UNIVERSIDADE FEDERAL FLUMINENSE
INSTITUTO DE FÍSICA
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA
DOUTORADO EM FÍSICA

DAVID ROSA JUNIOR

CONFINEMENT IN YANG-MILLS THEORY AS DUE TO PERCOLATING CENTER VORTICES

Niterói, 2022

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CONFINEMENT IN YANG-MILLS THEORY AS DUE TO PERCOLATING CENTER VORTICES

A thesis submitted to the Departamento de Física - Universidade Federal Fluminense in partial fulfillment of the requirements for the degree of Doctor in Sciences (Physics).

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Niterói
2022

Ficha catalográfica automática - SDC/BIF
Gerada com informações fornecidas pelo autor

J95c Junior, David Rosa
Confinement in Yang-Mills theory as due to percolating center vortices / David Rosa Junior ; Luis Esteban Oxman, orientador. Niterói, 2022.
129 f. : il.

Tese (doutorado)-Universidade Federal Fluminense, Niterói, 2022.

DOI: http://dx.doi.org/10.22409/PPGF.2022.d.12891685636

1. Confinamento de glúons. 2. Teoria quântica de campos. 3. Efeito Casimir. 4. Produção intelectual. I. Esteban Oxman, Luis, orientador. II. Universidade Federal Fluminense. Instituto de Física. III. Título.

CDD -

Bibliotecário responsável: Debora do Nascimento - CRB7/6368

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Tese apresentada ao Departamento de Física - Universidade Federal Fluminense como requisito parcial para obtenção do título de Doutor em Física.

Aprovado em 25/02/2022.

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Acknowledgments

Writing a thesis in pandemic times is an experience which involves many mixed feelings. As this is an ”acknowledgments” and not a ”complainings” section, I shall focus on the positive ones.

I am grateful to my advisor Luis Oxman, for his guidance and support during these years. I also thank Gustavo Simões for all the interesting physical insights and discussions during our many years of collaborations. I am also grateful to César Fosco, Diego Fiorentini, Hugo Reinhardt, Rodrigo Sobreiro for our collaborations.

I thank Davide Campagnari, Felix Spengler, Giuseppe Burgio, Hugo Reinhardt, and Markus Quandt for the receptiveness and interesting discussions during my stay in Tübingen.

Nivaldo Lemos is greatly acknowledged for his guidance and for teaching me the subjects that I was eager to learn in the first semesters of my undergraduate studies.

I thank my wife, Daianne, who I love so much, for existing, and for fighting bravely by my side even in the toughest times. I also thank my mother-in-law and father-in-law (Adelice and Arildo) for the support, especially during pandemic times.

I am grateful to my friends from Pouso Alegre: Alice, Danilo, Fernanda, Pedro, Rodrigo. I should also mention the people of the ”Capa da Solidão”: Daniel, Delano, Rodrigo, and Vittor.

I am grateful to the friends that I met thanks to Physics: Arthur, Bernardo, Cílio, Filipe, Guilherme, Luigi, Marcos, Mariana, Maron, Miguel. As for the friends from the Gragoatá, special thanks goes to Heloize for her refined humour, and to Juliana, for taking me into interesting lectures on social sciences in the Bloco E.

I thank my friends Otávio and Artur for the long lasting friendship. In this regard, I should also thank Tibia (and its variants) for providing me the opportunity of meeting these two, and for being such a good game.

I am grateful to my parents for the emotional and financial support, after having brought me into existence in this absurd and intriguing universe.

The CNPq and the DAAD are acknowledged for the financial support.

Last, but not least, I thank Dugi Otok.

Resumo

Nesta tese, iremos inicialmente discutir ensembles fenomenológicos de vórtices de centro. Mostraremos que, quando regras de fusão apropriadas e configurações não orientadas são incluídas, os ensembles são naturalmente representados por teorias de campos efetivas contendo soluções topológicas. Estas são paredes de domínio em 3 dimensões, e objetos unidimensionais estáticos em 4 dimensões. Nós discutiremos, então, como um ponto-sela baseado nessas soluções é capaz de capturar as propriedades assintóticas do tubo de fluxo confinante das teorias de Yang-Mills $SU(N)$, tanto em 3 como em 4 dimensões. Em seguida, revisaremos um novo procedimento de quantização para as teorias de Yang-Mills, baseado em uma condição de calibre que é local no espaço de configurações, e discutiremos como essa proposta é promissora não só para lidar com o problema de Gribov, mas também para mostrar um vislumbre de um caminho da teoria de Yang-Mills pura para os ensembles de vórtices de centro. Depois, estudamos a renormalizabilidade dessse procedimento no setor perturbativo e em setores rotulados por um número qualquer de vórtices de centro, estabelecendo portanto a calculabilidade desse ensemble de Yang-Mills. O cálculo explícito da contribuição de cada setor, a ser feito no futuro, envolve integrais de caminho de campos satisfazendo condições de contorno de Dirichlet nas superfícies de mundo dos vórtices, que têm codimensão dois. Nesse sentido, apresentamos o cálculo da energia de vácuo de um campo escalar satisfazendo condições de contorno em hipersuperfícies de diferentes codimensões. Discutimos as sutilezas que aparecem em cada caso, e mostramos que o caso de codimensão dois é o mais especial.

Abstract

In this thesis, we initially discuss phenomenological ensembles of center vortices in 3 and 4 dimensions. We show that, when appropriate matching rules and nonoriented configurations are included, the ensembles are naturally represented by effective field theories accommodating topological solutions. These are domain walls in 3 dimensions, and static one dimensional objects in 4 dimensions. We then discuss how a saddle-point based on these solutions is able to capture the asymptotic properties of the confining flux tube of $SU(N)$ Yang-Mills theories, both in 3 and 4 dimensions. Then, we review a novel quantization procedure for Yang-Mills theory, based on a gauge condition that is local in configuration space, and discuss how it is promising candidate not only to deal with the Gribov problem, but also to provide a glimpse of a path from pure Yang-Mills theory to ensembles of center vortices. Next, we study the renormalizability of this procedure in the perturbative sector and in sectors labeled by any number of center vortices, thus establishing the calculability of this Yang-Mills ensemble. The explicit computation of each sector's contribution, to be done in the future, involves the calculation of a path integral of fields satisfying Dirichlet boundary conditions in the vortices's worldsurfaces, which have codimension two. In this regard, we present the calculation of the vacuum energy for a scalar field satisfying boundary conditions along hypersurfaces of different codimensions. We discuss the subtleties that arise in each case, and show that codimension two is the most special one.

List of Publications

Published research papers

- D. Fiorentini, D. R. Junior, L. E. Oxman, G. M. Simões, R. F. Sobreiro. *Study of Gribov copies in a Yang-Mills ensemble.* Physical Review D, v. 103, p. 114010, 2021.
- D. R. Junior, L. E. Oxman, G. M. Simões. *3D Yang-Mills confining properties from a non-Abelian ensemble perspective.* Journal of High Energy Physics, v. 2020, p. 180, 2020.
- C. D. Fosco, D. R. Junior, L. E. Oxman. *Quantum effects due to a moving Dirichlet point.* Physical Review D, v. 101, p. 065014, 2020.
- D. Fiorentini, D. R. Junior, L. E. Oxman, R. F. Sobreiro. *Renormalizability of the center-vortex free sector of Yang-Mills theory.* Physical Review D, v. 101, p. 085007, 2020.
- D. R. Junior, L. E. Oxman, G. M. Simões.*BPS strings and the stability of the asymptotic Casimir law in adjoint flavor-symmetric Yang-Mills-Higgs models.* Physical Review D, v. 102, p. 074005, 2020.
- D. R. Junior, L. E. Oxman. *Geometry of the Shannon mutual information in continuum QFT.* Physical Review D, v. 95, p. 125005, 2017.

Published review papers

- D. R. Junior, L. E. Oxman, G. M. Simões. *From center-vortex ensembles to the confining flux tube.* Universe, v. 7, p. 253, 2021.

Preprints

- D. Fiorentini, D. R. Junior, L. E. Oxman, R. F. Sobreiro. *Renormalizability of a first principles Yang-Mills center-vortex ensemble.* arXiv:2008.01249, 2021.

Proceedings

- D. Fiorentini, D. R. Junior, L. E. Oxman, R. F. Sobreiro. *Renormalizability of center-vortex sectors in continuum Yang-Mills theory*. EPJ Web of Conferences 258, 02002 (2022).

This thesis is based on the results obtained in the first five published research papers of the above list, plus the one in preprint version.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Yang-Mills (YM) theory [1] is a very powerful framework to describe the fundamental interactions of nature for a wide range of energies. Electroweak [2, 3, 4] and strong interactions are currently described by the Standard Model, which consists of a gauge theory with group $SU(2) \times U(1) \times SU(3)$. These theories comprise the current theoretical basis for the understanding of high energy experiments such as those in the Large Hadron Collider (LHC). In fact, Quantum Electrodynamics (QED), the U(1) gauge theory that describes the electromagnetic interaction, was able to deliver a value for the anomalous magnetic dipole moment of the electron which agrees with experiment to within ten parts in a billion [5, 6, 7], the most accurate prediction in the history of science so far. Of course, such an agreement was only possible due to the existence of powerful and reliable calculational tools. For a wide range of energies, all couplings of the Standard model are small, and the standard (and very successful) method is perturbation theory. For very high energies, QED reaches a Landau pole[8, 9], which is usually interpreted as an indicator that the electroweak sector of the Standard Model must be an effective theory.

