(\Delta_{\alpha\beta} k^\alpha k^\beta\right)^\ell . \tag{C.23}$$

It is then necessary to derive relations analogous to Eqs. (C.19) for the moments $\rho_{-r}^{\mu_1\cdots\mu_\ell}$. Multiplying Eq. (C.15) by $\mathcal{F}_{ri}^{(\ell)}$, Eq. (C.22) can be recast as

$$\rho_{-r}^{\mu_1\cdots\mu_\ell} = \sum_{i=0}^{N_\ell} \mathcal{F}_{ri}^{(\ell)} \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}. \tag{C.24}$$

We then have

$$\frac{m^2}{3}\rho_{-r} = -\gamma_r^{(0)}\Pi + \frac{\chi_3^{(0)}}{\beta_3^{(0)}} \sum_{\substack{i=0 \\ (\neq 1,2)}}^{N_0} \mathcal{F}_{ri}^{(0)} \left(\zeta_r - \Omega_{r0}^{(0)}\zeta_0\right) X_3 = -\gamma_r^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}), \tag{C.25a}$$

$$\rho_{-r}^{\mu} = \gamma_r^{(1)} n^\mu + \frac{\chi_2^{(1)}}{\beta_2^{(1)}} \sum_{\substack{i=0 \\ (\neq 1)}}^{N_1} \mathcal{F}_{ri}^{(1)} \left(\kappa_r - \Omega_{r0}^{(1)}\kappa_0\right) X_2^\mu = \gamma_r^{(1)} n^\mu + \mathcal{O}(\mathrm{Kn}), \tag{C.25b}$$

$$\rho_{-r}^{\mu\nu} = \gamma_r^{(2)} \pi^{\mu\nu} + \frac{\chi_1^{(2)}}{\beta_1^{(2)}} \sum_{i=0}^{N_2} \mathcal{F}_{ri}^{(1)} \left(\eta_r - \Omega_{r0}^{(2)}\eta_0\right) X_1^{\mu\nu} = \gamma_r^{(2)} \pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}). \tag{C.25c}$$

Furthermore, we remark that moments of rank higher than 3 are at least of second order in Knudsen number.

After having also expressed the irreducible moments with negative powers of energy in terms of the 14 hydrodynamic moments that appear in the moment expansion, we are finally able to close the system of moment equations (3.16)-(3.18), effectively describing the system's dynamics using solely the aforementioned degrees of freedom. For this purpose, we diagonalize the collision term by multiplying the moment equations by the corresponding $\tau_{nr}^{(\ell)}$, cf. Eq. (C.21), and summing over r. In particular, the generalized collision term (C.3) thus becomes

$$\sum_{r=0}^{N_\ell} \tau_{nr}^{(\ell)} C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\rho_n^{\mu_1\cdots\mu_m} + \text{higher order terms}. \tag{C.26}$$

Then, employing Eqs. (3.22) to replace time derivatives of the hydrodynamic variables α_0, β_0 and u^μ in terms of space derivatives and further using Eq. (C.21) in the following form,

$$\sum_{j=0}^{N_\ell} \tau_{ij}^{(\ell)} \Omega_{jm}^{(\ell)} = \Omega_{im}^{(\ell)} \frac{1}{\chi_m^{(\ell)}}. \tag{C.27}$$

the moment equations can be expressed as

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R}, \tag{C.28a}$$
$$\tau_n \dot{n}^{\langle\mu\rangle} + n^\mu = \kappa_n I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu, \tag{C.28b}$$
$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}, \tag{C.28c}$$

The tensors $\mathcal{J}$, $\mathcal{J}^\mu$ and $\mathcal{J}^{\mu\nu}$ collect all terms of $\mathcal{O}(\mathrm{Kn}\,\mathrm{Re}^{-1})$,

$$\mathcal{J} = -\ell_{\Pi n}\nabla\cdot n - \tau_{\Pi n} n\cdot F - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n} n\cdot I + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$
$$\mathcal{J}^\mu = -n_\nu\omega^{\nu\mu} - \delta_{nn} n^\mu\theta - \ell_{n\Pi}\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\nabla_\lambda\pi^\lambda_\nu + \tau_{n\Pi}\Pi F^\mu - \tau_{n\pi}\pi^{\mu\nu}F_\nu \tag{C.29a}$$
$$- \lambda_{nn} n_\nu\sigma^{\mu\nu} + \lambda_{n\Pi}\Pi I^\mu - \lambda_{n\pi}\pi^{\mu\nu}I_\nu, \tag{C.29b}$$
$$\mathcal{J}^{\mu\nu} = 2\pi^{\langle\mu}_\lambda\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu}F^{\nu\rangle}$$
$$+ \ell_{\pi n}\nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu} I^{\nu\rangle}, \tag{C.29c}$$

where we have defined $I^\mu = \nabla^\mu\alpha_0$ and $F^\mu = \nabla^\mu P_0$. Furthermore, the tensors $\mathcal{K}$, $\mathcal{K}^\mu$ and $\mathcal{K}^{\mu\nu}$ contain all terms of $\mathcal{O}(\mathrm{Kn}^2)$,

$$\mathcal{K} = \zeta_1\,\omega_{\mu\nu}\omega^{\mu\nu} + \zeta_2\,\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta_3\,\theta^2 + \zeta_4\,I\cdot I + \zeta_5\,F\cdot F + \zeta_6\,I\cdot F + \zeta_7\,\nabla\cdot I + \zeta_8\,\nabla\cdot F,$$
$$\mathcal{K}^\mu = \kappa_1\sigma^{\mu\nu}I_\nu + \kappa_2\sigma^{\mu\nu}F_\nu + \kappa_3 I^\mu\theta + \kappa_4 F^\mu\theta + \kappa_5\omega^{\mu\nu}I_\nu + \kappa_6\Delta^\mu_\lambda\partial_\nu\sigma^{\lambda\nu} + \kappa_7\nabla^\mu\theta, \tag{C.30a}$$
$$\mathcal{K}^{\mu\nu} = \eta_1\omega^{\langle\mu}_\lambda\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \eta_4\sigma^{\langle\mu}_\lambda\omega^{\nu\rangle\lambda} \tag{C.30b}$$
$$+ \eta_5 I^{\langle\mu} I^{\nu\rangle} + \eta_6 F^{\langle\mu} F^{\nu\rangle} + \eta_7 I^{\langle\mu} F^{\nu\rangle} + \eta_8\nabla^{\langle\mu} I^{\nu\rangle} + \eta_9\nabla^{\langle\mu} F^{\nu\rangle}. \tag{C.30c}$$

Last, the tensors $\mathcal{R}$, $\mathcal{R}^\mu$ and $\mathcal{R}^{\mu\nu}$ contain all terms of second order in inverse Reynolds number

$$\mathcal{R} = \varphi_1\Pi^2 + \varphi_2 n\cdot n + \varphi_3\pi_{\mu\nu}\pi^{\mu\nu}, \tag{C.31a}$$
$$\mathcal{R}^\mu = \varphi_4 n_\nu\pi^{\mu\nu} + \varphi_5\Pi n^\mu, \tag{C.31b}$$
$$\mathcal{R}^{\mu\nu} = \varphi_6\Pi\pi^{\mu\nu} + \varphi_7\pi^{\lambda\langle\mu}\pi^{\nu\rangle}_\lambda + \varphi_8 n^{\langle\mu} n^{\nu\rangle}. \tag{C.31c}$$

Equations (C.28) depend only on the 14 hydrodynamic variables that appear in the conserved currents and are commonly regarded as the DNMR equations of relativistic fluid dynamics. We remark that non-hydrodynamic degrees of freedom, associated with tensors of rank higher than two are at least of second order in Knudsen and inverse Reynolds numbers. In particular, since these terms appear in the moment equations (3.16)-(3.18) either multiplied by other irreducible moments or being differentiated, they yield third-order terms and are identically disregarded in this derivation. Therefore, the resulting DNMR equations are a set of second-order equations of fluid dynamics.

APPENDIX D – Derivation of the equation of motion for the irreducible moments

In this appendix, we outline the details of the derivation of the equation of motion for the irreducible moments of arbitrary rank, given in Eq. (4.4). The starting point is

$$\dot{\varrho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\left\langle\frac{dE_{\mathbf{k}}^r}{d\tau}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle + \left\langle E_{\mathbf{k}}^r\frac{d}{d\tau}\left(k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right)\right\rangle + \left\langle E_{\mathbf{k}}^r k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\frac{df_{\mathbf{k}}}{d\tau}\right\rangle\right). \tag{D.1}$$

where we have employed the notation introduced in Ref. [2], $\langle\cdots\rangle \equiv \int dK(\cdots)f_{\mathbf{k}}$. In the following, we compute each term in the right-hand side individually.

D.1 First term

The first term is

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\frac{dE_{\mathbf{k}}^r}{d\tau}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle, \tag{D.2}$$

where

$$\frac{dE_{\mathbf{k}}^r}{d\tau} = rE_{\mathbf{k}}^{r-1}k^{\langle\mu\rangle}\dot{u}_\mu, \tag{D.3}$$

since the 4-momentum does not depend on spacetime. Furthermore, it is possible to replace k^μ by its orthogonal projection with respect to the 4-velocity, $k^{\langle\mu\rangle}$, since the 4-acceleration of the fluid is orthogonal to its 4-velocity, $u_\mu\dot{u}^\mu = 0$. Therefore,

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\frac{dE_{\mathbf{k}}^r}{d\tau}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle = r\dot{u}_{\nu_{\ell+1}}\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E_{\mathbf{k}}^{r-1}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}\right\rangle. \tag{D.4}$$

In order to obtain a result in terms of irreducible moments, it is necessary to compute the irreducible decomposition of the term $k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}$. First, we write

$$k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle} = \Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}k^{\langle\alpha_1\rangle}\cdots k^{\langle\alpha_{\ell+1}\rangle}. \tag{D.5}$$

Since the 4-momenta are all contracted with projection operators introduced in Eq. (3.3), we have the freedom to replace all of them by their orthogonal projection with respect to the fluid 4-velocity. Then,

$$\begin{aligned}&\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}k^{\langle\alpha_1\rangle}\cdots k^{\langle\alpha_{\ell+1}\rangle}\\ &= \Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}\left(k^{\langle\alpha_1}\cdots k^{\alpha_{\ell+1}\rangle} - \frac{C(\ell+1,1)}{N_{\ell+1,1}}\sum_{\mathcal{P}^{\ell+1}_\alpha\mathcal{P}^{\ell+1}_\beta}\Delta^{\alpha_\ell\alpha_{\ell+1}}\Delta_{\beta_\ell\beta_{\ell+1}}\Delta^{\alpha_1}_{\beta_1}\cdots\Delta^{\alpha_{\ell-1}}_{\beta_{\ell-1}}k^{\beta_1}\cdots k^{\beta_{\ell+1}}\right).\end{aligned} \tag{D.6}$$

Here we make use of Eq. (3.3) to write the irreducible 4-momenta of rank $\ell+1$, where only the first two terms provide non-zero contributions. In fact, all terms involving two or more doubly contravariant (and the same number of doubly covariant) rank 2 projection operators lead to vanishing terms, since there is always a contraction of the type

$$\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_i\alpha_j}=0,\quad \text{with}\ \ 0\leq(i,j)\leq\ell, \tag{D.7}$$

which is zero, given the traceless property of the projection operator, see Eq. (3.5).