Quantum Chromodynamics (QCD), on the other hand, is an asymptotically free theory, i.e., its coupling approaches zero for very high energies, while reaching a (perturbative) Landau pole in the infrared limit. This comes about as its β function is negative [10, 11, 12]. Perturbation theory is thus unapplicable in the low energy limit of QCD. Moreover, in this regime there is no gauge condition which is free of Gribov copies [13], thus invalidating the usual Faddeev-Popov quantization procedure for Yang-Mills theories [14]. However, many important physical phenomena, such as confinement, chiral symmetry breaking, and the formation of baryonic matter, take place in this regime of the theory, and it is therefore very important to overcome these practical and theoretical obstructions. A well-known successful approach is to discretize space-time and compute averages of observables through Monte Carlo simulations [15]. In this case the existence of Gribov copies is not a problem, as there is no need to fix the gauge. Most of the important results of low energy YM theory were obtained through this method.

However, these results sometimes come with the lack of understanding of the underlying physical mechanism. The most striking example of this is perhaps that of confinement: a rectangular Wilson loop in the $x^i - T$ plane, x^i being a spatial direction and T being the Euclidean time, follows an area law in the infrared regime of $SU(N)$ Yang-Mills theory [16]. This implies a linear potential between a static quark-antiquark pair. Moreover, lattice calculations point to the existence of a confining flux tube with very particular properties, such as a soliton-like chromoelectric field profile, and a string-like behaviour, as evidenced by the presence of the Lüscher term. Additionally, for asymptotic distances, the properties of the confining flux tube depend on the representation R of the static sources only by means of its $N-$ality, i.e. on how the center $Z(N)$ of $SU(N)$ is realized in the given representation. These properties of the flux tube are reviewed in chapter 2. There is currently no full theoretical understanding of the mechanism that leads to the formation of such a confining flux tube. In fact, this is related to one of the seven Millennium problems posed by the Clay Mathematics Institute [17]. Despite the lack of a complete understanding, some progress has been made in the last 40 years. A recent review of the different approaches that have been pursued can be found in e.g. Ref. [18]. One approach that has been developed is the idea that some degrees of freedom of Yang-Mills theory become predominant in the confining regime, giving rise to the possibility of describing the observed phenomena in terms of an effective description of these degrees [19, 20, 21, 22, 23, 24]. One important contribution in this regard was that of Polyakov in Refs. [25, 26, 27], where it was shown that compact QED in $2+1$ dimensions can be written as an ensemble of instantons, which in turn may be represented by an effective scalar theory. The Wilson loop could then be evaluated by means of a saddle-point of this effective theory. Moreover, this effective representation allowed him to show that the theory has a mass gap. This is a striking example of how the confining mechanism of a theory may be understood from first principles. It is also possible to make an analogy with the situation of superconductors, where the cooper pairs condense, giving rise to the Ginzburg-Landau effective description.

The first step in this framework is to identify what are the relevant degrees of freedom (d.o.f.) for describing the infrared regime of $SU(N)$ YM theories. In some proposals, these d.o.f are Abelian [28, 29, 30, 31, 32, 33, 34, 35]. However, some popular Abelian models such as the monopole plasma, dyon gas and dual Abelian Higgs models, are not consistent with observations that the string tension on a general representation R of $SU(N)$ depends only on the $N-$ality of R. In particular, these approaches predict a sum-of-areas law for double-winding Wilson Loops, in contrast to the observed difference-of-areas law [36]. Center vortices are the most promising configurations for accommodating these and many other observed properties of the confining string. In this thesis, we will focus on this approach. There are three main topics to be investigated in this framework: a) studying classical effective models where the confining flux tube is represented by a topological soliton; b) analyzing phenomenological ensembles of center vortices and deriving an effective field representation, c) pursuing a more direct connection between first principles Yang-Mills theory with the phenomenological approach. In chapter 3 we study these configurations in detail and review well-known numerical and analytical evidence of their importance in the confining regime. Then, we review and present different ensembles of these configurations which are able to capture various confining properties of YM theory. In particular, we show that these ensembles must contain the contribution of non-oriented vortex configurations in order to successfuly reproduce the asymptotic properties of the confining string. The most successful ensembles lead to effective field theories which accommodate topological solitons that are stable classically. This facilitates the computation of quantum averages, as a saddle-point calculation around these classical solutions may be performed. The relevant classical solutions are discussed both for the $2+1$d and for the $3+1$d case.

An important point to be understood is how such an ensemble of center vortices could be derived from a first principles calculation in continuum Yang-Mills theory. In this case there is an important obstacle, as the Yang-Mills partition function in the continuum is ill defined in the usual gauges due to the presence of Gribov copies in the infrared regime. In fact, Singer proved that any global gauge fixing condition will suffer of this problem [13]. As emphasized by Singer, quantization procedures relying on gauge fixings which are local in configuration space can in principle be well-defined. In chapter 4 we review one possibility in this regard, where the configuration space $\{A_\mu\}$ is divided into disjoint sectors $\mathcal{V}(S_0)$ labeled by center vortices, and then the gauge is fixed by a sector-dependent gauge condition. More specifically, in this framework the Yang-Mills partition function is written as a sum over the partial contributions of the sectors $\mathcal{V}(S_0)$, as follows

$$Z_{YM} = \sum_{S_0} Z_{YM}^{S_0} \quad , \quad Z_{YM}^{S_0} = \int_{\vartheta(S_0)} [DA] e^{-S_{YM}} \; . \tag{1.1}$$

Then, an appropriate identity is inserted in each of the partial contributions $Z_{YM}^{S_0}$, in order to implement the Fadeev-Popov procedure locally:

$$Z_{YM}^{S_0} = \int_{\vartheta(S_0)} [DA] e^{-S_{YM}} \delta(f_{S_0}(A)) \text{Det} \frac{\delta f_{S_0}(A^U)}{\delta U}\bigg|_{U=I} \; . \tag{1.2}$$

Here, $f_{S_0}(A)$ is a sector-dependent gauge fixing condition. In chapter 4 we discuss the procedure carefully and show that it not only has the potential of dealing with the Gribov problem, but also leads naturally to a first principles Yang-Mills center vortex ensemble. In chapter 5 we present a proof of the renormalizability of this Yang-Mills ensemble using the algebraic method, thus establishing its calculability.

As discussed in chapter 5.8, the explicit computation of the partial contributions $Z_{YM}^{S_0}$ will necessarily involve path-integrals in $d+1$ dimensions with boundary conditions in hypersurfaces of dimension $d-1$. This is in contrast with usual Casimir energy calculations, which are problems with boundary conditions in surfaces of codimension 1 with respect to the full spacetime. It is therefore important to understand the role of the codimension in these type of problems. In this regard, in chapter 6 we compute the vacuum energy of a scalar field with boundary conditions in hypersurfaces of different codimensions. We give special emphasis to the codimension 2 case.

Chapter 2

Some ideas and results regarding confinement

An usual setup which is used to describe confinement in QCD is that of a quark that is initially close to an antiquark (both of mass m), and then the pair is separated by a large distance L. The energy of this configuration grows linearly with L, i.e. $E = \sigma L$, for some constant σ. When $\sigma L >> 2mc^2$, it will be energetically favorable for the flux tube to break, giving rise to a new quark-antiquark pair. As a consequence, it is not possible to observe an isolated quark in this confining regime. Since quarks are colored particles, the spectrum consists only of color singlets. Even though the formation of a flux tube is intimately connected with the absence of colored particles in the spectrum in this case, it is important to understand that, in general, these are separate concepts. Confinement, a priori, is related to the existence of a flux tube. In section 2.1 we briefly review the continuum formulation of YM theories. In section 2.2, following Ref. [18], we introduce lattice gauge theory and the observables which allow for the study of flux tubes. In section 2.3 we review the Wilson Loop in pure YM theory, and discuss its possible behaviours and implications. In section 2.4, we review, very briefly, a theory with no flux tubes whose spectrum consists only of color singlets.

2.1 Quantizing YM theory in the continuum

The primary example of a classical Yang-Mills theory is that of electrodynamics, which is described by the Maxwell Lagrangian, which reads, in Euclidean conventions,

$$L_{Max} = -\frac{1}{4g^2} F_{\mu\nu}^2 \; , \tag{2.1}$$

The field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is invariant under the $U(1)$ gauge transformations

$$A_\mu \to A_\mu + \frac{i}{g} e^{i\alpha(x)} \partial_\mu e^{-i\alpha(x)} \;, \tag{2.2}$$

where $\alpha(x)$ is a smooth function. More general classical Yang-Mills theories are generalizations of Maxwell's theory that are invariant by local gauge transformations that belong to a general gauge group G, usually a simple Lie group. We will restrict our attention to the case $G = SU(N)$.

In this case, the gauge field is an element of the Lie algebra of G, i.e. $A_\mu = A^a_\mu T^a$, $a = 1, \ldots, N^2 - 1$. The Lagrangian of this theory turns out to be (see Appendix A for conventions)

$$L_{YM} = \frac{1}{4}(F_{\mu\nu}, F_{\mu\nu}) \;, \tag{2.3}$$

where the non-Abelian field strength is defined by

$$\begin{aligned} F_{\mu\nu} &= \tfrac{i}{g}[D_\mu, D_\nu] \;, \\ D_\mu &= \partial_\mu - ig[A_\mu,] \;. \end{aligned} \tag{2.4}$$

This action is invariant under the local gauge transformation

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \;, U \in SU(N) \;. \tag{2.5}$$

Because of the non-Abelian character of the group, this action contains interaction terms for the different color components A^a_μ, as well as self-interactions. To quantize these theories, an usual approach is that of the path-integral formalism, where the fundamental object is the partition function

$$Z_{YM} = \int [DA] e^{-S_{\text{YM}}} \;. \tag{2.6}$$

The usual path connecting the partition function with the real world is established trough the LSZ reduction formula. The typical setup is an experiment where n_i particles are initially prepared with momenta p^j, $j = 1, \ldots, n_i$, and then interact through some process. One then may obtain the probability amplitude that the final state contains n_f particles with momenta k^a, $a = 1, \ldots, n_f$ by means of the vacuum expectation value of the time-ordered product of the corresponding fields. This product is a local composite operator. Expectation values of general operators O of this type may be evaluated in the path-integral formalism as

$$\langle O \rangle = \frac{\int [D\text{fields}]\, O\, e^{-S}}{\int [D\text{fields}]\, e^{-S}} \;, \tag{2.7}$$

and thus the importance of evaluating the partition function. Here, we have switched to Euclidean space by means of a Wick rotation. The issue here is that the list of known functional integrals is quite short. In fact, it contains only one entry: the Gaussian path-integral. For a set of real bosonic fields ϕ_I and a differential operator O_{IJ},

$$\int [D\phi]\, e^{-\phi_I O_{IJ}\phi_J + J_I\phi_I} = \frac{\mathcal{N}}{\sqrt{\det O}} e^{\frac{1}{4}J_I O^{-1}_{IJ} J_J} \ , \tag{2.8}$$

where $\mathcal{N}$ is a normalization factor. The labels I, J stand for internal and space-time indices, i.e., there is an integration and sum over repeated labels. Similar formulas hold for complex and Grassman fields

$$\int [D\phi][D\bar{\phi}]\, e^{-\phi_I O_{IJ}\bar{\phi}_J + \bar{J}_I\bar{\phi}_I + J_I\phi_I} = \frac{\mathcal{N}}{\det O}\, e^{J_I O^{-1}_{IJ}\bar{J}_J} \ , \tag{2.9}$$

$$\int [D\theta][D\theta^*]\, e^{-\theta_I O_{IJ}\theta^*_J + \eta^*_I\theta_I + \eta_I\theta^*_I} = \det O\, e^{\eta^*_I O^{-1}_{IJ}\eta_J} \ , \tag{2.10}$$

where the bar stands for complex conjugation, and the star for Grassman conjugate. The first obstacle is thus that Yang-Mills is an interacting theory, and thus the path integral is not Gaussian. The usual way around this is to separate the Gaussian from the interacting part, i.e.,

$$S_{YM} = S_0 + S_{int} \ , \tag{2.11}$$

$$S_0 = \frac{1}{2}\int d^4x\, A_\mu(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu)A_\nu \ . \tag{2.12}$$

Then, one defines

$$Z_0[J] = \int [DA] e^{-S_0 + \int d^4x\, J_\mu(x)A_\mu(x)} \ , \tag{2.13}$$

because the following useful formula holds

$$\int [DA] f(A) e^{-S_0(A)} = \int [DA] f(\delta/\delta J) e^{-S_0(A) + \int d^4x\, J_\mu(x)A_\mu(x)}|_{J=0} = f(\delta/\delta J) Z_0[J]|_{J=0} \ . \tag{2.14}$$

This allows for the perturbative evaluation of Z_{YM} and then of any observable by taking appropriate derivatives of $Z_0[J]$, which is a Gaussian functional integral. The problem of this approach to YM theory is that even $Z_0[J]$ is ill-defined, as the operator $O_{\mu\nu} = -\partial^2 + \partial_\mu\partial_\nu$ has zero modes. This can be seen more evidently in momentum space, since

$$(k^2\delta_{\mu\nu} - k_\mu k_\nu)k_\nu = 0 \ . \tag{2.15}$$

Formula (2.8) may therefore not be applied, as the inverse of $O_{\mu\nu}$ does not exist. This is to be expected, as the kernel of $O_{\mu\nu}$ is given by

$$O_{\mu\nu} A_\nu = 0 \ . \tag{2.16}$$

Due to the gauge invariance of the theory, this equation has as many solutions as there are gauge transformations, even in the presence of appropriate boundary conditions. To solve this problem, the typical approach is to find a condition $f(A) = 0$ that selects one representative at each gauge orbit, and then introduce an identity in the partition function

$$1 = \int [DU] \delta(f(A)) \det M(A) \ , \tag{2.17}$$

where the differential operator $M(A)$ is defined as

$$M(A) = \tfrac{\delta f(A^U)}{\delta \alpha}|_{\alpha=0} \ , \tag{2.18}$$

$$U = e^{i\alpha^a T^a} \ . \tag{2.19}$$

However, Singer proved that, for $SU(N)$ YM theories, every continuous, global condition $f(A)$ will not be able to select an unique representative for all gauge orbits. This implies that Gribov copies, which are elements of the same orbit that satisfy $f(A) = 0$, will necessarily exist. Thus, the identity of eq. (2.17) is necessarily ill-defined, as the operator $M(A)$ contains zero modes. In section 4.21 we will come back to this issue, and discuss possible ways around it.

The above discussion seems to imply that the usual FP approach is useless for YM theory. However, it turns out that this formalism works quite well in the perturbative regime, giving results consistent with experiments. Let us review the usual explanation for this. Consider a particular gauge-fixing condition, the Landau gauge

$$f(A) = \partial_\mu A_\mu = 0 \ . \tag{2.20}$$

For a Gribov copy to exist, it is necessary that there is an $\mathcal{A}$ such that $f(\mathcal{A}) = f(\mathcal{A}^{\mathcal{U}}) = 0$, for a nontrivial $U \in SU(N)$. For an infinitesimal gauge transformation $U = e^{i\alpha^a T^a}$, the variation of the gauge field is given by

$$\delta \mathcal{A}^a_\mu = \frac{\partial_\mu \alpha^a}{g} + f^{abc} \alpha^b \mathcal{A}^c_\mu \ . \tag{2.21}$$

Thus, there will be an infinitesimal copy if there exist $N^2 - 1$ scalar functions α^a satisfying

$$(-\partial^2 \delta^{ac} + g f^{abc} \mathcal{A}^b_\mu \partial_\mu) \alpha^c = M^{ac} \alpha^c = 0 \ , \tag{2.22}$$

with the boundary condition $\alpha^a \to 0$ as $|x| \to \infty$. We see that, for $g = 0$, the only solution is $\alpha^a = 0$, as in this case $M^{ac} = -\partial^2 \delta^{ac}$. Therefore, for small values of the product $g\mathcal{A}$, one expects that no zero modes will be produced.

2.2 Lattice gauge theory

In Quantum Field Theory, we are generally interested in the computation of averages of observables O

$$\langle O(\phi) \rangle = \frac{\int [D\phi] O(\phi) e^{-S[\phi]}}{\int [D\phi] e^{-S[\phi]}} , \tag{2.23}$$

where ϕ denotes all the fields of the theory. As discussed in the previous section, the usual continuum methods for dealing with these objects only work for theories with small coupling constants. When this is not the case, a possibility is to perform a full computation of the path integrals using a computer. In this section we shall review very briefly the main elements of this approach. A more complete introduction can be found in e.g. Ref. [37]. For the purpose of performing these path integrals in a computer, it is necessary to discretize d dimensional space-time, transforming it in a hypercubic lattice, where the distance between neighboring points is a. Accordingly, the action $S[\phi]$ must be discretized, thus becoming $S_l[\phi, a]$, such that

$$\lim_{a \to 0} S_l[\phi, a] = S[\phi] . \tag{2.24}$$

Scalar fields $\phi_S(x)$ are discretized straightforwardly by assigning a variable $\phi_S(x_i)$ for each lattice point x_i. Gauge fields $A_\mu(x)$, on the other hand, become link variables $U_\mu(x_i) = e^{iagA_\mu(x_i)}$, $\mu = 0, \ldots, d$. A commonly used discretization of Yang-Mills action is given by the Wilson action

$$S_W = -\frac{\beta}{2N} \sum_{x,\mu<\nu} \left(Tr(U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)) + c.c. \right) . \tag{2.25}$$

In the limit $a \to 0$, by expanding the link variables and retaining only linear terms, it is possible to show that it reduces, upon identifying $\beta = \frac{1}{g^2}$, to the Yang-Mills action. This discretization is of course not unique [37]. This can be easily understood: any action $S' = S_W + aV(U)$, where V is independent of a, will also reproduce the YM action in the $a \to 0$ limit. The Wilson Action is invariant under the gauge transformations

$$U_\mu(x) \to G(x) U_\mu(x) G^\dagger(x+\mu) \; , G \in SU(N) . \tag{2.26}$$

In fact, the plaquettes, which are the building blocks of the Wilson action, are gauge invariant. These are defined as the product of the link variables along the smallest closed paths of the lattice:

$$P_{\mu\nu}(x) = Tr(U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)) \ . \tag{2.27}$$

Matter fields may be included naturally as well. For instance, the action of an $SU(N)$ scalar field in the representation R is

$$\begin{aligned} S &= S_W + S_M \ , \\ S_M &= -\gamma \sum_{x,\mu} \left(\phi^\dagger(x)U_\mu^{\mathsf{R}}(x)\phi(x+\mu) + c.c.\right) + \sum_x (m^2 + 2(d+1))\phi^\dagger(x)\phi(x) \ , \end{aligned} \tag{2.28}$$

where U_μ^{R} stands for the link variable in the R representation of the group. The scalar fields transforms as $\phi(x) \to G(x)\phi(x)$ to assure the gauge invariance of the full action.