Next, it is necessary to compute the number of non-vanishing permutations accounted in the sums in Eq. (D.6). First, we look at the doubly contravariant projection operator. One of its indices must necessarily be $\alpha_{\ell+1}$, regardless of the other, otherwise leading to zero given Eq. (D.7). This index can be paired with ℓ other indices, which corresponds to a total of ℓ combinations. Second, the doubly covariant projection operators are contracted with the corresponding pair of 4-momenta, without restrictions to be imposed on these indices. In this case, the number of different combinations is simply a permutation of $\ell+1$ elements taken in pairs, i.e., $\frac{(\ell+1)!}{2!(\ell-1)!}$. Last, the $\ell-1$ remaining 4-momenta are contracted with projection operators with one covariant and one contravariant index. For a given a covariant (contravariant) index, there are $\ell-1$ possible contravariant (covariant) indices it can be paired with. Naturally, a different covariant (contravariant) index can now be paired with $\ell-2$ contravariant (covariant) indices, and so on. Therefore, the number of permutations for these projectors is simply $(\ell-1)!$, thus leading to

$$\begin{aligned}
k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle} &= k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle}-\ell\times\frac{(\ell+1)!}{2!(\ell-1)!}\times(\ell-1)!\times\frac{C(\ell+1,1)}{N_{\ell+1,1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\\
&\quad\times\sum_{\mathcal{P}^{\ell+1}_\alpha\mathcal{P}^{\ell+1}_\beta}\Delta^{\alpha_\ell\alpha_{\ell+1}}\Delta_{\beta_\ell\beta_{\ell+1}}\Delta^{\alpha_1}_{\beta_1}\cdots\Delta^{\alpha_{\ell-1}}_{\beta_{\ell-1}}k^{\beta_1}\cdots k^{\beta_{\ell+1}}\\
&= k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle}+\Delta_{\mathbf{kk}}\frac{\ell}{2\ell+1}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}k^{\langle\alpha_1\rangle}\cdots k^{\langle\alpha_{\ell-1}\rangle}\\
&= k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle}+\Delta_{\mathbf{kk}}\frac{\ell}{2\ell+1}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle},
\end{aligned} \tag{D.8}$$

which is exactly Eq. (4.5b) from the main text. Note that we are able to replace $k^{\langle\alpha_1\rangle}\cdots k^{\langle\alpha_{\ell-1}\rangle}$ by $k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle}$ in the last equality since this term is contracted with $\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}$. Finally, plugging this result in Eq. (D.4), we obtain

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\frac{dE^r_{\mathbf{k}}}{d\tau}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle=r\left[\varrho^{\mu_1\cdots\mu_{\ell+1}}_{r-1}\dot{u}_{\mu_{\ell+1}}+\frac{\ell}{2\ell+1}\left(m^2\varrho^{\langle\mu_1\cdots\mu_{\ell-1}}_{r-1}-\varrho^{\langle\mu_1\cdots\mu_{\ell-1}}_{r+1}\right)\dot{u}^{\mu_\ell\rangle}\right], \tag{D.9}$$

where we have used $\Delta_{\mathbf{kk}}=m^2-E^2_{\mathbf{k}}$.

D.2 Second term

The second term on the right-hand side of Eq. (D.1) is

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^r_{\mathbf{k}}\frac{d}{d\tau}\left(k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right)\right\rangle = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{d}{d\tau}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\left\langle E^r_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_\ell}\right\rangle, \tag{D.10}$$

The task here is to calculate the time derivative of the 2ℓ-index projection operator,

$$\begin{aligned}&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{d}{d\tau}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\\&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{d}{d\tau}\left(\frac{C(\ell,0)}{N_{\ell,0}}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_\ell}_{\alpha_\ell}+\frac{C(\ell,1)}{N_{\ell,1}}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1\nu_2}\Delta_{\alpha_1\alpha_2}\Delta^{\nu_3}_{\alpha_3}\cdots\Delta^{\nu_\ell}_{\alpha_\ell}+\cdots\right).\end{aligned} \tag{D.11}$$

It can readily be seen that, except for the first term, there will always be contractions either of the type given by Eq. (D.7), or

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}u^{\nu_i},\quad\text{with}\quad 0\leq i\leq\ell, \tag{D.12}$$

which are identically zero since the projection operator is traceless and orthogonal to the fluid 4-velocity by construction. Therefore, the only non-vanishing contribution comes from the first term. Analogously to the discussion developed in the previous subsection, the sums account for $\ell!$ possibilities to arrange the indices of the remaining projection operators. In particular, all permutations lead to the same result when contracted with $k^{\alpha_1}\cdots k^{\alpha_\ell}$, this yielding a factor ℓ. Furthermore, since the covariant derivative of the metric is zero, we have

$$\frac{d}{d\tau}\Delta^\mu_\nu = -\dot{u}^\mu u_\nu - u^\mu\dot{u}_\nu, \tag{D.13}$$

and thus

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{d}{d\tau}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right) = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{C(\ell,0)}{N_{\ell,0}}\frac{d}{d\tau}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_\ell}_{\alpha_\ell} = -\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\dot{u}^{\nu_\ell}u_{\alpha_\ell}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_{\ell-1}}_{\alpha_{\ell-1}}. \tag{D.14}$$

Using this result in Eq. (D.10), we obtain

$$\begin{aligned}\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^r_{\mathbf{k}}\frac{d}{d\tau}\left(k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right)\right\rangle &= -\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r+1}_{\mathbf{k}}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_\ell\rangle}\right\rangle\dot{u}^{\nu_{\ell-1}}\\&= -\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r+1}_{\mathbf{k}}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle\dot{u}^{\nu_{\ell-1}}\\&= -\ell\varrho^{\langle\mu_1\cdots\mu_{\ell-1}}_{r+1}\dot{u}^{\mu_\ell\rangle},\end{aligned} \tag{D.15}$$

where we have used Eq. (4.5a) to obtain the second equality, whose derivation is discussed in Appendix E.

D.3 Third term

The last term on the right-hand side of Eq. (D.1) is

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK E^r_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\frac{df_{\mathbf{k}}}{d\tau} \tag{D.16}$$

At this point, it is convenient to decompose the 4-momentum in terms of its longitudinal and transverse components with respect to the fluid 4-velocity as $k^\mu = u^\mu E_{\mathbf{k}} - k^{\langle\mu\rangle}$. Then, from the Boltzmann equation, Eq. (2.18), it is possible to express the time derivative of the single-particle distribution function as

$$\frac{df_{\mathbf{k}}}{d\tau} = E^{-1}_{\mathbf{k}} C[f] - E^{-1}_{\mathbf{k}} k^{\langle\mu\rangle}\nabla_\mu f_{\mathbf{k}}. \tag{D.17}$$

Therefore, Eq. (D.16) can be written as

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK E^r_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\frac{df_{\mathbf{k}}}{d\tau}\\
&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\int dK E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} C[f] - \int dK E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\mu_{\ell+1}\rangle}\nabla_{\mu_{\ell+1}} f_{\mathbf{k}}\right).
\end{aligned} \tag{D.18}$$

The first term on the right-hand side of this equation can be immediately identified as the generalized collision term, see Eq. (3.25),

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} C[f] = \mathcal{C}^{\langle\mu_1\cdots\mu_\ell\rangle}_{r-1}. \tag{D.19}$$

We now look at the last term term on the right-hand side of Eq. (D.18).

$$\begin{aligned}
&-\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\nu_{\ell+1}\rangle}\nabla_{\nu_{\ell+1}} f_{\mathbf{k}}\\
&= -\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\nu_{\ell+1}}\left\langle E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\nu_{\ell+1}\rangle}\right\rangle + \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\left(\nabla_{\nu_{\ell+1}} E^{r-1}_{\mathbf{k}}\right) k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\nu_{\ell+1}\rangle}\right\rangle\right.\\
&\left.+\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r-1}_{\mathbf{k}}\nabla_{\nu_{\ell+1}}\left(k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\nu_{\ell+1}\rangle}\right)\right\rangle\right).
\end{aligned} \tag{D.20}$$

In the following, we work each of these terms individually.

Part I

Using Eq. (D.8), the first term on the right-hand of Eq. (D.20) can be written as

$$\begin{aligned}
&-\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\left\langle E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} k^{\langle\nu_{\ell+1}\rangle}\right\rangle\\
&= -\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\left(\left\langle E^{r-1}_{\mathbf{k}} k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle}\right\rangle + \frac{\ell}{2\ell+1}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}\left\langle\Delta_{\mathbf{kk}} E^{r-1}_{\mathbf{k}} k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle}\right\rangle\right)\\
&= -\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\varrho^{\nu_1\cdots\nu_{\ell+1}}_{r-1} - \frac{\ell}{2\ell+1}\Delta^{\mu_1\cdots\mu_\ell}_{\alpha_1\cdots\alpha_\ell}\nabla^{\alpha_\ell}\left\langle\Delta_{\mathbf{kk}} E^{r-1}_{\mathbf{k}} k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle}\right\rangle\\
&= -\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\varrho^{\nu_1\cdots\nu_{\ell+1}}_{r-1} - \frac{\ell}{2\ell+1}\nabla^{\langle\mu_1}\left(m^2\varrho^{\mu_2\cdots\mu_\ell\rangle}_{r-1} - \varrho^{\mu_2\cdots\mu_\ell\rangle}_{r+1}\right),
\end{aligned} \tag{D.21}$$

where we have used $\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\nabla_{\nu_{\ell+1}}\Delta^{\alpha_\ell\nu_{\ell+1}} = 0$ and $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla^{\alpha_\ell}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right) = 0$, since every term carries contractions either of the type given by Eq. (D.7) or by Eq. (D.12).

Part II

We proceed to analyze the second term on the right-hand side of Eq. (D.20). First, we note that

$$\nabla_\mu E_{\mathbf{k}}^{r-1} = (r-1)E_{\mathbf{k}}^{r-2}k^{\langle\nu\rangle}\left(\nabla_\mu u_\nu\right). \tag{D.22}$$

Then, we use the relation

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\theta + \omega_{\mu\nu} \tag{D.23}$$

to write

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\left(\nabla_{\nu_{\ell+1}}E_{\mathbf{k}}^{r-1}\right)k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}\right\rangle\\
&= (r-1)\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\frac{\theta}{3}\left\langle b_{\mathbf{k}}E_{\mathbf{k}}^{r-2}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\right\rangle + \left\langle E_{\mathbf{k}}^{r-2}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}k^{\langle\nu_{\ell+2}\rangle}\right\rangle\sigma_{\nu_{\ell+1}\nu_{\ell+2}}\right)\\
&= \frac{r-1}{3}\left(m^2\varrho_{r-2}^{\mu_1\cdots\mu_\ell} - \varrho_r^{\mu_1\cdots\mu_\ell}\right)\theta + (r-1)\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E_{\mathbf{k}}^{r-2}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}k^{\langle\nu_{\ell+2}\rangle}\right\rangle\sigma_{\nu_{\ell+1}\nu_{\ell+2}}.
\end{aligned} \tag{D.24}$$

The vorticity is contracted with a symmetric tensor, thus leading to zero. In order to express the second term on the right-hand side of Eq. (D.24) in terms of irreducible moments, we successively use Eq. (D.8),

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E_{\mathbf{k}}^{r-2}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}k^{\langle\nu_{\ell+2}\rangle}\right\rangle\sigma_{\nu_{\ell+1}\nu_{\ell+2}}\\
&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E_{\mathbf{k}}^{r-2}\left(k^{\langle\nu_1}\cdots k^{\nu_{\ell+1}\rangle}k^{\langle\nu_{\ell+2}\rangle} + \frac{\ell}{2\ell+1}\Delta_{\mathbf{kk}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}\rangle}k^{\langle\nu_{\ell+2}\rangle}\right)\right\rangle\\
&\times\sigma_{\nu_{\ell+1}\nu_{\ell+2}}\\
&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\sigma_{\nu_{\ell+1}\nu_{\ell+2}}\left\langle E_{\mathbf{k}}^{r-2}\left[k^{\langle\nu_1}\cdots k^{\nu_{\ell+2}\rangle} + \frac{\ell+1}{2\ell+3}\Delta_{\mathbf{kk}}\Delta^{\nu_1\cdots\nu_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}\Delta^{\beta_{\ell+1}\nu_{\ell+2}}k^{\langle\beta_1}\cdots k^{\beta_\ell\rangle}\right.\right.\right.\\
&\left.\left.\left.\frac{\ell}{2\ell+1}\Delta_{\mathbf{kk}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\alpha_\ell\nu_{\ell+1}}\left(k^{\langle\alpha_1}\cdots k^{\alpha_{\ell-1}}k^{\nu_{\ell+2}\rangle} + \frac{\ell-1}{2\ell-1}\Delta_{\mathbf{kk}}\Delta^{\alpha_1\cdots\alpha_{\ell-1}}_{\lambda_1\cdots\lambda_{\ell-1}}\Delta^{\lambda_{\ell-1}\nu_{\ell+2}}k^{\langle\lambda_1}\cdots k^{\lambda_{\ell-2}\rangle}\right)\right]\right\rangle\\
&= \varrho_{r-2}^{\mu_1\cdots\mu_{\ell+2}}\sigma_{\mu_{\ell+1}\mu_{\ell+2}} + \frac{\ell(\ell-1)}{4\ell^2-1}\left(m^4\varrho_{r-2}^{\langle\mu_1\cdots\mu_{\ell-2}} - 2m^2\varrho_r^{\langle\mu_1\cdots\mu_{\ell-2}} + \varrho_{r+2}^{\langle\mu_1\cdots\mu_{\ell-2}}\right)\sigma^{\mu_{\ell-1}\mu_\ell\rangle}\\
&+ \frac{\ell+1}{2\ell+3}\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1\cdots\nu_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}\left(m^2\varrho_{r-2}^{\beta_1\cdots\beta_\ell} - \varrho_r^{\beta_1\cdots\beta_\ell}\right)\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}}\\
&+ \frac{\ell}{2\ell+1}\left(m^2\varrho_{r-2}^{\alpha\langle\mu_1\cdots\mu_{\ell-1}} - \varrho_r^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}\right)\sigma_\alpha^{\mu_\ell\rangle}.
\end{aligned} \tag{D.25}$$

Using Eq. (3.3), it is possible to write the term $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1\cdots\nu_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}}$ as

$$
\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1\cdots\nu_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}} \\
&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left[\frac{C(\ell+1,0)}{\mathcal{N}_{\ell+1,0}}\sum_{\mathcal{P}^{\ell+1}_\nu\mathcal{P}^{\ell+1}_\beta}\Delta^{\nu_1}_{\beta_1}\cdots\Delta^{\nu_{\ell+1}}_{\beta_{\ell+1}}\right. \\
&\left.+\frac{C(\ell+1,1)}{\mathcal{N}_{\ell+1,1}}\sum_{\mathcal{P}^{\ell+1}_\nu\mathcal{P}^{\ell+1}_\beta}\Delta^{\nu_\ell\nu_{\ell+1}}\Delta_{\beta_\ell\beta_{\ell+1}}\Delta^{\nu_1}_{\beta_1}\cdots\Delta^{\nu_{\ell-1}}_{\beta_{\ell-1}}+\cdots\right]\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}},
\end{aligned}
\tag{D.26}
$$

where the ellipsis denote all terms containing at least two covariant (and contravariant) projection operators, which are identically zero since they involve contractions similar to Eq. (D.7).

We analyze each term inside square brackets separately, starting with the first. All permutations containing the projector $\Delta^{\nu_{\ell+1}}_{\beta_{\ell+1}}$ are identically zero, since the shear tensor is traceless and orthogonal to the 4-velocity, and thus $\Delta^{\nu_{\ell+1}}_{\beta_{\ell+1}}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}} = 0$. Therefore, in order to account exclusively the non-vanishing terms, we calculate the number of total permutations and exclude those that contain the aforementioned operator. Hence, the number of contributing terms in the sum is simply

$$
(\ell+1)! - \ell! = \ell\ell!. \tag{D.27}
$$

Then, the first term inside square brackets reduces to

$$
\begin{aligned}
&\frac{C(\ell+1,0)}{\mathcal{N}_{\ell+1,0}}\sum_{\mathcal{P}^{\ell+1}_\nu\mathcal{P}^{\ell+1}_\beta}\Delta^{\nu_1}_{\beta_1}\cdots\Delta^{\nu_{\ell+1}}_{\beta_{\ell+1}}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}} \\
&= \frac{C(\ell+1,0)}{\mathcal{N}_{\ell+1,0}}\times\ell\ell!\times\Delta^{\nu_1}_{\beta_{\ell+1}}\Delta^{\nu_2}_{\beta_2}\cdots\Delta^{\nu_\ell}_{\beta_\ell}\Delta^{\nu_{\ell+1}}_{\beta_1}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}} \\
&= \frac{\ell}{\ell+1}\Delta^{\nu_1}_{\beta_{\ell+1}}\Delta^{\nu_2}_{\beta_2}\cdots\Delta^{\nu_\ell}_{\beta_\ell}\Delta^{\nu_{\ell+1}}_{\beta_1}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}} \\
&= \frac{\ell}{\ell+1}\varrho^{\alpha\nu_1\cdots\nu_{\ell-1}}\sigma^{\nu_\ell}_{\alpha}.
\end{aligned}
\tag{D.28}
$$

The next step is to compute the number of non-zero permutations in the second term in square brackets. First, we note that the doubly contravariant projection operator must be $\Delta^{\nu_i\nu_{\ell+1}}$, with $0 \leq i \leq \ell$, which accounts for ℓ different possibilities, otherwise leading to vanishing terms analogous to Eq. (D.7). Similarly, the doubly covariant projection operator can only be of the type $\Delta_{\beta_i\beta_{\ell+1}}$, once again corresponding to ℓ different combinations. In particular, since the indices $\beta_{\ell+1}$ and $\nu_{\ell+1}$ were already used in constructing these projectors, there are no restrictions to be imposed on the projectors with one covariant and one contravariant indices. Therefore, the first contravariant (covariant) index can be paired with any of the remaining $\ell-1$ covariant (contravariant) indices. The next can be paired with $\ell-2$ indices and so on. Wherefore, the

number of permutations in this case is $(\ell-1)\times(\ell-2)\times\cdots\times 1=(\ell-1)!$, and thus

$$\begin{aligned}
&\frac{C(\ell+1,1)}{\mathcal{N}_{\ell+1,1}}\sum_{\mathcal{P}_\nu^{\ell+1}\mathcal{P}_\beta^{\ell+1}}\Delta^{\nu_\ell\nu_{\ell+1}}\Delta_{\beta_\ell\beta_{\ell+1}}\Delta^{\nu_1}_{\beta_1}\cdots\Delta^{\nu_{\ell-1}}_{\beta_{\ell-1}}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}}\\
&=\frac{C(\ell+1,1)}{\mathcal{N}_{\ell+1,1}}\times\ell\times\ell\times(\ell-1)!\times\Delta^{\nu_\ell\nu_{\ell+1}}\Delta_{\beta_\ell\beta_{\ell+1}}\Delta^{\nu_1}_{\beta_1}\cdots\Delta^{\nu_{\ell-1}}_{\beta_{\ell-1}}\varrho^{\beta_1\cdots\beta_\ell}\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}}\\
&=-\frac{2\ell}{(\ell+1)(2\ell+1)}\varrho^{\alpha\nu_1\cdots\nu_{\ell-1}}\sigma^{\nu_\ell}_{\alpha}.
\end{aligned}\tag{D.29}$$

Then, from Eqs. (D.28) and (D.29), the last term on the right-hand side of Eq. (D.25) becomes

$$\begin{aligned}
&\frac{\ell+1}{2\ell+3}\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1\cdots\nu_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}\left(m^2\varrho^{\beta_1\cdots\beta_\ell}_{r-2}-\varrho^{\beta_1\cdots\beta_\ell}_{r}\right)\sigma^{\beta_{\ell+1}}_{\nu_{\ell+1}}\\
&=\frac{\ell(2\ell-1)}{(2\ell+1)(2\ell+3)}\left(m^2\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_{r-2}-\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_{r}\right)\sigma^{\mu_\ell\rangle}_{\alpha}.
\end{aligned}\tag{D.30}$$

Finally, from Eqs. (D.24), (D.25) and (D.30), the second term on the right-hand side of Eq. (D.20) is given by

$$\begin{aligned}
&(r-1)\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle\left(\nabla_{\nu_{\ell+1}}E^{r-1}_{\mathbf{k}}\right)k^{\langle\nu_1}\cdots k^{\nu_\ell}k^{\langle\nu_{\ell+1}\rangle}\right\rangle\\
&=(r-1)\varrho^{\mu_1\cdots\mu_{\ell+2}}_{r-2}\sigma_{\mu_{\ell+1}\mu_{\ell+2}}+\frac{r-1}{3}\left(m^2\varrho^{\mu_1\cdots\mu_\ell}_{r-2}-\varrho^{\mu_1\cdots\mu_\ell}_{r}\right)\theta\\
&+(r-1)\frac{2\ell}{2\ell+3}\left(m^2\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_{r-2}-\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_{r}\right)\sigma^{\mu_\ell\rangle}_{\alpha}\\
&+(r-1)\frac{\ell(\ell-1)}{4\ell^2-1}\left(m^4\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r-2}-2m^2\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r}+\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r+2}\right)\sigma^{\mu_{\ell-1}\mu_\ell\rangle}.
\end{aligned}\tag{D.31}$$

Part III

The final step is to compute the last term on the right-hand side of Eq. (D.20). We have

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r-1}_{\mathbf{k}}\nabla_{\nu_{\ell+1}}\left(k^{\langle\nu_1}\cdots k^{\nu_\ell}k^{\langle\nu_{\ell+1}\rangle}\right)\right\rangle=\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle,\tag{D.32}$$

where

$$\nabla_{\nu_{\ell+1}}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}\right)=\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}-\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}u_{\alpha_{\ell+1}}\theta.\tag{D.33}$$

Therefore, Eq. (D.32) becomes

$$\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\left(\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\Delta^{\nu_{\ell+1}}_{\alpha_{\ell+1}}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle\\
&=\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle-\Delta^{\mu_1\cdots\mu_\ell}_{\alpha_1\cdots\alpha_\ell}\left\langle E^{r}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_\ell}\right\rangle\theta\\
&=\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle-\varrho^{\mu_1\cdots\mu_\ell}_{r}\theta.
\end{aligned}\tag{D.34}$$

The next step is to compute the term $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)$,

$$\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)=\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\alpha_{\ell+1}}\left[\frac{C(\ell,0)}{\mathcal{N}_{\ell,0}}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_\ell}_{\alpha_\ell}+\cdots\right]. \tag{D.35}$$

Once again, the ellipsis denote all vanishing terms. In fact, terms containing at least one doubly covariant (and one corresponding doubly contravariant) projection operators are identically zero, since they always involve contractions analogous to Eqs. (D.7) and/or (D.12). Furthermore, given the symmetry of the projection operator being contracted, all terms yield the same result,

$$\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{C(\ell,0)}{\mathcal{N}_{\ell,0}}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_{\ell-1}}_{\alpha_{\ell-1}}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_\ell}_{\alpha_\ell}\right), \tag{D.36}$$

where $\nabla_{\alpha_{\ell+1}}\Delta^{\nu_i}_{\alpha_j}=-\left(u^{\nu_i}\nabla_{\alpha_{\ell+1}}u_{\alpha_j}+u_{\alpha_j}\nabla_{\alpha_{\ell+1}}u^{\nu_i}\right)$, with $\{i,j\}\in[1,\ldots,\ell]$. The $\ell-1$ remaining projection operators that are not being derived are contracted with $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$. We remark that this is a symmetric tensor under the exchange of its covariant and contravariant indices, and therefore all these terms are identical, hence the factor ℓ, which comes from the product rule. As it was already discussed, the number of different ways we can arrange the indices of the projectors $\Delta^{\nu_i}_{\alpha_j}$ is simply $\ell!$, thus leading to

$$\begin{aligned}&\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\frac{C(\ell,0)}{\mathcal{N}_{\ell,0}}\sum_{\mathcal{P}^\ell_\nu\mathcal{P}^\ell_\alpha}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_{\ell-1}}_{\alpha_{\ell-1}}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_\ell}_{\alpha_\ell}\right)\\&=-\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_{\ell-1}}_{\alpha_{\ell-1}}u_{\alpha_\ell}\left(\sigma^{\nu_\ell}_{\alpha_{\ell+1}}+\frac{1}{3}\Delta^{\nu_\ell}_{\alpha_{\ell+1}}\theta+\omega_{\alpha_{\ell+1}}{}^{\nu_\ell}\right),\end{aligned} \tag{D.37}$$

where we have used Eq. (D.23) to obtain the last equality. Using this result in Eq. (D.34),

$$\begin{aligned}&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle\\&=-\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\Delta^{\nu_1}_{\alpha_1}\cdots\Delta^{\nu_{\ell-1}}_{\alpha_{\ell-1}}u_{\alpha_\ell}\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle\left(\sigma^{\nu_\ell}_{\alpha_{\ell+1}}+\frac{1}{3}\Delta^{\nu_\ell}_{\alpha_{\ell+1}}\theta+\omega_{\alpha_{\ell+1}}{}^{\nu_\ell}\right)-\varrho^{\mu_1\cdots\mu_\ell}_r\theta\\&=-\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r}_{\mathbf{k}}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_{\ell-1}\rangle}k^{\langle\nu_{\ell+1}\rangle}\right\rangle\left(\sigma^{\nu_\ell}_{\nu_{\ell+1}}+\frac{1}{3}\Delta^{\nu_\ell}_{\nu_{\ell+1}}\theta+\omega_{\nu_{\ell+1}}{}^{\nu_\ell}\right)-\varrho^{\mu_1\cdots\mu_\ell}_r\theta\\&=-\left(1+\frac{\ell}{3}\right)\varrho^{\mu_1\cdots\mu_\ell}_r\theta-\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^{r}_{\mathbf{k}}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_{\ell-1}\rangle}k^{\langle\nu_{\ell+1}\rangle}\right\rangle\left(\sigma^{\nu_\ell}_{\nu_{\ell+1}}+\omega_{\nu_{\ell+1}}{}^{\nu_\ell}\right).\end{aligned} \tag{D.38}$$