Then, the path-integral (eq. (2.23)) is usually evaluated by Monte-Carlo methods, where an initial discretized field configuration is generated randomly, and then updated according to an algorithm which selects the best ones [37]. One possibility is the Metropolis algorithm. In this case, an update in the configuration is accepted if it lowers the energy, and refused with a probability proportional to $e^{-\beta\Delta E}$ if it increases the energy. This is done until a thermalized configuration is reached. The complicated path-integral is then computed by the sum of the contribution of these configurations and, with a sufficiently large number of them, this sum will provide a good approximation for the exact result.

2.3 Observables that probe confining properties

Consider an operator which creates two scalar particles in representation R separated by a distance L in the v (a normalized vector) direction at a given time t

$$Q(t) = \phi^\dagger(0,t)\left(\prod_{n=0}^{L-1} U_v^{\mathsf{R}}(nv,t)\right)\phi(Lv,t) \ . \tag{2.29}$$

The operators $\phi, \phi^\dagger$ create the antiparticle and particle, respectively, while the link variables ensure the gauge-invariance of the operator. Now, consider the following expectation value

$$\langle Q^\dagger(T)Q(0)\rangle = \langle 0|Q^\dagger(T)Q(0)|0\rangle = \langle 0|Q^\dagger e^{-HT}Q|0\rangle \ , \tag{2.30}$$

where in the last equality we have switched from the Heisenberg to the Schrödinger picture. Now, we insert the completeness relation of energy eigenstates

$$\begin{aligned}\langle 0|Q^\dagger e^{-HT} Q|0\rangle &= \textstyle\sum_{n,m} \langle 0|Q^\dagger|n\rangle\langle n|e^{-HT}|m\rangle\langle m|Q|0\rangle \\ &= \textstyle\sum_n e^{-E_n T}\langle 0|Q^\dagger|n\rangle\langle n|Q|0\rangle \ .\end{aligned} \tag{2.31}$$

On the other hand,

$$\langle Q^\dagger(T)Q(0)\rangle = \frac{\int[DU]\int[D\phi]Q^\dagger(T)Q(0)e^{-S}}{\int[DU]\int[D\phi]e^{-S}} \ . \tag{2.32}$$

In the large mass limit, the term proportional to γ in S (eq. (2.28)) may be treated as a perturbation, and this path-integral yields, to lowest order in m^{-1} [18],

$$\mathcal{N}\mathcal{W}_{\mathsf{R}}(LT)[U] \ , \tag{2.33}$$

where $\mathcal{N}$ is a constant, and the object $\mathcal{W}_{\mathsf{R}}(LT)[U]$ is the Wilson Loop associated to this path. It is the trace of the product of the link variables along a rectangle of length L in the spatial v direction, and of length T in the temporal direction. Then, from eqs. (2.31) and (2.33), we get

$$\mathcal{W}_{\mathsf{R}}(LT) = \frac{1}{\mathcal{N}}\sum_n e^{-E_n T}\langle 0|Q^\dagger|n\rangle\langle n|Q|0\rangle \ . \tag{2.34}$$

It should be noted that only the energy eigenstates $|n\rangle$ with the same quantum numbers as Q contribute to this sum. Then, in the $T \to \infty$ limit, only the term $n = 0$ survives, yielding

$$e^{-E_0 T} \propto \mathcal{W}_{\mathsf{R}}(LT) \ . \tag{2.35}$$

Therefore, it is possible to obtain the energy of the ground state containing a very massive quark-antiquark pair in representation R from the corresponding Wilson Loop. In particular, an area law for this observable, i.e. $\mathcal{W}_{\mathsf{R}}(LT) \propto e^{-\sigma LT}$, implies a linear static potential. We thus arrive at a possible definition of confinement in pure YM: a theory is said to be confining if arbitrarily large Wilson Loops follow an area law in the fundamental representation. In fact, this is the observed behaviour for SU(N) pure gauge theories, and is a reasonable definition in this case. On the other hand, when dynamical matter is considered, the confining string will break at very large quark-antiquark separations, as the formation of additional particle-antiparticle pairs will become energetically favorable [18]. The Wilson loop is therefore not an order parameter for confinement in QCD. See [38] for a recent discussion of this subject. In that work, the authors propose a weaker criterium for confinement, and argue that it reduces to the area law for pure YM theory.

2.3.1 Some facts regarding the Wilson Loop in Yang-Mills theories

As discussed in the previous section, asymptotic Wilson Loops in the fundamental representation follow an area law in pure $SU(N)$ YM theories. There are, however, other important properties of the confining string that may be probed with this observable. Regarding the dependence of the string tension on the representation D of the Wilson Loop, the observed behaviour is a Casimir scaling at intermediate distances [39]-[47], where the string tension is proportional to the quadratic Casimir of D. At asymptotic distances, the results vary depending on the dimension of spacetime. In 3d, lattice simulations show that the ratio of σ_{R} and σ, the string tension in representation R and in the fundamental, respectively, is given by [48]

$$\frac{\sigma^{(3)}_{\mathsf{R}}}{\sigma^{(3)}} = \frac{k(N-k)}{N-1} \ . \tag{2.36}$$

The number k is known as the $N-$ality of the representation R, and indicates how the center of $SU(N)$ is realized in the representation. The center of a group is the set of elements that commute with all elements of the group. In the case of $SU(N)$, the center is given by $Z(N)$, which has $N-1$ elements, i.e. $k = 1, \ldots, N-1$. More precisely, the $N-$ality is defined by the formula

$$\mathsf{R}(e^{\frac{i2\pi}{N}} I) = e^{\frac{i2\pi k}{N}} I_{\mathsf{R}} \ , \tag{2.37}$$

I, I_{R} being the $N \times N$ and $\mathcal{D} \times \mathcal{D}$ identity matrices ($\mathcal{D}$ is the dimension of the representation R). The behaviour indicated by Eq. (2.36) is known as a Casimir Law, as the quantity $k(N-K)$ corresponds to the quadratic Casimir of the $k-$Antisymmetric representation. This is the most stable representation with a given $n-$ality k. This law is among the possibilities in 4d as well. However, in this case, the lattice data can't distinguish between a Casimir or a Sine law [49]

$$\frac{\sigma^{(4)}_{\mathsf{R}}}{\sigma^{(4)}} = \frac{k(N-k)}{N-1} \quad \text{vs.} \quad \frac{\sigma^{(4)}_{\mathsf{R}}}{\sigma^{(4)}} = \frac{\sin k\pi/N}{\sin \pi/N} \ . \tag{2.38}$$

Regarding the subleading terms in the static quark-antiquark potential, there is evidence [50, 51] for an universal $1/L$ correction at asymptotic distances, i.e.

$$V_{\mathsf{R}}(L) = \sigma_{\mathsf{R}} L - \frac{\pi(d-2)}{24}\frac{1}{L} + O(L^{-2}) \ . \tag{2.39}$$

This contribution is known as a ”Lüscher term”. This same exact term also arises in the vacuum energy of a quantum Nambu-Goto string with its endpoints fixed at a distance L, due to its transverse fluctuations [52, 53]. This ”string character” of the confining potential has been confirmed by comparing the spectrum of its excited states to that of string theory [54, 55]. Moreover, with the choice of an appropriate closed loop $\mathcal{C}_\mathrm{e}$, the observable $W_{\mathcal{C}_\mathrm{e}}$ gives information about the chromoelectric field distribution around the confining string. In $3+1$d, this was studied in Refs. [56, 57, 58] and references therein. The observed profiles turn out to be consistent with that of a Nielsen-Olesen vortex. These results indicate that a soliton-like flux tube is formed between the static quark-antiquark pair.

2.3.2 Wilson Loop and magnetic center symmetry

Despite the fact that we had to introduce matter in the theory to arrive at eq. (2.35), it should be noted that the Wilson Loop is a perfectly well-defined observable for pure Yang Mills theory, as it only depends on the link variables. Therefore, it also makes sense to compute it along more general closed curves C, which may be spatial, temporal, or a combination of both. An interesting question is: what else does it measure in pure YM theory, besides the potential between external sources? This was answered by t' Hooft in his seminal paper [19], which we will review in this section.