The final step is to determine the second term on the right-hand side of Eq. (D.38). At this point, we make use of Eq. (4.5a) to write

$$\begin{aligned}&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_{\ell-1}\rangle}k^{\langle\nu_{\ell+1}\rangle}\\&=\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}k^{\langle\nu_1}\cdots k^{\nu_{\ell-1}}k^{\nu_{\ell+1}\rangle}+\frac{\ell-1}{2\ell-1}\Delta_{\mathbf{kk}}\Delta^{\nu_{\ell-1}\nu_{\ell+1}}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_{\ell-2}\rangle}+\cdots.\end{aligned} \tag{D.39}$$

As usual, the ellipsis contain all the terms that are zero when contracted with $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$. In particular, we note that the doubly contravariant projector must be $\Delta^{\nu_i\nu_{\ell+2}}$, with $i\in[1,\ldots,\ell]$, corresponding to $\ell-1$ different permutations, otherwise leading to contractions analogous

to Eq. (D.7). Therefore, the last term on the right-hand side of Eq. (D.38) can be cast in the following form

$$
\begin{aligned}
&-\ell\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left\langle E^r_{\mathbf{k}}k^{\langle\nu_1}\cdots k^{\langle\nu_{\ell-1}\rangle}k^{\langle\nu_{\ell+1}\rangle}\right\rangle\left(\sigma^{\nu_\ell}_{\nu_{\ell+1}}+\omega_{\nu_{\ell+1}}{}^{\nu_\ell}\right)\\
&=-\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\ell\left\langle E^r_{\mathbf{k}}k^{\langle\nu_1}\cdots k^{\nu_{\ell-1}}k^{\nu_{\ell+1}\rangle}\right\rangle+\frac{\ell(\ell-1)}{2\ell-1}\Delta^{\nu_{\ell-1}\nu_{\ell+1}}\left\langle E^r_{\mathbf{k}}\Delta_{\mathbf{kk}}k^{\langle\nu_1\rangle}\cdots k^{\langle\nu_{\ell-2}\rangle}\right\rangle\right)\\
&\times\left(\sigma^{\nu_\ell}_{\nu_{\ell+1}}+\omega_{\nu_{\ell+1}}{}^{\nu_\ell}\right)\\
&=-\ell\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_r\left(\sigma^{\mu_\ell\rangle}_{\alpha}-\omega^{\mu_\ell\rangle}{}_{\alpha}\right)-\frac{\ell(\ell-1)}{2\ell-1}\left(m^2\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_r-\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r+2}\right)\sigma^{\mu_{\ell-1}\mu_\ell\rangle}.
\end{aligned}
\tag{D.40}
$$

Note that we have exchanged the order of the covariant and contravariant indices in the vorticity tensor, thus leading to a minus sign, since this is an antisymmetric object, which further leads to $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\omega^{\nu_{\ell-1}\nu_\ell}=0$, since the projection operator is symmetric by construction. Then, using Eq. (D.40), we can write Eq. (D.38) as

$$
\begin{aligned}
&\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\left(\nabla_{\alpha_{\ell+1}}\Delta^{\nu_1\cdots\nu_\ell}_{\alpha_1\cdots\alpha_\ell}\right)\left\langle E^{r-1}_{\mathbf{k}}k^{\alpha_1}\cdots k^{\alpha_{\ell+1}}\right\rangle\\
&=-\left(1+\frac{\ell}{3}\right)\varrho^{\mu_1\cdots\mu_\ell}_r\theta-\ell\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_r\left(\sigma^{\mu_\ell\rangle}_{\alpha}-\omega^{\mu_\ell\rangle}{}_{\alpha}\right)\\
&-\frac{\ell(\ell-1)}{2\ell-1}\left(m^2\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_r-\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r+2}\right)\sigma^{\mu_{\ell-1}\mu_\ell\rangle}.
\end{aligned}
\tag{D.41}
$$

From Eqs. (D.21), (D.31), and (D.41), it is possible to write Eq. (D.20) as

$$
\begin{aligned}
&-\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK\,E^{r-1}_{\mathbf{k}}k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}k^{\langle\nu_{\ell+1}\rangle}\nabla_{\nu_{\ell+1}}f_{\mathbf{k}}\\
&=(r-1)\varrho^{\mu_1\cdots\mu_{\ell+2}}_{r-2}\sigma_{\mu_{\ell+1}\mu_{\ell+2}}-\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\nabla_{\nu_{\ell+1}}\varrho^{\nu_1\cdots\nu_{\ell+1}}_{r-1}+\ell\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_r\omega^{\mu_\ell\rangle}{}_{\alpha}\\
&+\frac{\ell}{2\ell+3}\left[(2r-2)m^2\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_{r-2}-(2r+2\ell+1)\varrho^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}_r\right]\sigma^{\mu_\ell\rangle}_{\alpha}\\
&+\frac{\ell(\ell-1)}{4\ell^2-1}\left[(r-1)m^4\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r-2}-(2r+2\ell-1)m^2\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_r+(r+2\ell)\varrho^{\langle\mu_1\cdots\mu_{\ell-2}}_{r+2}\right]\sigma^{\mu_{\ell-1}\mu_\ell\rangle}\\
&+\frac{1}{3}\left[(r-1)m^2\varrho^{\mu_1\cdots\mu_\ell}_{r-2}-(r+\ell+2)\varrho^{\mu_1\cdots\mu_\ell}_r\right]\theta-\frac{\ell}{2\ell+1}\nabla^{\langle\mu_1}\left(m^2\varrho^{\mu_2\cdots\mu_\ell\rangle}_{r-1}-\varrho^{\mu_2\cdots\mu_\ell\rangle}_{r+1}\right).
\end{aligned}
\tag{D.42}
$$

We are finally in position to obtain the final form of the equation of motion for the irreducible moments of rank ℓ. Combining Eqs. (D.9), (D.15), (D.19) and (D.42), it is possible to write Eq. (D.1) as

$$
\begin{aligned}
&\dot{\varrho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} \\
&= C_{r-1}^{\mu_1\cdots\mu_\ell} + r\varrho_{r-1}^{\mu_1\cdots\mu_{\ell+1}}\dot{u}_{\mu_{\ell+1}} - \Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell}\nabla_{\nu_{\ell+1}}\varrho_{r-1}^{\nu_1\cdots\nu_{\ell+1}} + (r-1)\varrho_{r-2}^{\mu_1\cdots\mu_{\ell+2}}\sigma_{\mu_{\ell+1}\mu_{\ell+2}} \\
&+ \ell\varrho_r^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}\omega^{\mu_\ell\rangle}{}_\alpha + \frac{\ell}{2\ell+1}\left[rm^2\varrho_{r-1}^{\langle\mu_1\cdots\mu_{\ell-1}} - (r+2\ell+1)\varrho_{r+1}^{\langle\mu_1\cdots\mu_{\ell-1}}\right]\dot{u}^{\mu_\ell\rangle} \\
&+ \frac{1}{3}\left[(r-1)m^2\varrho_{r-2}^{\mu_1\cdots\mu_\ell} - (r+\ell+2)\varrho_r^{\mu_1\cdots\mu_\ell}\right]\theta - \frac{\ell}{2\ell+1}\nabla^{\langle\mu_1}\left(m^2\varrho_{r-1}^{\mu_2\cdots\mu_\ell\rangle} - \varrho_{r+1}^{\mu_2\cdots\mu_\ell\rangle}\right) \\
&+ \frac{\ell}{2\ell+3}\left[(2r-2)m^2\varrho_{r-2}^{\alpha\langle\mu_1\cdots\mu_{\ell-1}} - (2r+2\ell+1)\varrho_r^{\alpha\langle\mu_1\cdots\mu_{\ell-1}}\right]\sigma_\alpha^{\mu_\ell\rangle} \\
&+ \frac{\ell(\ell-1)}{4\ell^2-1}\left[(r-1)m^4\varrho_{r-2}^{\langle\mu_1\cdots\mu_{\ell-2}} - (2r+2\ell-1)m^2\varrho_r^{\langle\mu_1\cdots\mu_{\ell-2}} + (r+2\ell)\varrho_{r+2}^{\langle\mu_1\cdots\mu_{\ell-2}}\right]\sigma^{\mu_{\ell-1}\mu_\ell\rangle},
\end{aligned}
\tag{D.43}
$$

which is exactly Eq. (4.4) from the main text.

APPENDIX E – Useful identities for the irreducible momenta

In this appendix, we derive the identity (4.5a) for the irreducible momenta that are employed in the calculations in the main text.

We have

$$
\begin{aligned}
&k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} \\
&= \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} k^{\nu_1}\cdots k^{\nu_\ell} \\
&= \left[\sum_{q=0}^{[\ell/2]} \frac{C(\ell,q)}{\mathcal{N}_{\ell,q}} \sum_{\mathcal{P}^\ell_\mu\mathcal{P}^\ell_\nu} \Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2q-1}\mu_{2q}}\Delta_{\nu_1\nu_2}\cdots\Delta_{\nu_{2q-1}\nu_{2q}}\Delta^{\mu_{2q+1}}_{\nu_{2q+1}}\cdots\Delta^{\mu_\ell}_{\nu_\ell}\right] k^{\nu_1}\cdots k^{\nu_\ell}.
\end{aligned}
\tag{E.1}
$$

We must analyze all different contractions between the $2-$index projection operators and the 4-momenta. First, we note that the doubly covariant projectors, $\Delta_{\nu_i\nu_j}$, are always contracted with a pair of 4-momenta $k^{\nu_i}k^{\nu_j}$, resulting in $\Delta_{\alpha\beta}k^\alpha k^\beta = b_{\mathbf{k}}$, where $i,j = 1,2,\ldots,2q$. Naturally, since q corresponds to the number of doubly covariant (as well as the number of doubly contravariant) projection operators, it will also indicate the power of $b_{\mathbf{k}}$ of each term of the sum. It is then necessary to compute the number of different configurations among the indices of aforementioned projectors. The result is simply

$$
\begin{aligned}
\sum_{\mathcal{P}^\ell_\nu} \Delta_{\nu_1\nu_2}\Delta_{\nu_3\nu_4}\cdots\Delta_{\nu_{2q-1}\nu_{2q}} &= \frac{\ell!}{2!(\ell-2)!}\times\frac{(\ell-2)!}{2!(\ell-4)!}\times\cdots\times\frac{(\ell-2q+2)!}{2!(\ell-2q)!} \\
&= \frac{\ell!}{2^q(\ell-2q)!}\Delta_{\nu_1\nu_2}\Delta_{\nu_3\nu_4}\cdots\Delta_{\nu_{2q-1}\nu_{2q}}.
\end{aligned}
\tag{E.2}
$$

Furthermore, these projectors commute. Therefore, in order to avoid counting the same term more than once, we must then multiply this result by the inverse number of different forms that these projectors can be arranged, which is given simply by $1/q!$.

On the other hand, projectors of the type $\Delta^{\mu_m}_{\nu_n}$ are contracted with the remaining 4-momenta, k^{ν_n}, with $m,n = 2q+1, 2q+2,\cdots,\ell$. Therefore, each covariant (contravariant) index μ_m can be paired with a total of $\ell-2q$ contravariant (covariant) indices ν_n, thus resulting in a total of $\ell-2q$ possibilities. In the next projector, the covariant (contravariant) index can now be paired with one of the $\ell-2q-1$ remaining contravariant (covariant) indices, and so on. Consequently, the number of possible configurations to arrange these projectors is $(\ell-2q)!$.

Then, the irreducible momenta can be expressed as

$$
k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} = \sum_{q=0}^{[\ell/2]} \frac{C(\ell,q)}{\mathcal{N}_{\ell,q}} \sum_{\mathcal{P}^\ell_\mu} b^q_{\mathbf{k}} \times \frac{\ell!}{2^q(\ell-2q)!} \times \frac{1}{q!} \times (\ell-2q)! \times k^{\langle\mu_{2q+1}\rangle}\cdots k^{\langle\mu_\ell\rangle}. \tag{E.3}
$$

Finally, we obtain

$$k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} = k^{\langle\mu_1\rangle}\cdots k^{\langle\mu_\ell\rangle} + \sum_{q=1}^{[\ell/2]} b_{\mathbf{k}}^q \frac{\ell!}{2^q q!}\frac{C(\ell,q)}{\mathcal{N}_{\ell,q}}\sum_{\mathcal{P}_\mu^\ell}\Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_{2q-1}\mu_{2q}}k^{\langle\mu_{2q+1}\rangle}\cdots k^{\langle\mu_\ell\rangle}, \tag{E.4}$$

which is exactly Eq. (4.5a) from the main text.

APPENDIX F – Analytical treatment of the divergence

In this appendix, we provide an analytical calculation to address the source of the divergence of the moment expansion. In particular, we show that the moments $\chi_{n,\ell}$ grow factorially at intermediate times, and that the series for the distribution function becomes (and stays) asymptotic after a certain time.

F.1 Evolution of the irreducible moments

We start from the original equations for the irreducible moments, which take the form [58, 83] – see Eq. (4.25) from the main text,

$$\frac{d\varrho_{n,\ell}}{d\tau} = -\frac{1}{\tau_R}\left(\varrho_{n,\ell} - \varrho^{\text{eq}}_{n,\ell}\right) - \mathcal{P}_{n,\ell}\frac{\varrho_{n,\ell-1}}{\tau} - \mathcal{Q}_{n,\ell}\frac{\varrho_{n,\ell}}{\tau} - \mathcal{R}_{n,\ell}\frac{\varrho_{n,\ell+1}}{\tau}, \tag{F.1}$$

where we reproduce the following coefficients below for convenience

$$\mathcal{P}_{n,\ell} = 2\ell\frac{(n+2\ell)(2\ell-1)}{(4\ell+1)(4\ell-1)}, \tag{F.2a}$$

$$\mathcal{Q}_{n,\ell} = \frac{2\ell(2\ell+1)+n(24\ell^2+12\ell-3)}{3(4\ell-1)(4\ell+3)} + \frac{2}{3}, \tag{F.2b}$$

$$\mathcal{R}_{n,\ell} = (n-2\ell-1)\frac{(2\ell+1)(2\ell+2)}{(4\ell+1)(4\ell+3)}. \tag{F.2c}$$

Introducing the vectors

$$\vec{\varrho}_m = \{\varrho_{m,0}, \varrho_{m,1}, \cdots\}, \quad \vec{\varrho}^{\text{eq}}_m = \{\varrho^{\text{eq}}_{m,0}, \varrho^{\text{eq}}_{m,1}, \cdots\}, \tag{F.3}$$

we can rewrite the equations of motion as

$$\partial_\tau \vec{\varrho}_m + \mathbb{A}(m)\vec{\varrho}_m = \frac{1}{\tau_R}\vec{\varrho}^{\text{eq}}_m\,, \tag{F.4}$$

where we defined the matrix

$$[\mathbb{A}(m)]_{\ell k} = \left[\frac{1}{\tau_R} + \frac{\mathcal{Q}_{m,\ell}}{\tau}\right]\delta_{\ell k} + \frac{\mathcal{P}_{m,\ell}}{\tau}\delta_{\ell-1,k} + \frac{\mathcal{R}_{m,\ell}}{\tau}\delta_{\ell+1,k}. \tag{F.5}$$

At this point, we make several assumptions to simplify the system of equations (F.4). First, the equilibrium term on the right-hand side provides an asymptotic value for the irreducible moments, and is not expected to be the cause of the pathological behavior explored in Chapter 5. Thus, in the following, we simply set it to zero, thus causing the irreducible moments to relax

to zero at asymptotically long times. Furthermore, we assume a constant relaxation time. In summary, we consider the system of first-order ODEs

$$\partial_{\hat{\tau}} \vec{\varrho}_m + \left[\mathbb{1} + \frac{1}{\hat{\tau}}\mathbb{B}(m)\right] \vec{\varrho}_m = 0, \tag{F.6}$$

where

$$[\mathbb{B}(m)]_{\ell k} = \mathcal{Q}_{m,\ell}\delta_{\ell k} + \mathcal{P}_{m,\ell}\delta_{\ell-1,k} + \mathcal{R}_{m,\ell}\delta_{\ell+1,k}, \tag{F.7}$$

and $\hat{\tau} = \tau/\tau_R$, as in the main text. The system (F.6) is straightforwardly solved by

$$\vec{\varrho}_m(\hat{\tau}) = e^{-(\hat{\tau}-\hat{\tau}_0)} \exp\left[-\ln\left(\hat{\tau}/\hat{\tau}_0\right)\mathbb{B}(m)\right] \vec{\varrho}_m(\hat{\tau}_0). \tag{F.8}$$

In order to proceed, we have to simplify the matrix $\mathbb{B}(m)$. Since we observe that the divergences happen for large m at any $\ell > 0$, let us first assume that $m \gg 1$,

$$\mathcal{P}_{m,\ell} = m\frac{2\ell(2\ell-1)}{(4\ell+1)(4\ell-1)} + \mathcal{O}(1), \tag{F.9a}$$

$$\mathcal{Q}_{m,\ell} = m\frac{8\ell^2+4\ell-1}{(4\ell-1)(4\ell+3)} + \mathcal{O}(1), \tag{F.9b}$$

$$\mathcal{R}_{m,\ell} = m\frac{(2\ell+1)(2\ell+2)}{(4\ell+1)(4\ell+3)} + \mathcal{O}(1). \tag{F.9c}$$

Then, the respective moments evolve as

$$\vec{\varrho}_m(\hat{\tau}) \simeq e^{-(\hat{\tau}-\hat{\tau}_0)} \exp\left[\ln\left(\hat{\tau}_0^m/\hat{\tau}^m\right)\mathbb{D}\right] \vec{\varrho}_m(\hat{\tau}_0)\ , \tag{F.10}$$

where we defined

$$\mathbb{D}_{\ell k} = \frac{8\ell^2+4\ell-1}{(4\ell-1)(4\ell+3)}\delta_{\ell k} + \frac{2\ell(2\ell-1)}{(4\ell-1)(4\ell+1)}\delta_{\ell-1,k} + \frac{(2\ell+1)(2\ell+2)}{(4\ell+1)(4\ell+3)}\delta_{\ell+1,k}, \tag{F.11}$$

which is now independent of m. To further simplify this matrix, we assume that $4\ell \gg 3$, such that we can approximate

$$\mathbb{D}_\ell \simeq \frac{1}{2}\delta_{\ell k} + \frac{1}{4}\left(\delta_{\ell-1,k} + \delta_{\ell+1,k}\right). \tag{F.12}$$

Naturally, this assumption fails for $\ell = 0, 1$, but becomes better quickly. We have now reduced the problem to finding the exponential of a tridiagonal symmetric Toeplitz matrix, which has been investigated in Ref. [162]. Such an exponential can be approximated in the following manner,

$$\left[\exp\begin{pmatrix} b & z & 0 & 0 & \cdots \\ z & b & z & 0 & \cdots \\ 0 & z & b & z & \cdots \\ 0 & 0 & z & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\right]_{\ell k} \simeq e^b\left[I_{\ell-k}(2z) - I_{\ell+k+2}(2z)\right]\ , \tag{F.13}$$

where $I_n(x)$ denotes the modified Bessel function of the first kind.[1] Now, we can approximately express the irreducible moments as

$$\vec{\varrho}_m(\hat{\tau}) \simeq e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{m/2} \mathbb{E}(m)\vec{\varrho}_m(\hat{\tau}_0), \tag{F.14}$$

where we defined

$$[\mathbb{E}(m)]_{\ell k} = I_{\ell-k}\left(\ln \frac{\hat{\tau}_0^{m/2}}{\hat{\tau}^{m/2}}\right) - I_{\ell+k+2}\left(\ln \frac{\hat{\tau}_0^{m/2}}{\hat{\tau}^{m/2}}\right). \tag{F.15}$$

Since we initialize the system in thermal equilibrium,

$$\varrho_{m,\ell}(\hat{\tau}_0) = [\vec{\varrho}_m]_\ell(\hat{\tau}_0) = e^{\alpha} T_0^{m+2} \frac{(m+1)!}{2\pi^2}\delta_{\ell 0}, \tag{F.16}$$

we obtain

$$\begin{aligned}\varrho_{m,\ell}(\hat{\tau}) &\simeq e^{\alpha} T_0^{m+2} \frac{(m+1)!}{2\pi^2} e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{m/2} \left[I_\ell\left(\ln \frac{\hat{\tau}_0^{m/2}}{\hat{\tau}^{m/2}}\right) - I_{\ell+2}\left(\ln \frac{\hat{\tau}_0^{m/2}}{\hat{\tau}^{m/2}}\right)\right] \\ &= e^{\alpha} T_0^{m+2} \frac{(m+1)!}{2\pi^2} e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{m/2} \frac{4(\ell+1)(-1)^\ell}{m \ln(\hat{\tau}/\hat{\tau}_0)} I_{\ell+1}\left(\ln \frac{\hat{\tau}^{m/2}}{\hat{\tau}_0^{m/2}}\right).\end{aligned} \tag{F.17}$$

In Fig. 22, we compare the exact solution for the irreducible moments of the simplified system, Eq. (F.8), with the approximation of large m and ℓ, Eq. (F.17). We observe that, even for *moderately* large m and $\ell \geq 1$, these solutions display a rather good qualitative agreement. Quantitatively, compared to the exact solution there are relative factors of orders $\mathcal{O}(1) - \mathcal{O}(10)$.

F.2 Factorial divergence of the normalized moments

In order to move on to the normalized moments $\chi_{m,\ell}$, defined as

$$\chi_{m,\ell} = \frac{\varrho_{m,\ell}}{\varrho_{m,0}^{\text{eq}}}, \tag{F.18}$$

it is first necessary to estimate the equilibrium moments. For this, we assume the temperature satisfies the equation of motion for an ideal fluid,

$$\frac{dT}{d\tau} + \frac{T}{3\tau} = 0, \tag{F.19}$$

that is, disregarding the term proportional to $\chi_{2,1}$ that originally appears in Eq. (4.31). The temperature of the system is then given by

$$T(\hat{\tau}) \simeq T_0 \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{1/3}. \tag{F.20}$$

[1] Note that, in contrast to the notation of Ref. [162], the indices ℓ and k start from 0.

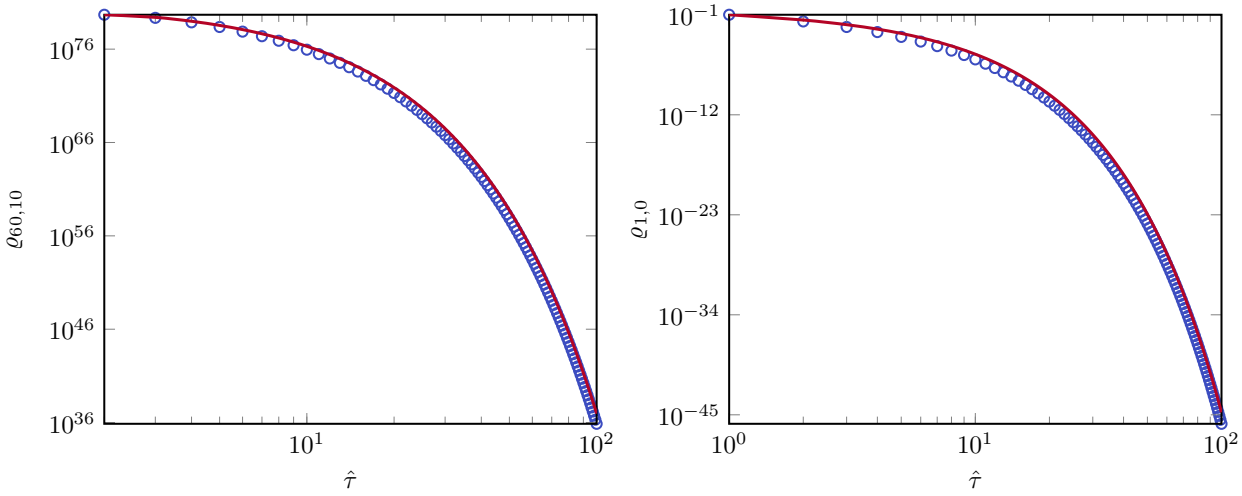

Figure 22 – Evolution of different moments (blue circles) in comparison to the analytical approximation (red lines). Left: $(m = 60, \ell = 10)$, right: $(m = 1, \ell = 0)$. The chemical potential and the initial temperature are taken as $\alpha = 0$ and $T_0 = 1$. The initial time is taken to be $\hat{\tau}_0 = 1$.

Hence, the equilibrium moments read

$$\varrho^{\mathrm{eq}}_{m,0}(\hat{\tau}) \simeq e^{\alpha} \frac{(m+1)!}{2\pi^2} T_0^{m+2} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{(m+2)/3}, \tag{F.21}$$

and the normalized moments are

$$\chi_{m,\ell}(\hat{\tau}) = \frac{\varrho_{m,\ell}(\hat{\tau})}{\varrho^{\mathrm{eq}}_{m,0}(\hat{\tau})} \simeq e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{(m-4)/6} \frac{4(\ell+1)(-1)^{\ell}}{m \ln(\hat{\tau}/\hat{\tau}_0)} I_{\ell+1}\left(\ln \frac{\hat{\tau}^{m/2}}{\hat{\tau}_0^{m/2}}\right). \tag{F.22}$$

We can obtain an estimate of the maximum value of the rescaled moments. Using the expansion of the Bessel function for large arguments,

$$I_{\nu}(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \tag{F.23}$$

which is justified at not too small times since m is large, we can approximate that, up to logarithmic corrections, the rescaled moments behave as

$$\chi_{m,\ell}(\hat{\tau}) \sim e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(m+2)/3}. \tag{F.24}$$

This suggests that the maximum of $\chi_{m,\ell}(\hat{\tau})$ lies at $\hat{\tau}_{\max}/\hat{\tau}_0 \simeq (m+2)/3$. We can then estimate that the magnitude of the maximum grows as

$$\chi_{m,\ell}(\hat{\tau}_{\max}) \sim \left(\frac{m/3}{e}\right)^{m/3} \sim (m/3)!, \tag{F.25}$$

showing a factorial divergence.