The starting point is to consider pure Yang-Mills theory in 3 spacetime dimensions in the Weyl gauge $A_0 = 0$. Then, define spatial operators $\hat{V}(\vec{x_0})$ at fixed time that implement singular gauge transformations $S_0^{\vec{x_0}}$

$$\hat{V}(\vec{x_0})|A\rangle = |A^{S_0^{\vec{x_0}}}\rangle \, . \tag{2.40}$$

$S_0^{\vec{x_0}}$ is defined in such a way that it changes by a center element when going around a closed loop that encircles $\vec{x_0}$, and is thus associated to the introduction of a thin center vortex on the gauge field configuration. In Ref. [59], an explicit representation for this local operator was given. The correlation functions

$$\langle T(\hat{V}(x_1)\dots\hat{V}(x_n)\hat{V}^\dagger(y_1)\dots\hat{V}^\dagger(y_m))\rangle \tag{2.41}$$

would be computed formally by introducing appropriate Dirac strings between the points $x_1\dots x_n$ and $y_1\dots y_m$ in the Yang-Mills path integral. In principle, there are nontrivial contributions for all (n,m) satisfying $n = m(\mathrm{mod}N)$. Then, the proposal is to effectively generate the most important correlation functions through the Lagrangian

$$\mathcal{L} = \partial^\mu\bar{V}\,\partial_\mu V + m^2\,\bar{V}V + \frac{\lambda}{2}\,(\bar{V}V)^2 + \xi\,(V^N + \bar{V}^N)\,, \tag{2.42}$$

V being a complex scalar field. This description includes quadratic and quartic terms, which are the most relevant for the understanding of formation of condensates, as well as the $N-$th order terms that capture the center character of the singular transformations $S_0^{\vec{x_0}}$, representing the possibility that N center vortices may annihilate. Now, consider the possible results for the 3d Euclidean Green's function $\langle \bar{V}(y)V(x)\rangle$. In a Higgs phase ($m^2 > 0$), an exponential decay is expected for $|x-y| \to \infty$, which implies, by the clustering property, that $\langle \hat{V} \rangle = 0$. When $m^2 < 0$, a condensate is formed, and the $Z(N)$ magnetic symmetry is spontaneously broken, i.e. $\langle \hat{V} \rangle \neq 0$.

To understand the implications of these results to the confining properties of the theory, it will be useful to consider the algebra of $\hat{V}$ with the Wilson Loop $\hat{W}_C$ along a spatial curve C:

$$\hat{W}_C \hat{V}(\vec{x_0}) = \hat{V}(\vec{x_0}) \hat{W}_C \, e^{i\frac{2\pi L(C,\vec{x_0})}{N}} \; , \tag{2.43}$$

$L(C, \vec{x}_0)$ being the linking number between C and $\vec{x_0}$, which is one if $\vec{x_0}$ lies within C and zero otherwise. Now, consider the basis $|V\rangle$ where the operator $\hat{V}$ is diagonal. Then,

$$\hat{V}(\vec{x})\hat{W}_C|V\rangle = e^{-i\frac{2\pi L(C,\vec{x})}{N}} V(\vec{x})\hat{W}_C|V\rangle \; , \tag{2.44}$$

i.e. the Wilson Loop operator implements a magnetic $Z(N)$ transformation in the interior $I(C)$ of the curve C. In a phase where the magnetic $Z(N)$ symmetry is spontaneously broken, the vacuum $|\Omega\rangle$ is by definition not invariant under the action of the Wilson Loop operator, and it is thus clear that $\langle \Omega|\hat{W}_C|\Omega\rangle$ will be the overlap of two inequivalent states for $\vec{x} \in I(C)$. In this case, it is thus reasonable to expect an effect proportional to the area of the minimal surface whose border is the Wilson Loop. On the other hand, if $m^2 > 0$, the magnetic $Z(N)$ symmetry is not spontaneously broken, and $\hat{W}_C$ leaves $|\Omega\rangle$ invariant, implying a perimeter law. We can reverse the argument to conclude that an area (resp. perimeter) law for the Wilson loop implies a phase of YM where the magnetic center symmetry is (resp. not) spontaneously broken. Finally, it is important to emphasize that this discussion shows that the 't Hooft operator $\hat{V}(x)$ serves as an alternative (dis)order parameter for confinement. The generalization of the 't Hooft operator to higher dimensions is done by means of Eq. (2.43). In particular, in 4 dimensions it is defined along a loop $\mathcal{C}'$. As argued in 't Hooft's paper, in the absence of massless particles in the spectrum, this operator follows a perimeter law in the confining phase, and an area law otherwise.

2.4 Color confinement x flux tube

It is well-known that the observed spectrum of QCD consists only of particles with neutral color charge. In fact, this holds more generally for a gauge-Higgs theory, as discussed in Ref. [60]. In the following we will, following Ref. [18], briefly review some ideas to illustrate why this is true. Consider SU(2) Yang-Mills-Higgs theory, where the gauge field is coupled to scalars ϕ in the fundamental representation. Moreover, let the modulus of the Higgs field be fixed. The lattice action of this theory is [61]

$$S = \beta \sum_{p} Tr[UUU^{\dagger}U^{\dagger}] + \gamma \sum_{x,\mu} \frac{1}{2} Tr[\phi^{\dagger}(x) U_{\mu}(x) \phi(x+\hat{\mu})] \ . \qquad (2.45)$$

The first term is just the discretized YM action. The second contains the coupling of the gauge field with the discretized scalar field $\phi(x)$, and its corresponding kinetic term. For sufficiently large γ this is a theory that is very similar to that of the weak interactions, with massive vector bosons, implying a Yukawa potential. For small γ, however, the dynamics is resemblant of ordinary QCD, with the formation of flux tubes between the color sources, and string breaking for large separations. Osterwalder and Seiler proved [62] that, for any two points of the parameter space (β, γ), there is a path for which all local, gauge-invariant observables of the quantum theory of (2.45) vary smoothly along the path. In particular, this implies that there is such a path connecting pure YM theory, which is confining in the sense that the Wilson Loops follow area laws, to a theory with massive bosons, with Yukawa interactions. This theorem implies, as emphasized in Ref. [60], that if color confinement exists in the limit of small γ, the spectrum of the theory in the Higgs phase is also composed solely of colorless states. An interesting implication of this result is that the absence of colorful particles in the spectrum does not imply the existence of a flux tube. The mechanism responsible for the color confinement is completely different in these two limits. For small γ, a confining flux tube is formed between the colored sources, which is broken at sufficiently large distances if the creation of an additional particle-antiparticle pair is energetically favorable. In the Higgs-like limit, it is the short-rangedness of the Yukawa interactions that make the color field undetectable far from the source.

Chapter 3

Ensembles of center vortices

Center vortices are gauge field configurations a_μ defined by their contribution to the Wilson Loop along a curve $\mathcal{C}_{\mathsf{e}}$, in a given representation R of $SU(N)$:

$$\mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[a_\mu] = \frac{1}{\mathcal{D}}\,\mathrm{tr}\,\left[\mathsf{R}\left(e^{i\frac{2\pi}{N}}I\right)\right]^{\mathsf{L}(\omega,\mathcal{C}_{\mathsf{e}})} = e^{i\frac{2\pi k}{N}\mathsf{L}(\omega,\mathcal{C}_{\mathsf{e}})}\ , k = 1,\ldots,N\ , \tag{3.1}$$

where $\mathsf{L}(\omega, \mathcal{C}_{\mathsf{e}})$ is the total linking number between $\mathcal{C}_{\mathsf{e}}$ and ω, the vortex guiding center. That is, the contribution of a center-vortex to the Wilson loop is an element of the center $Z(N)$ of $SU(N)$ raised to the linking number between ω and $\mathcal{C}_{\mathsf{e}}$. This definition has important implications:

1. In d space-time dimensions, these objects must be located at $d-2$ dimensional closed hypersurfaces, as these are able to link curves.

2. These objects are automatically compatible with the observed $N-$ality of the Wilson Loop [63].

In particular, they are concentrated in closed worldsurfaces in $d = 3+1$ dimensions, or in closed lines in $d = 2+1$. These objects were primarily defined and studied in the lattice, where they may be identified after fixing the gauge to Direct Maximal Center Gauge [64] (there are other possibilities, see [18]). The gauge-fixed link variables $U_\mu(x)$ are then decomposed as $U_\mu(x) = Z_\mu(x)V_\mu(x)$, where $Z_\mu(x)$ is the center element of $SU(N)$ that is closest to $U_\mu(x)$. A plaquette is then said to be pierced by a thin vortex (or $p-$ vortex) if the product of its center elements is nontrivial. With these concepts, it is possible to identify vortex configurations in the lattice and investigate their role on the confining properties of YM theory. In particular, the Wilson Loop may be computed by considering only the contribution from center vortices, and the result is an area law with the correct string tension σ [64, 65, 66]. As a consistency check, this same observable may be computed by removing the vortices, and the result is a perimeter law (associated to a nonconfining theory). There is also strong numerical evidence that these field configurations are key to explain the high temperature deconfinement transition [67, 68, 69, 70, 71, 72, 73], chiral symmetry breaking [74, 75], and the topological susceptibility [75, 76]. These evidences motivate the idea that YM theory in the infrared regime could be described by an ensemble of center vortices. In the following sections we present some ensembles and discuss the various physical properties that they are able to describe.

3.1 The simplest vortex ensemble

The simplest realization of a vortex ensemble is to compute a planar Wilson Loop of area A in the presence of V vortices in the percolating regime (very large, straight vortices everywhere) in SU(2) gauge theory[18]. Also, let us imagine that spacetime is a box of length L. Then, the probability that i of the vortices will link the Wilson Loop is

$$P(i) = \frac{V!}{i!(V-i)!}\left(\frac{A}{L^2}\right)^i\left(1-\frac{A}{L^2}\right)^{V-i} . \tag{3.2}$$

For each linking point there is a contribution of -1 to the Wilson Loop. Therefore,

$$\mathcal{W}(\mathcal{C}_{\mathsf{e}}) = \sum_{i=0}^{V}(-1)^i P(i) = \left(1-\frac{2A}{L^2}\right)^V , \tag{3.3}$$

where the absence of a superscript means that R is the fundamental representation. Assuming that the vortex density $\rho = \frac{V}{L^2}$ is fixed in the limit $V \to \infty$, we arrive at

$$\mathcal{W}(\mathcal{C}_{\mathsf{e}}) = e^{-2\rho A} , \tag{3.4}$$

which is an area law for the Wilson Loop. Of course, this is an extremely simplified model, as no action for the vortices is considered. In particular, this simplified picture is not able to explain Casimir Scaling, or the emergence of a fluctuating flux tube.

3.2 Intermediate Casimir Scaling

In this section, we review an improved version of the simplest ensemble presented in the previous section. The idea is to consider the possibility of center vortex thickness, where the singular core of these configurations is smoothed out in a finite region. This approach was introduced in Refs. [77, 78, 79] (see also [80]). The idea is to consider the existence of various vortex domains with total flux $z_j = e^{i2\pi j/N}$, as measured by a fundamental Wilson Loop. The contribution of such an object is assumed to be captured by the insertion of a group element G^j in the holonomy. More precisely, in the lattice the Wilson Loop along a curve $\mathcal{C}_{\mathsf{e}}$ in representation R is given by

$$\mathsf{Tr}_{\mathsf{R}}(U\dots U)\ , \tag{3.5}$$

where $U\dots U$ represents the ordered product of the link variables along the closed curve $\mathcal{C}_{\mathsf{e}}$. These link variables are assumed to be in a general irreducible representation R of $SU(N)$. Then, the contribution of a domain of type j is modeled by the replacement

$$\begin{aligned}&\mathsf{Tr}_{\mathsf{R}}(U\dots U)\to\mathsf{Tr}_{\mathsf{R}}(U\dots G^j\dots U)\ ,\\&G^j(x,S)=\frac{1}{\mathcal{D}}Se^{i\theta_C(x)\beta^j\cdot T}S^\dagger\ ,\end{aligned} \tag{3.6}$$

where $\theta_{\mathcal{C}_{\mathsf{e}}}(x)\in[0,2\pi]$ measures how much of the domain pierces $S(\mathcal{C}_{\mathsf{e}})$, the minimal surface whose border is $\mathcal{C}_{\mathsf{e}}$. Specifically, $\theta_{\mathcal{C}_{\mathsf{e}}}$ is equal to 2π if the domain is fully contained within $S(\mathcal{C}_{\mathsf{e}})$, and it is zero if the domain has zero overlap with $S(\mathcal{C}_{\mathsf{e}})$. Moreover, if the overlap between $S(\mathcal{C}_{\mathsf{e}})$ and the domain is partial, $\theta_{\mathcal{C}_{\mathsf{e}}}$ will be a number between 0 and 2π, proportional to the overlap size. The different β^j, $j=1,\dots,N$ are proportional to the weights of the fundamental representation, and label the N possible types of center vortices. The generators T^a and the link variables U are assumed to be in the representation R of $SU(N)$, whose dimension we will denote by d_R. Moreover, the product $\beta\cdot T$ denotes $\beta|_q T_q$, where T_q are the generators of the Cartan subalgebra. Then, the next assumption is that the group orientation given by S is random, and should be averaged. In this case, the full contribution of a domain of type j is given by

$$\begin{aligned}&\bar{G}^j(x)=\frac{1}{\mathcal{D}}\int dS\,Se^{i\theta_C(x)\beta^j\cdot T}S^\dagger=\frac{1}{\mathcal{D}}\mathsf{Tr}\,e^{i\theta_C(x)\beta^j\cdot T}I_{\mathcal{D}}=\mathcal{G}^j_{\mathsf{R}}(x)I_{\mathcal{D}}\ .\\&\mathcal{G}^j_{\mathsf{R}}(x)\equiv\frac{1}{\mathcal{D}}\mathsf{Tr}\,e^{i\theta_C(x)\beta^j\cdot T}\end{aligned} \tag{3.7}$$

To obtain this result, the orthogonality relation

$$\int d\mu(g)D^{(i)}(g)|_{ab}D^{(j)}(g^{-1})|_{cd}=\delta_{ij}\delta_{ad}\delta_{bc} \tag{3.8}$$

was used, where $d\mu(g)$ is the Haar measure of $SU(N)$, and $D^{(i)}$ stands for the $i-$ irreducible representation of this group. Then, the contribution of m domains of type e.g. 1, centered at $x_1,\dots,x_m$ amounts to

$$\mathsf{Tr}_{\mathsf{R}}(U\dots U)\to\mathcal{G}^1_{\mathsf{R}}(x_1)\dots\mathcal{G}^1_{\mathsf{R}}(x_m)\mathsf{Tr}_{\mathsf{R}}(U\dots U)\ . \tag{3.9}$$

This allowed the authors to approximate the average of the Wilson Loop in the presence of these domains as

$$\langle W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}};\{x_j\})\rangle \approx \prod_i \mathcal{G}^1_{\mathsf{R}}(x_i)\langle W_0(\mathcal{C}_{\mathsf{e}})\rangle \ , \tag{3.10}$$

where $\langle W_0(\mathcal{C}_{\mathsf{e}})\rangle$ is the average of configurations which do not contain vortices, or a perturbative average, following a perimeter law. The final assumption is to consider that the probabilities of finding domains in any two different plaquettes is independent. Then, an estimate for $\langle W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})\rangle$ is obtained from the formula

$$\langle W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})\rangle \approx \langle W_{\mathsf{R},0}(\mathcal{C}_{\mathsf{e}})\rangle \prod_x ((1-\sum_{j=1}^{N} f_j) + \sum_{j=1}^{N} f_j \mathcal{G}^j_{\mathsf{R}}(x)) \ , \tag{3.11}$$

where f_j is the probability that a given plaquette is pierced by a domain of type j, and the product runs over all sites of the plane A' of the lattice that contains the loop $\mathcal{C}_{\mathsf{e}}$. In the general case, Eq. (3.11) may be written as

$$\langle W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})\rangle \approx \langle W_{\mathsf{R},0}(\mathcal{C}_{\mathsf{e}})\rangle \ \exp \sum_x \left[\ln(1-\sum_{j=1}^{N} f_j(1-\mathcal{G}^j_{\mathsf{R}}(x))\right] = \langle W_{\mathsf{R},0}(\mathcal{C}_{\mathsf{e}})\rangle \ \exp\left[-\sigma^d_C A\right] \ ,$$

$$\sigma^{\mathsf{R}}_{\mathcal{C}_{\mathsf{e}}} \equiv -\sum_x \frac{1}{A} \ln(1-\sum_{j=1}^{N} f_j(1-\mathcal{G}^j_{\mathsf{R}}(x)) \ . \tag{3.12}$$

This is not quite an area law, as the quantities $\mathcal{G}^j_{\mathsf{R}}(x)$ depend on the loop $\mathcal{C}_{\mathsf{e}}$ through the angle $\theta_{\mathcal{C}_{\mathsf{e}}}(x)$. For small and intermediate loops $\mathcal{C}_{\mathsf{e}}$, the authors argued that both f_j and $\theta_{\mathcal{C}_{\mathsf{e}}}(x)$ will be very small, so that the following approximation may be considered

$$\mathcal{G}^j_{\mathsf{R}}(x) \approx 1 - \frac{1}{2\mathcal{D}}\beta^{(j)}_q\beta^{(j)}_p(\theta_C(x))^2 \mathrm{Tr}(T_qT_p) = 1 - \frac{C^{(2)}_{\mathsf{R}}}{2(N^2-1)}(\beta^{(j)})^2(\theta_{\mathcal{C}_{\mathsf{e}}}(x))^2 \ , \tag{3.13}$$

where the formula

$$\mathrm{Tr}\, T_qT_p = \mathcal{D}\,\delta_{qp}\frac{C^{(2)}_{\mathsf{R}}}{N^2-1} \ , \tag{3.14}$$

$C^{(2)}_{\mathsf{R}}$ being the quadratic Casimir of the representation R, was used. With these approximations, Eq. (3.12) becomes

$$\sigma^{\mathsf{R}}_{\mathcal{C}_{\mathsf{e}}} \approx \frac{1}{A}\left[\sum_x \sum_{j=1}^{N-1} \frac{f_j}{2(N^2-1)}(\beta^{(j)})^2\theta^2_{\mathcal{C}_{\mathsf{e}}}(x)\right] C^{(2)}_{\mathsf{R}} \ . \tag{3.15}$$

This formula should be studied carefully, as it only makes sense to interpret $\sigma^d_{\mathcal{C}_\mathsf{e}}$ as a string tension if the term in brackets is proportional to the area A of $S(\mathcal{C}_\mathsf{e})$. This will strongly depend on the profile used for the function $\theta_{\mathcal{C}_\mathsf{e}}(x)$[79]. Choosing the profiles appropriately, the authors showed that the ratio of the string tensions in two different representations R, R' is

$$\frac{\sigma_C^{\mathsf{R}}}{\sigma_C^{\mathsf{R}'}} = \frac{C^{(2)}_{\mathsf{R}}}{C^{(2)}_{\mathsf{R}'}} , \tag{3.16}$$

which is the expected Casimir scaling at intermediate distances.