In Fig. 23, we display the rescaled irreducible moments $\chi_{m,0}$ and $\chi_{m,1}$ for several values of m, considering the temperature of an ideal fluid (F.22). It can be clearly seen that these moments diverge with increasing m. In the same plots, we also display the maximum values of the corresponding moments, which are achieved at a time $\hat{\tau} = (m+2)/3$.

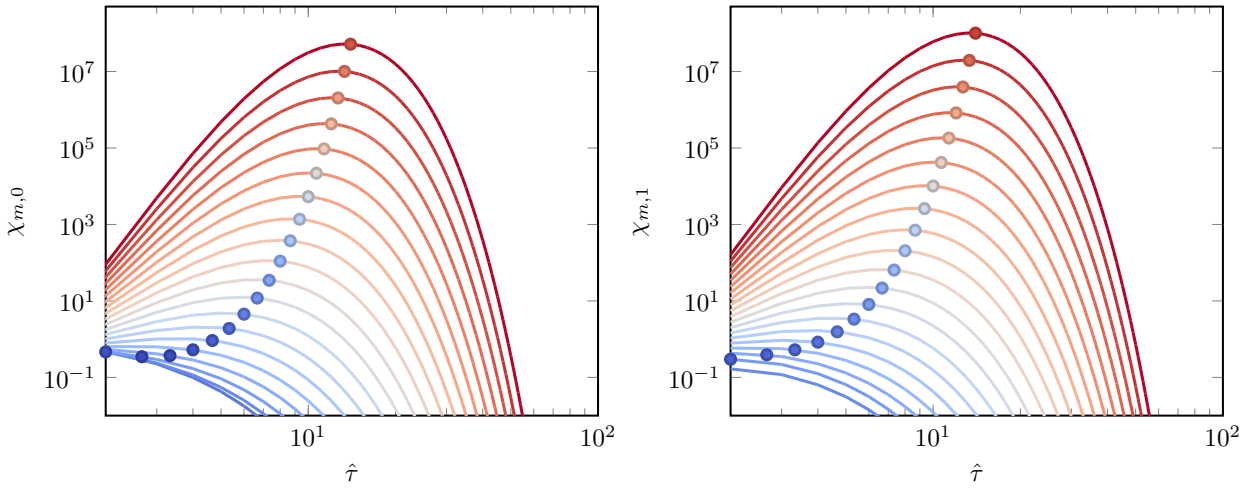

Figure 23 – The analytic evolution of the normalized moments $\chi_{m,0}$ and $\chi_{m,1}$ for $m = 2, 4, \ldots, 40$. The points mark the value of $\chi_{m,\ell}$ at the time $\hat{\tau} = (m+2)/3$. The initial time is taken to be $\hat{\tau}_0 = 1$.

F.3 Summing the distribution function

Using the results of the previous section, we are now in a position to analyze what the growth of the moments $\chi_{m,\ell}$ implies for the distribution function. First we compute the coefficients

$$\begin{aligned} c_{n,\ell}(\hat{\tau}) &= (4\ell+1)n! \sum_{m=0}^{n} \frac{(-1)^m\,(m+2\ell+1)!}{(n-m)!(m+4\ell+1)!m!} \chi_{m+2\ell,\ell}(\hat{\tau}) \\ &= (4\ell+1)4(\ell+1)(-1)^\ell e^{-(t-t_0)} \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{2/3} \frac{1}{\ln\left(\hat{\tau}/\hat{\tau}_0\right)} \\ &\quad \times \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{(m+2\ell+1)!}{(m+4\ell+1)!} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{(m+2\ell)/6} \frac{1}{m+2\ell} I_{\ell+1}\left(\frac{m+2\ell}{2} \ln \frac{\hat{\tau}}{\hat{\tau}_0}\right) . \end{aligned} \tag{F.26}$$

The ℓ-th multipole moment of the distribution function (truncated at order N, which we later send to infinity) is given by

$$\mathcal{F}_{\mathbf{k}}^{(\ell)}(N,\hat{\tau}) = \frac{2}{4\ell+1} f_{0\mathbf{k}}(\beta E_{\mathbf{k}})^{2\ell} \sum_{n=0}^{N} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) c_{n,\ell}(\hat{\tau}), \tag{F.27}$$

thus we need to evaluate the following expression,

$$\begin{aligned}
&\sum_{n=0}^{N} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) c_{n,\ell}(\hat{\tau}) \\
&= (4\ell+1)4(\ell+1)(-1)^{\ell} e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(2-\ell)/3} \frac{1}{\ln\left(\hat{\tau}/\hat{\tau}_0\right)} \\
&\times \sum_{n=0}^{N} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \sum_{m=0}^{n} \binom{n}{m} \left(-\sqrt[6]{\frac{\hat{\tau}_0}{\hat{\tau}}}\right)^m \frac{(m+2\ell+1)!}{(m+4\ell+1)!} \frac{1}{m+2\ell} I_{\ell+1}\left(\frac{m+2\ell}{2} \ln \frac{\hat{\tau}}{\hat{\tau}_0}\right) \\
&= (4\ell+1)4(\ell+1)(-1)^{\ell} e^{-(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(2-\ell)/3} \frac{1}{\ln\left(\hat{\tau}/\hat{\tau}_0\right)} \\
&\times \sum_{m=0}^{N} \frac{\left(-\sqrt[6]{\hat{\tau}_0/\hat{\tau}}\right)^m}{m!} \frac{(m+2\ell+1)!}{(m+4\ell+1)!} \frac{1}{m+2\ell} I_{\ell+1}\left(\frac{m+2\ell}{2} \ln \frac{\hat{\tau}_0}{\hat{\tau}}\right) \sum_{n=m}^{N} \frac{n!}{(n-m)!} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}).
\end{aligned} \tag{F.28}$$

Invoking the integral representation of the associated Laguerre polynomials

$$L_n^{(\alpha)}(x) = e^x x^{-\alpha/2} \frac{1}{n!} \int_0^{\infty} \mathrm{d}s\, e^{-s} s^{n+\alpha/2} J_{\alpha}\left(2\sqrt{sx}\right), \tag{F.29}$$

we have

$$\begin{aligned}
&\sum_{n=m}^{N} \frac{n!}{(n-m)!} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \\
&= e^{\beta E_{\mathbf{k}}} (\beta E_{\mathbf{k}})^{-2\ell-1/2} \int_0^{\infty} \mathrm{d}s\, e^{-s} s^{2\ell+1/2} J_{4\ell+1}(2\sqrt{s\beta E_{\mathbf{k}}}) \sum_{n=m}^{N} \frac{s^n}{(n-m)!} \\
&= e^{\beta E_{\mathbf{k}}} \frac{(\beta E_{\mathbf{k}})^{-2\ell-1/2}}{(N-m)!} \int_0^{\infty} \mathrm{d}s\, s^{2\ell+1/2+m} J_{4\ell+1}(2\sqrt{s\beta E_{\mathbf{k}}}) \Gamma(N-m+1, s).
\end{aligned} \tag{F.30}$$

Writing the incomplete Gamma function as

$$\Gamma(n, a) = a^n \int_1^{\infty} \mathrm{d}u\, e^{-au} u^{n-1}, \tag{F.31}$$

and switching the order of integration, we find

$$\begin{aligned}
&\sum_{n=m}^{N} \frac{n!}{(n-m)!} L_n^{(4\ell+1)}(\beta E_{\mathbf{k}}) \\
&= e^{\beta E_{\mathbf{k}}} \frac{(\beta E_{\mathbf{k}})^{-2\ell-1/2}}{(N-m)!} \int_1^{\infty} \mathrm{d}u\, u^{N-m} \int_0^{\infty} \mathrm{d}s\, e^{-su} s^{2\ell+3/2+N} J_{4\ell+1}(2\sqrt{s\beta E_{\mathbf{k}}}) \\
&= e^{\beta E_{\mathbf{k}}} \frac{(N+1)!}{(N-m)!} \int_0^1 \mathrm{d}v\, e^{-v\beta E_{\mathbf{k}}} v^{m+4\ell+1} L_{N+1}^{(4\ell+1)}(v\beta E_{\mathbf{k}}) \\
&= e^{\beta E_{\mathbf{k}}} \frac{(m+4\ell+1)!}{(4\ell+1)!} \binom{N+4\ell+2}{N-m} {}_2F_2\left(\begin{matrix} N+4\ell+3,\ m+4\ell+2 \\ 4\ell+2,\ m+4\ell+3 \end{matrix} \middle| -\beta E_{\mathbf{k}}\right),
\end{aligned} \tag{F.32}$$

where we employed Eq. (F.29) and substituted $v = 1/u$. The hypergeometric function ${}_2F_2$ arises because of the connection between Laguerre polynomials and confluent hypergeometric functions

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\middle| x\right) = e^x \binom{n+\alpha}{n} {}_1F_1\left(\begin{matrix}n+\alpha+1\\ \alpha+1\end{matrix}\middle| -x\right), \tag{F.33}$$

and the integral identity [163]

$${}_2F_2\left(\begin{matrix}a,\ b\\ c,\ d\end{matrix}\middle| x\right) = \frac{(d-1)!}{(b-1)!(d-b)!}\int_0^1 \mathrm{d}v\, v^{b-1}(1-v)^{d-b-1}\, {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle| xv\right), \tag{F.34}$$

which can be combined (with $a \equiv N+4\ell+3$, $b \equiv m+4\ell+2$, and $c \equiv 4\ell+2$) to yield

$${}_2F_2\left(\begin{matrix}a,\ b\\ c,\ b+1\end{matrix}\middle| -\beta E_{\mathbf{k}}\right) = \frac{b(a-c)!(c-1)!}{(a-1)!}\int_0^1 \mathrm{d}v\, e^{-v\beta E_{\mathbf{k}}} v^{b-1} L_{a-c}^{(c-1)}(v\beta E_{\mathbf{k}})\,. \tag{F.35}$$

Inserting the result (F.32) into Eq. (F.27), we arrive at

$$\begin{aligned}\mathcal{F}_{\mathbf{k}}^{(\ell)}(N,\hat{\tau}) = 8e^{\alpha}(\ell+1)(-1)^{\ell}e^{-(\hat{\tau}-\hat{\tau}_0)}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(2-\ell)/3}\frac{(\beta E_{\mathbf{k}})^{2\ell}}{\ln(\hat{\tau}/\hat{\tau}_0)}\frac{(N+4\ell+2)!}{(4\ell+1)!N!}\\ \times\sum_{m=0}^{N}\binom{N}{m}\left(-\sqrt[6]{\frac{\hat{\tau}_0}{\hat{\tau}}}\right)^m\frac{(m+2\ell+1)!}{(m+4\ell+2)!}\frac{I_{\ell+1}\left(\frac{m+2\ell}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right)}{m+2\ell}\\ \times{}_2F_2\left(\begin{matrix}N+4\ell+3,\ m+4\ell+2\\ 4\ell+2,\ m+4\ell+3\end{matrix}\middle| -\beta E_{\mathbf{k}}\right).\end{aligned} \tag{F.36}$$

By repeatedly using the relation

$$\frac{\partial}{\partial x}{}_2F_2\left(\begin{matrix}a,\ b\\ c,\ d\end{matrix}\middle| x\right) = \frac{ab}{cd}{}_2F_2\left(\begin{matrix}a+1,\ b+1\\ c+1,\ d+1\end{matrix}\middle| x\right), \tag{F.37}$$

we can rewrite $\mathcal{F}_{\mathbf{k}}^{(\ell)}$ as

$$\begin{aligned}\mathcal{F}_{\mathbf{k}}^{(\ell)}(N,\hat{\tau}) = 8e^{\alpha}(\ell+1)(-1)^{\ell}e^{-(\hat{\tau}-\hat{\tau}_0)}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(2-\ell)/3}\frac{(\beta E_{\mathbf{k}})^{2\ell}}{\ln(\hat{\tau}/\hat{\tau}_0)}(N+1)(N+2)\\ \times\frac{\partial^{4\ell}}{\partial(\beta E_{\mathbf{k}})^{4\ell}}\sum_{m=0}^{N}\binom{N}{m}\left(-\sqrt[6]{\frac{\hat{\tau}_0}{\hat{\tau}}}\right)^m\frac{(m+2\ell+1)!}{(m+4\ell+1)!}\frac{I_{\ell+1}\left(\frac{m+2\ell}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right)}{(m+2\ell)(m+2)}\\ \times{}_2F_2\left(\begin{matrix}N+3,\ m+2\\ 2,\ m+3\end{matrix}\middle| -\beta E_{\mathbf{k}}\right).\end{aligned} \tag{F.38}$$