3.3 More general center-vortex ensembles

The ensembles presented in the previous sections rely on very simple and powerful ideas regarding the definition of center vortices which are independent of the dimension of space-time. Despite being very useful for providing a simple understanding for the emergence of an area law for the Wilson Loop and a possible explanation for the Casimir Scaling of the string tension at intermediate distances, they are not sufficient to describe all the confining properties of YM theory. Among the properties which are not contemplated are: the emergence of a confining flux tube, the asymptotic scaling law of the string tension, the transition to a deconfined phase for higher temperatures, the existence of a nonvanishing chiral condensate when fermions are included, and the topological susceptibility of the vacuum. For this purpose, it is necessary to consider a more general setup, which includes not only the center element contribution from the vortex configurations, but also an effective vortex weight. That is, we are interested in approximating Wilson Loop averages in a general representation R by expressions of the form

$$\langle W_{\mathsf{R}}(\mathcal{C}_\mathsf{e})\rangle \approx \mathcal{N}\sum_{\Omega} e^{-S(\Omega)}\frac{1}{d_{\mathsf{R}}}\mathsf{Tr}\left[R(e^{i\frac{2\pi}{N}}I)\right]^{L(\Omega,\mathcal{C}_\mathsf{e})} , \tag{3.17}$$

where Ω is the guiding center of the vortex, which is located in a closed $d-2$ dimensional hypersurface in d spacetime dimensions. In principle all surfaces should be considered, regardless of orientation. The choice of the effective vortex weight $e^{-S(\omega)}$ is made based on phenomenological inputs and on properties of the vortex configurations observed in the lattice. In particular, terms proportional to the area (tension term) and to the extrinsic curvature (stiffness term) of ω should be included in $S(\omega)$ for $d = 3, 4$ [81, 82]. For example, in $d = 4$ an ensemble of fluctuating vortex closed worldsurfaces was introduced in the lattice in Ref. [76], with an action

$$S_{latt}(\omega) = \mu A(\omega) + cN_p \ , \tag{3.18}$$

where $A(\omega)$ is the area of the surface ω and N_p is the number of pairs of neighboring plaquettes which lie on different planes, and is thus a lattice version of a stiffness term. This model was initially introduced for $SU(2)$, and later generalized for $SU(3)$ [83], and is able to describe the fundamental string tension with a reasonable accuracy, and also other important confining properties such as the order of the deconfinement transition, however the scaling of the string tension in different representations was not studied. When it comes to center-vortex ensembles in the continuum, the inclusion of stiffness is again important in order to ensure a well-defined limit when the monomer size goes to zero in $d = 3$ [84, 85], and to avoid a branched polymer phase in $d = 4$ [86, 87]. In this regard, some interesting proposals have been made, both in $d = 3$ and $d = 4$. The different proposals differ mainly in the inclusion or not of non-oriented configurations, and turn out to be more complete and satisfactory in the latter case. In the following subsections we will present the different proposals in both $3d$ and $4d$.

3.3.1 Center vortices in the continuum

The starting point for the construction of the vortex ensemble in the continuum is the identification of these variables. In this regard, explicit expressions for the gauge field configurations of center vortices in the continuum for arbitrary spacetime dimension were given for the first time in [66]. A typical center-vortex is given by $a^{\Omega}_{\mu} = \beta \cdot T\partial_{\mu}\chi$, $\beta = 2N\omega$, where ω is a weight of the fundamental representation of $SU(N)$ and χ is an angle that is multivalued with respect to the vortex closed hypersurface Ω. The weights w are tuples with $N-1$ components containing the eigenvalues of the Cartan generators T_q, $q = 1, \dots N$ (see Appendix A for details regarding our Lie Algebra conventions). Moreover, we defined $\beta \cdot T \equiv \beta_q T_q$. This gauge field is locally (but no globally) equivalent to a pure gauge with the phase

$$S = e^{i\chi\beta\cdot T} \ . \tag{3.19}$$

In order to see that this indeed a center-vortex configuration, we must show that it has the following expression for the Wilson Loop along a curve $\mathcal{C}_{\mathsf{e}}$

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[a^{\Omega}_{\mu}] = \mathsf{Tr}_{\mathsf{R}} P\left(e^{i\oint_C a^{\Omega}_{\mu} dx_{\mu}}\right) = \frac{1}{d_{\mathsf{R}}}\mathsf{tr}\left[R(e^{i\frac{2\pi}{N}}I)\right]^{L(\Omega,\mathcal{C}_{\mathsf{e}})} \ , \tag{3.20}$$

where P stands for path-ordering, R is a general representation of $SU(N)$, and $L(\Omega, \mathcal{C}_{\mathsf{e}})$ is the linking number between the vortex hypersurface Ω and $\mathcal{C}_{\mathsf{e}}$. For this purpose, notice that, as said before, locally we may write

$$a_\mu^\Omega = \frac{i}{g} S \partial_\mu S^{-1} \ . \tag{3.21}$$

Then, notice the following property of an holonomy along a general curve C, which starts at x_i and ends at x_f:

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[A_\mu^S] = S(x_f) W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[A_\mu] S^{-1}(x_i) \ . \tag{3.22}$$

Applying this relation for the case of the closed loop $\mathcal{C}_{\mathsf{e}}$, with $x_i = x_f = x$ being some arbitrarily chosen point along the curve, we get:

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[a_\mu^\Omega] = S(x(\sigma_f)) S^{-1}(x(\sigma_i)) \ , \tag{3.23}$$

where $x(\sigma)$ is a parametrization of the closed loop $\mathcal{C}_{\mathsf{e}}$. Now, if the vortex line does not link the Wilson Loop, the angle χ will not go through nontrivial changes upon completion of the path $\mathcal{C}_{\mathsf{e}}$, and the contribution of a_μ^Ω will be trivial. However, if l links $\mathcal{C}_{\mathsf{e}}$, the multivalued phase will change by 2π for each time that the hypersurface Ω links the curve $\mathcal{C}_{\mathsf{e}}$, and we will have

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[a_\mu^\Omega] = S(\chi = 2\pi L(\Omega, \mathcal{C}_{\mathsf{e}})) S^{-1}(\chi = 0) = S(\chi = 2\pi L(\Omega, \mathcal{C}_{\mathsf{e}})) = e^{i 2\pi \beta \cdot T L(\Omega, \mathcal{C}_{\mathsf{e}})} \ , \tag{3.24}$$

Then, notice

$$e^{i 2\pi \beta \cdot T} = e^{i 2\pi 2N\omega \cdot T} = e^{-i\frac{2\pi}{N}} I \ , \tag{3.25}$$

where we have used that two fundamental weights ω_q, ω_p satisfy

$$\omega_q \cdot \omega_p = \frac{N\delta_{qp} - 1}{2N^2} \ . \tag{3.26}$$

Therefore, we have showed that Eq. (3.20) holds for a_μ^l, and hence that this is a center-vortex configuration. In order to get a more concrete picture of this configuration, let us consider the specific case for $d = 4$ where the vortex guiding center is located along the infinite $x_0 = 0, x_3 = 0$ plane. Then, the angle χ is simply the polar angle φ, and we may represent the vortex configuration a_μ locally as

$$a_\mu = \frac{1}{g} \partial_\mu \varphi \beta \cdot T \ . \tag{3.27}$$

It should be emphasized that this representation is valid only locally, i.e. the δ singularity in the derivative of φ should be disregarded. A representation that makes this explicit is

$$a_\mu = \frac{i}{g} Ad(S)\partial_\mu Ad(S^{-1}) \; , \tag{3.28}$$

where Ad stands for the adjoint representation. The representations (3.27) and (3.28) are equivalent. In order to better understand this configuration, let us compute its field strength:

$$F_{\mu\nu} = \frac{1}{g}\beta \cdot T[\partial_\mu, \partial_\nu]\varphi \tag{3.29}$$

where we cannot assume that $[\partial_\mu, \partial_\nu]\varphi = 0$, as this is a multivalued function. Indeed,

$$F_{12} = -F_{21} = -\frac{2g}{x_1^2 + x_2^2}\beta \cdot T \; , \tag{3.30}$$

and all other components vanish. This expression confirms again our claim that the vortex configuration, in spite of Eq. (3.21), is not a trivial configuration, as the $SU(N)$ valued transformation $S(x)$ is not regular, and is thus not a gauge transformation. In fact, this vortex configuration is singular for $x_1 = x_2 = 0$. This type of singularity may be smoothed by considering a more general vortex configuration which satisfies Eq. (3.21) only asymptotically, while being accompanied by some smooth profile that vanishes at the vortex guiding center in order to eliminate possible singularities. In this regard, the ”bare-bone” configurations which satisfy Eq. (3.21) are named thin center vortices, while the smoothed ones are known as thick center vortices. It is then natural to wonder whether a center-vortex ensemble should consider all possible vortex configurations with all possible smoothings for a given guiding-center location, or if it is sufficient to consider just one representative. For now we will simply assume the latter, and on chapter 4, we will provide a possible justification for this.

3.4 Non-abelian Center-vortex ensembles in 3d

Center vortices in $2+1$ dimensions are gauge field configurations a_μ^l localized on closed loops l. It is therefore natural to expect that ensembles of these configurations, in the sense of Eq. (3.17), will be described by effective field theories. In the following we present one possible ensemble that was recently proposed in Ref. [88], derive the corresponding effective field theory and discuss its physical consequences.