The crucial ingredient is the sum that appears in the second line of the equation above, which we will analyze more closely in the following. Using the integral representation of the Bessel function

$$I_{\nu}(x) = \frac{(x/2)^{\nu}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\pi}\mathrm{d}\vartheta\,(\sin\vartheta)^{2\nu}\,e^{x\cos\vartheta}\,, \tag{F.39}$$

as well as

$${}_2F_2\left(\begin{matrix}N+3,\ m+2\\ 2,\ m+3\end{matrix}\middle| -\beta E_{\mathbf{k}}\right) = (m+2)\int_0^1\mathrm{d}v\, v^{m+1}\,{}_1F_1\left(\begin{matrix}N+3\\ 2\end{matrix}\middle| -v\beta E_{\mathbf{k}}\right), \tag{F.40}$$

we can rewrite the sum in $\mathcal{F}_{\mathbf{k}}^{(\ell)}$ as

$$
\begin{aligned}
&(N+1)(N+2)\sum_{m=0}^{N}\binom{N}{m}\left(-\sqrt[6]{\frac{\hat{\tau}_0}{\hat{\tau}}}\right)^m\frac{(m+2\ell+1)!}{(m+4\ell+1)!}\frac{I_{\ell+1}\left(\frac{m+2\ell}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right)}{(m+2\ell)(m+2)}\\
&\times {}_2F_2\left(\begin{matrix}N+3,\ m+2\\ 2,\ m+3\end{matrix}\middle|-\beta E_{\mathbf{k}}\right)\\
&=(N+1)(N+2)\frac{(\ln\hat{\tau}/\hat{\tau}_0)^{\ell+1}}{4^{\ell+1}\sqrt{\pi}\Gamma\left(\ell+\frac{3}{2}\right)}\int_0^\pi \mathrm{d}\vartheta\,(\sin\vartheta)^{2\ell+2}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{\ell\cos\vartheta}\\
&\times\sum_{m=0}^{N}\binom{N}{m}\left[-\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{\cos\vartheta/2-1/6}\right]^m\frac{(m+2\ell+1)!}{(m+4\ell+1)!}\frac{(m+2\ell)^\ell}{m+2}{}_2F_2\left(\begin{matrix}N+3,\ m+2\\ 2,\ m+3\end{matrix}\middle|-\beta E_{\mathbf{k}}\right)\\
&=(N+1)(N+2)\frac{(\ln\hat{\tau}/\hat{\tau}_0)^{\ell+1}}{4^{\ell+1}\sqrt{\pi}\Gamma\left(\ell+\frac{3}{2}\right)}\int_0^\pi \mathrm{d}\vartheta\,(\sin\vartheta)^{2\ell+2}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{\ell\cos\vartheta}\\
&\times\int_0^1\mathrm{d}v\,{}_1F_1\left(\begin{matrix}N+3\\ 2\end{matrix}\middle|-v\beta E_{\mathbf{k}}\right)\sum_{m=0}^{N}\binom{N}{m}\left[-v\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{\cos\vartheta/2-1/6}\right]^m\frac{(m+2\ell+1)!}{(m+4\ell+1)!}(m+2\ell)^\ell\,.
\end{aligned}
\tag{F.41}
$$

In the following, we will first specialize to the case $\ell = 0$, and subsequently treat the more involved case $\ell > 0$, which works in a similar way, but is technically more complicated.

F.3.1 Case $\ell = 0$

From Eq. (F.41), we obtain in the case where $\ell = 0$,

$$
\begin{aligned}
&(N+1)(N+2)\sum_{m=0}^{N}\binom{N}{m}\left(-\sqrt[6]{\frac{\hat{\tau}_0}{\hat{\tau}}}\right)^m\frac{I_1\left(\frac{m}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right)}{m(m+2)}{}_2F_2\left(\begin{matrix}N+3,\ m+2\\ 2,\ m+3\end{matrix}\middle|-\beta E_{\mathbf{k}}\right)\\
&=(N+1)(N+2)\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\int_0^\pi\mathrm{d}\vartheta\sin^2\vartheta\int_0^1\mathrm{d}v\,{}_1F_1\left(\begin{matrix}N+3\\ 2\end{matrix}\middle|-v\beta E_{\mathbf{k}}\right)[1-vy(\vartheta)]^N\\
&=-(N+2)\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\int_0^\pi\mathrm{d}\vartheta\sin^2\vartheta\frac{\partial}{\partial y(\vartheta)}\int_0^1\mathrm{d}v\,{}_1F_1\left(\begin{matrix}N+3\\ 2\end{matrix}\middle|-v\beta E_{\mathbf{k}}\right)[1-vy(\vartheta)]^{N+1}\,,
\end{aligned}
\tag{F.42}
$$

where we defined $y(\vartheta) = (\hat{\tau}/\hat{\tau}_0)^{\cos\vartheta/2-1/6}$. By looking at the integrand in Eq. (F.42), we can identify the crucial ingredient for the observed divergence of the distribution function: as long as the integration interval is such that $y \in (0, 2)$ always holds, the integrand converges for $N \to \infty$. Conversely, if $y \leq 0$ or $y \geq 2$ at some point inside the integral, the limit $N \to \infty$ does not exist, implying that the series for the distribution function becomes asymptotic.

At the upper limit of the ϑ-integration, we have $y(\pi) = (\hat{\tau}/\hat{\tau}_0)^{-2/3}$, which converges to zero from above for large times and is thus well behaved. Conversely, at the lower limit of the ϑ-integration, we have $y(0) = (\hat{\tau}/\hat{\tau}_0)^{1/3}$, which leads to divergences as soon as $\hat{\tau}/\hat{\tau}_0 > 8$. The edge case $\hat{\tau}/\hat{\tau}_0 = 8$ is still well behaved, since then the divergence only occurs at $\vartheta = 0$, and

that point carries zero weight due to the factors of $\sin\vartheta$. Thus, we can conclude that the series for the distribution function is asymptotic for times $\hat{\tau}/\hat{\tau}_0 > 8$.

In the following, we will assume that $\hat{\tau}/\hat{\tau}_0 \leq 8$, such that the limit $N \to \infty$ exists. In order to perform the v-integral, we note that for $N \gg 1$

$$
\begin{aligned}
{}_1F_1\left(\begin{matrix} N+3 \\ 2 \end{matrix}\middle| -x\right) &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\frac{(N+3)_k}{(2)_k} \\
&\approx \sum_{k=0}^{\infty} \frac{[-(N+2)x]^k}{k!}\frac{1}{(2)_k} = {}_0F_1\left(\begin{matrix} - \\ 2 \end{matrix}\middle| -(N+2)x\right) = \frac{J_1\left(2\sqrt{(N+2)x}\right)}{\sqrt{(N+2)x}} \,.
\end{aligned}
\tag{F.43}
$$

Using this relation and substituting $u = 2\sqrt{(N+2)v\beta E_{\mathbf{k}}}$, Eq. (F.42) becomes

$$
\begin{aligned}
&-(N+2)\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\partial}{\partial y(\vartheta)} \int_0^1 \mathrm{d}v\, {}_1F_1\left(\begin{matrix} N+3 \\ 2 \end{matrix}\middle| -v\beta E_{\mathbf{k}}\right)[1-vy(\vartheta)]^{N+1} \\
&= -\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi\beta E_{\mathbf{k}}}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\partial}{\partial y(\vartheta)} \int_0^{2\sqrt{(N+2)\beta E_{\mathbf{k}}}} \mathrm{d}u\, J_1(u)\left[1-\frac{y(\vartheta)}{4(N+2)\beta E_{\mathbf{k}}}u^2\right]^{N+1} \\
&\approx -\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi\beta E_{\mathbf{k}}}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\partial}{\partial y(\vartheta)} \int_0^{\infty} \mathrm{d}u\, J_1(u)\exp\left[-\frac{y(\vartheta)}{4\beta E_{\mathbf{k}}}u^2\right] \,,
\end{aligned}
\tag{F.44}
$$

where we used that $N \gg 1$ in the second step and expanded $\ln(1-x) \approx -x$ for small x. Note that, at this point, all dependence on the truncation N is gone, which is due to the fact that we put the upper limit of integration to infinity [and approximated $(N+1)/(N+2) \approx 1$]. This procedure is permissible as long as $\hat{\tau}/\hat{\tau}_0 \leq 8$; for larger values of $\hat{\tau}/\hat{\tau}_0$ it amounts to neglecting an infinite contribution. Employing the integral [cf. Eq. (10.22.54) of Ref. [163]]

$$
\int_0^{\infty} du\, J_1(u) e^{-au^2} = 1 - e^{-1/(4a)} \,,
\tag{F.45}
$$

we find

$$
\begin{aligned}
&-\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi\beta E_{\mathbf{k}}}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\partial}{\partial y(\vartheta)} \int_0^{\infty} \mathrm{d}u\, J_1(u)\exp\left[-\frac{y(\vartheta)}{4\beta E_{\mathbf{k}}}u^2\right] \\
&= \frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\exp\left[-\frac{\beta E_{\mathbf{k}}}{y(\vartheta)}\right]}{y(\vartheta)^2} \,.
\end{aligned}
\tag{F.46}
$$

Using the definition of $y(\vartheta)$, the Taylor series of the exponential function, and the integral representation (F.39) finally gives

$$
\begin{aligned}
&\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \frac{\exp\left[-\frac{\beta E_{\mathbf{k}}}{y(\vartheta)}\right]}{y(\vartheta)^2} \\
&= \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/3}\frac{\ln\hat{\tau}/\hat{\tau}_0}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!}\left[-\beta E_{\mathbf{k}}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/6}\right]^k \int_0^{\pi} \mathrm{d}\vartheta \sin^2\vartheta \exp\left[-\cos\vartheta \ln\frac{\hat{\tau}}{\hat{\tau}_0}\left(1+\frac{k}{2}\right)\right] \\
&= \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/3}\sum_{k=0}^{\infty}\frac{1}{k!(k+2)}\left[-\beta E_{\mathbf{k}}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/6}\right]^k I_1\left(\frac{k+2}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right) \,.
\end{aligned}
\tag{F.47}
$$

Thus, making use of Eq. (F.38), we find the zeroth multipole moment of the distribution function to be (for $\hat{\tau}/\hat{\tau}_0 \leq 8$)

$$\begin{aligned}\mathcal{F}_{\mathbf{k}}^{(0)}(\hat{\tau}) &\equiv \lim_{N\to\infty} \mathcal{F}_{\mathbf{k}}^{(0)}(N,\hat{\tau}) \\ &= 8e^{\alpha}\frac{\hat{\tau}}{\hat{\tau}_0}\frac{e^{-(\hat{\tau}-\hat{\tau}_0)}}{\ln\hat{\tau}/\hat{\tau}_0}\sum_{k=0}^{\infty}\frac{1}{k!(k+2)}\left[-\beta E_{\mathbf{k}}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/6}\right]^k I_1\left(\frac{k+2}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right) ,\end{aligned} \tag{F.48}$$

whereas the limit does not exist in the standard sense for $\hat{\tau}/\hat{\tau}_0 > 8$.

In closing, let us note two special cases: First, for $\hat{\tau}/\hat{\tau}_0 = 1$, we may use the expansion

$$I_\nu(x) \approx \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu \tag{F.49}$$

to obtain

$$\mathcal{F}_{\mathbf{k}}^{(0)}(\hat{\tau}_0) = 2e^{\alpha}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\beta E_{\mathbf{k}}\right)^k = 2e^{\alpha-\beta E_{\mathbf{k}}} , \tag{F.50}$$

as expected, since we consider the system to be initially in an equilibrium state. Second, for $\beta E_{\mathbf{k}} = 0$, only the $k = 0$-term contributes and we have

$$\mathcal{F}_{\mathbf{0}}^{(0)}(\hat{\tau}) = 4e^{\alpha}\frac{\hat{\tau}}{t_0}\frac{e^{-(t-\hat{\tau}_0)}}{\ln\hat{\tau}/\hat{\tau}_0}I_1\left(\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right) . \tag{F.51}$$

We display the zeroth and first multipoles of the single-particle distribution function for $\beta E_{\mathbf{k}} = 3$ as functions of the rescaled proper time in Fig. 24. We observe that the convergence of multipoles of higher order requires the inclusion of more terms in the expansion. Moreover, it can be clearly seen that the expansion indeed becomes asymptotic for times $\hat{\tau}/\hat{\tau}_0 > 8$.

F.3.2 Case $\ell > 0$

The argumentation for $\ell > 0$ works in the same way as shown before. The main complication lies in evaluating the sum in Eq. (F.41), which can be done by noting that

$$\begin{aligned}&\sum_{m=0}^{N}\binom{N}{m}(-y)^m\frac{(m+2\ell+1)!}{(m+4\ell+1)!}(m+2\ell)^\ell \\ &= \lim_{a\to1}\frac{\partial^\ell}{\partial\ln a^\ell}\sum_{m=0}^{N}\binom{N}{m}\frac{(-y)^m a^{m+2\ell}}{(m+2\ell+2)\cdots(m+4\ell)(m+4\ell+1)} \\ &= \lim_{a\to1}\frac{\partial^\ell}{\partial\ln a^\ell}a^{-2\ell-1}\int(\mathrm{d}a)^{2\ell}\sum_{m=0}^{N}\binom{N}{m}(-y)^m a^{m+2\ell+1} \\ &= \lim_{a\to1}\frac{\partial^\ell}{\partial\ln a^\ell}a^{-2\ell-1}\int(\mathrm{d}a)^{2\ell}\ a^{2\ell+1}(1-ay)^N ,\end{aligned} \tag{F.52}$$

where the notation $\int(\mathrm{d}a)^{2\ell}$ means that one has to integrate 2ℓ times with respect to a, and the limit $a \to 1$ has to be taken at the end. From this expression, it is already clear that the fact that the series for the distribution function becomes asymptotic for $\hat{\tau}/\hat{\tau}_0 > 8$ is not changed. In order

to obtain the finite contribution in the case $\hat{\tau}/\hat{\tau}_0 \leq 8$, we can repeat the same steps as outlined above [with $y(\vartheta)$ replaced by $ay(\vartheta)$] and find from Eq. (F.38)

$$\mathcal{F}_{\mathbf{k}}^{(\ell)}(\hat{\tau}) = 8e^{\alpha}(\ell+1)(-1)^{\ell}e^{-(\hat{\tau}-\hat{\tau}_0)}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(2-\ell)/3}\frac{(\beta E_{\mathbf{k}})^{2\ell}}{\ln\left(\hat{\tau}/\hat{\tau}_0\right)}g_{\ell}\left(\beta E_{\mathbf{k}},\hat{\tau}\right)\ , \tag{F.53}$$

where we defined

$$\begin{aligned} g_{\ell}\left(\beta E_{\mathbf{k}},\hat{\tau}\right) &= \frac{\partial^{4\ell}}{\partial(\beta E_{\mathbf{k}})^{4\ell}}\lim_{a\to 1}\frac{\partial^{\ell}}{\partial\ln a^{\ell}}a^{-2\ell-1}\int(\mathrm{d}a)^{2\ell}\,\frac{a^{2\ell-1}\left(\ln\hat{\tau}/\hat{\tau}_0\right)^{\ell+1}}{4^{\ell+1}\sqrt{\pi}\Gamma\left(\ell+\frac{3}{2}\right)} \\ &\quad\times\int_0^{\pi}\mathrm{d}\vartheta\left(\sin\vartheta\right)^{2\ell+2}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{\ell\cos\vartheta}\frac{\exp\left[-\frac{\beta E_{\mathbf{k}}}{ay(\vartheta)}\right]}{y(\vartheta)^2} \\ &= \left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{(1+2\ell)/3}\frac{\left(\ln\hat{\tau}/\hat{\tau}_0\right)^{\ell+1}}{4^{\ell+1}\sqrt{\pi}\Gamma\left(\ell+\frac{3}{2}\right)}\int_0^{\pi}\mathrm{d}\vartheta\left(\sin\vartheta\right)^{2\ell+2}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{-(\ell+1)\cos\vartheta}h_{\ell}\left(\beta E_{\mathbf{k}},\vartheta\right)\ . \end{aligned} \tag{F.54}$$

The function h_{ℓ} is given by

$$\begin{aligned} h_{\ell}\left(\beta E_{\mathbf{k}},\vartheta\right) &= \lim_{a\to 1}\frac{\partial^{\ell}}{\partial\ln a^{\ell}}a^{-2\ell-1}\int(\mathrm{d}a)^{2\ell}\,a^{-2\ell-1}\exp\left[-\frac{\beta E_{\mathbf{k}}}{ay(\vartheta)}\right] \\ &= \lim_{a\to 1}\frac{\partial^{\ell}}{\partial\ln a^{\ell}}\frac{\left[y(\vartheta)\right]^{2\ell}}{a^2\left(\beta E_{\mathbf{k}}\right)^{2\ell}}\exp\left[-\frac{\beta E_{\mathbf{k}}}{ay(\vartheta)}\right] \\ &= (-1)^{\ell}\left[\frac{y(\vartheta)}{\beta E_{\mathbf{k}}}\right]^{2\ell}\sum_{k=0}^{\infty}\frac{(k+2)^{\ell}}{k!}\left[-\frac{\beta E_{\mathbf{k}}}{y(\vartheta)}\right]^{k}\ , \end{aligned} \tag{F.55}$$

where we used

$$\int(\mathrm{d}a)^{2\ell}\,a^{-2\ell-1}e^{-\frac{r}{a}} = \frac{a^{2\ell-1}}{r^{2\ell}}e^{-\frac{r}{a}}\ , \tag{F.56}$$

which can be proven by induction. Finally, inserting Eq. (F.55) into Eq. (F.54) and subsequently into Eq. (F.53), we obtain

$$\mathcal{F}_{\mathbf{k}}^{(\ell)}(\hat{\tau}) = 8e^{\alpha}(\ell+1)\frac{\hat{\tau}}{\hat{\tau}_0}\frac{e^{-(\hat{\tau}-\hat{\tau}_0)}}{\ln\left(\hat{\tau}/\hat{\tau}_0\right)}\sum_{k=0}^{\infty}\frac{1}{k!(k+2)}\left[-\beta E_{\mathbf{k}}\left(\frac{\hat{\tau}}{\hat{\tau}_0}\right)^{1/6}\right]^{k}I_{\ell+1}\left(\frac{k+2}{2}\ln\frac{\hat{\tau}}{\hat{\tau}_0}\right)\ , \tag{F.57}$$

where we made use of the integral representation (F.39). When comparing Eq. (F.57) to Eq. (F.48), we observe that the ℓ-th multipole moment of the distribution function is very similar to the zeroth one, the main difference being the order of the modified Bessel function.

Lastly, we have to make a remark on the quantitative accuracy of the distribution function derived here. Although the analytical form obtained in Eq. (F.57) is rather elegant, the quantitative agreement with the full distribution function constructed from the solution (F.8) is not good, as can be seen in Fig. 25. The reason for this is the approximation made in the solution for the moments (F.17), which captures the important qualitative aspects (such as the expansion becoming asymptotic), but is off by factors of up to order $\mathcal{O}(10)$, which manifests itself in the distribution function. Nevertheless, the calculations performed here shed light on the origin of the observed pathological behavior of the moment expansion.

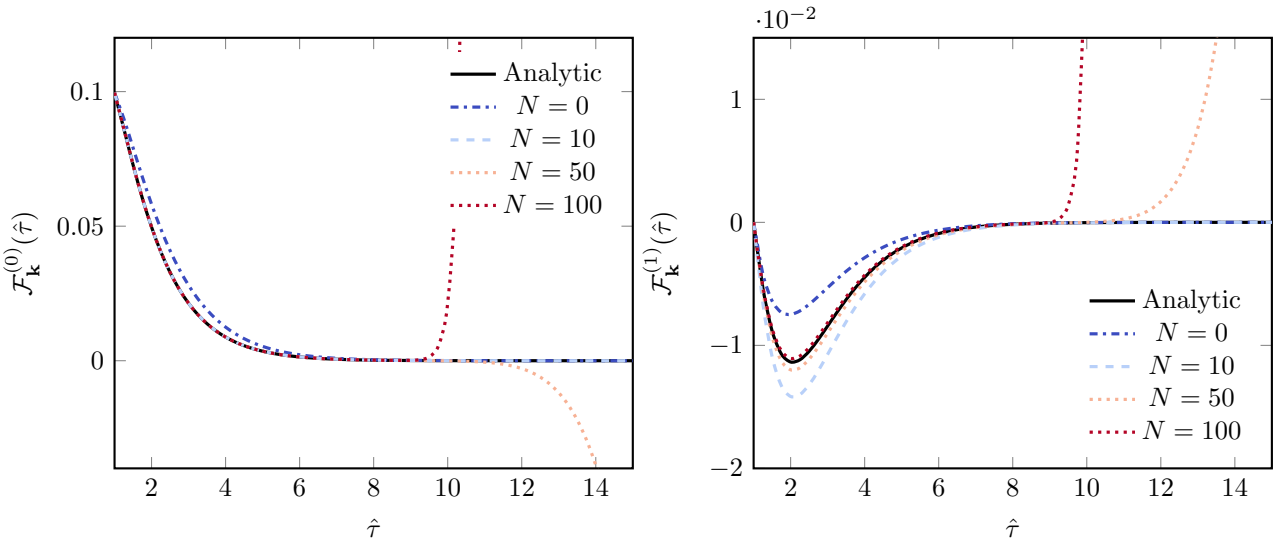

Figure 24 – The zeroth (left panel) and first (right panel) multipole moments of the distribution function at energy $\beta E_{\mathbf{k}} = 3$ as functions of time for different values of N. The lines for finite N correspond to the evaluation of Eq. (F.38), while the black line denotes the analytical solution given by Eq. (F.57). In both cases, the initial time is taken to be $\hat{\tau}_0 = 1$.

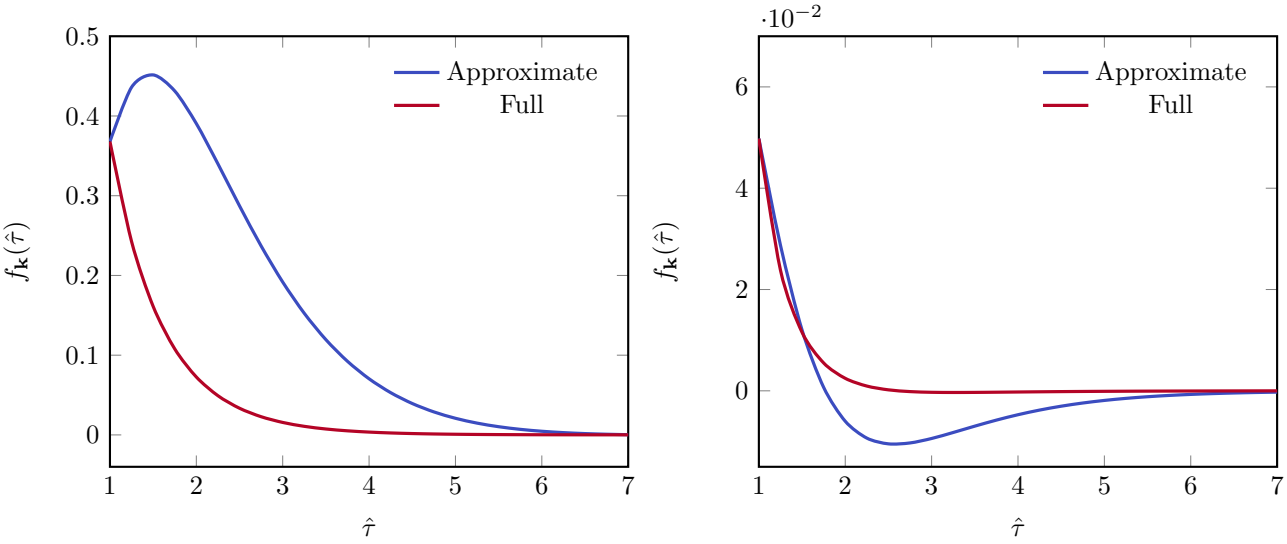

Figure 25 – The analytic distribution function computed by summing Eq. (F.57) for ℓ up to 4 and the full solution computed with Eq. (F.8) for $\beta E_{\mathbf{k}} = 1$ (left panel) and $\beta E_{\mathbf{k}} = 3$ (right panel) as functions of time. The initial time is taken to be $\hat{\tau}_0 = 1$, and we set $\Theta = 0$.

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