In $d = 3$, the Wilson Loop of these configurations satisfy

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[a_\mu^l] = \left(e^{i2\pi\beta\cdot\omega_e}\right)^{L(l,\mathcal{C}_{\mathsf{e}})} \; , \tag{3.31}$$

where ω_e is a weight of the quark representation, and $L(l, \mathcal{C}_\mathrm{e})$ is the linking number between the closed vortex line l and the curve $\mathcal{C}_\mathrm{e}$. Also, as the fundamental weights satisfy Eq. (3.26), the vortex contribution to the Wilson Loop may be rewritten as

$$W_R(\mathcal{C}_\mathrm{e})[a^l_\mu] = W_l[j^{\mathcal{C}_e}_\mu] = \frac{1}{N}\mathrm{Tr}P\left(e^{i\oint j^{\mathcal{C}_\mathrm{e}}_\mu}\right) \,, \tag{3.32}$$

with $j^{\mathcal{C}_\mathrm{e}}_\mu \equiv 2\pi\beta_e \cdot T s_\mu$, s_μ being a source concentrated on a surface $S(\mathcal{C}_\mathrm{e})$ whose border is $\mathcal{C}_\mathrm{e}$, given explictly by

$$s_\mu(x) = \frac{1}{2}\int d\tau_1 d\tau_2 \epsilon_{\mu\nu\rho}\frac{\partial x^\nu}{\partial\tau_1}\frac{\partial x^\rho}{\partial\tau_2}\delta(y(\tau_1,\tau_2)-x) \,, \tag{3.33}$$

$y(\tau_1,\tau_2)$ being a parametrization of the surface $S(\mathcal{C}_\mathrm{e})$. The source s_μ is defined so as to assure that

$$\oint_l s_\mu dx_\mu = I(S(\mathcal{C}_\mathrm{e}), l) = L(\mathcal{C}_\mathrm{e}, l) \,, \tag{3.34}$$

where $I(S(\mathcal{C}_\mathrm{e}), l)$ is the intersection number between the surface $S(C)$ and the closed vortex worldline l. The representation (3.32) allows the inclusion of an action for the vortices in the Wilson average in a natural way, a typical contribution being

$$e^{-\int_0^L \frac{1}{2\kappa}\dot{u}_\mu\dot{u}_\mu + \mu} W_l[j^{\mathcal{C}_\mathrm{e}}_\mu] \,, \tag{3.35}$$

L is the length of the loop. The parameters κ and μ introduce stiffness and tension for the vortex worldlines, respectively. These are phenomenological parameters introduced to implement properties that are observed on the lattice [81, 82]. We also defined the unit tangent vector to the vortex worldline

$$\dot{u}_\mu(s) = \frac{du_\mu}{ds} \,,\; u_\mu(s) = \frac{dx_\mu}{ds} \,. \tag{3.36}$$

Then, the Wilson average is computed in the ensemble by summing the contribution from any number of vortices V of all possible sizes and locations in space-time

$$\begin{aligned} Z_l[j^{\mathcal{C}_\mathrm{e}}_\mu] =& \langle W_{loops}[j^{\mathcal{C}_\mathrm{e}}_\mu]\rangle = \sum_{V=0}^{\infty}\prod_{i=1}^{V}\int_0^\infty \frac{dL_k}{L_k}\int dx_k \int du_k \times \\ & \times \int [Dx^{(k)}]^{L_k}_{x_k,u_k} e^{-\int_0^{L_k} ds_k\left(\frac{1}{2\kappa}\dot{u}^{\mu(k)}_k\dot{u}^{\mu(k)}_k + \mu\right)} W_{l_k}[j^C_\mu] \,. \end{aligned} \tag{3.37}$$

The measure $[Dx^{(k)}]^{L_k}_{x_k,u_k}$ path-integrates over all loops of length L_K starting and ending at x_k with unit tangent vector u_k. The object $Z_l[j_\mu]$ may also be written as

$$Z_l[j_\mu] = e^{\int_0^\infty \frac{dL}{L} \int dx \int du\, \mathrm{tr}\, Q(x,u;x,u;L)} ,$$
$$Q(x,u;x_0,u_0;L) = \int [Dx]^L_{x,u;x_0,u_0} e^{-\int_0^L ds\left(\frac{1}{2\kappa}\dot{u}_\mu \dot{u}_\mu + \mu\right)} \Gamma_\gamma[j_\mu] = P\left(e^{i\int_\gamma j_\mu(x) dx_\mu}\right) . \quad (3.38)$$

That is, the ensemble of vortex loops may be written in terms of the building block $Q(x,u;x_0,u_0,L)$, which is simply the probability amplitude for a line to start at x_0 with orientation u_0, and end at x with final tangent vector u. Next, by using the techniques of Refs [89, 90], it is possible to derive a diffusion equation for this building block

$$\left(\partial_L - \frac{\kappa}{2}\hat{L}_u^2 + \mu + u_\mu(\partial_\mu - ij_\mu)\right) Q(x,u;x_0,u_0;L) = 0 , \quad (3.39)$$

where $\hat{L}_u^2$ is the Laplace operator in the unit sphere. This is just a Schrödinger-like equation where the time is replaced by the length of the curve. It is to be solved with the initial condition $Q(x,u;x_0,u_0,0) = \delta^{(3)}(x-x_0)\delta^{(2)}(u-u_0)I_N$. In the small κ limit, where the loops are very flexible, the solution may be approximated as

$$Q(x,u;x_0,u_0;L) \approx \langle x|e^{-LO}|x_0\rangle \ , \ O = -\frac{1}{3\kappa}(\partial_\mu - ij_\mu)^2 + \mu I_N . \quad (3.40)$$

Then,

$$Z_l[j_\mu] \approx e^{-\mathrm{Tr}\ln O} = (\det O)^{-1} = \int [D\varphi] e^{-\int d^3x\, \varphi^\dagger O \varphi} , \quad (3.41)$$

φ being a complex field in the fundamental representation of $SU(N)$.

3.4.1 Including vortex correlations in the ensemble

Up to this point, only noninteracting vortex loops were included in the ensemble. A more complete description may be obtained by including natural correlations of these objects. One possibility is to include the creation of N vortices at some initial point x_0, which propagate along open paths γ_i, and then annihilate at a common final point x_f. This is possible as the N different weights of the fundamental representation of $SU(N)$ satisfy $\sum_{i=1}^N \omega_i = 0$. This property allows us to visualize this as a configuration of $N-1$ center vortices with guiding centers along the loops l_j, $j = 1,\ldots,N-1$, which are formed by the composition of γ_i with γ_N^{-1} (see Fig. 3.1). These vortex loops carry weights $\beta_1,\ldots,\beta_{N-1}$. This implies that the corresponding gauge field is $A_\mu^{NV} = \sum_{i=1}^{N-1} \beta_i \cdot T \partial_\mu \chi_i$, where χ_i is a multivalued angle when going around the vortex loop l_i. Its contribution to the Wilson Loop is

$$W_{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})[A_\mu^{NV}] = \left(e^{\frac{i2\pi k}{N}}\right)^{L(S(C),l_1)+\cdots+L(S(C),l_{N-1})} . \quad (3.42)$$

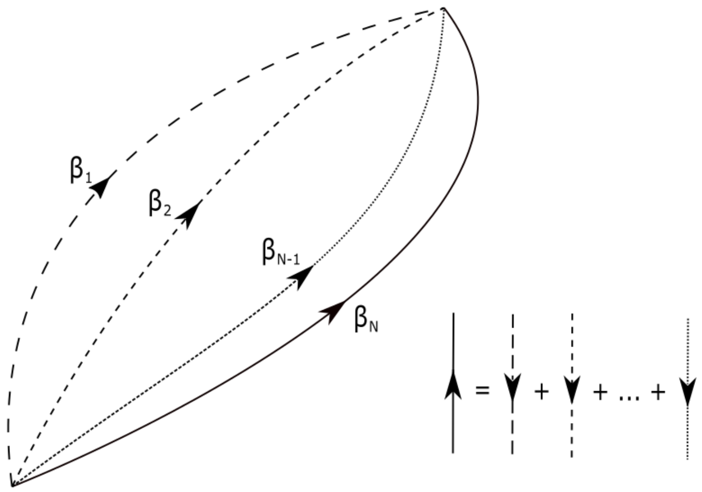


Figure 3.1: The N center-vortex creation-annihilation process . The lines γ_i are labeled by the corresponding weights β_i.

In analogy to (3.32), this can be rewritten in terms of holonomies Γ^{γ_i} along the curves γ_i, as follows

$$W_{\mathrm{R}}(\mathcal{C}_{\mathrm{e}})[A_\mu^{NV}] = \frac{1}{N!}\epsilon_{i_1\dots i_N}\epsilon_{i'_1\dots i'_N}\Gamma^{\gamma_1}_{i_1 i'_1}\dots\Gamma^{\gamma_N}_{i_N i'_N} \ . \tag{3.43}$$

Then, the contribution C_N of this correlation to the Wilson average is obtained by adding the phenomenological action cost for each vortex line

$$C_N \propto \int d^3x_0\, d^3x \pi_{i=1}^N \int dL_i du^i du_0^i \int [Dx^{(i)}]^{L_i}_{x_0,u_0;x,u} e^{-\int_0^{L_i}\left(\frac{1}{2\kappa}\dot{u}_\mu^{(i)}\dot{u}_\mu^{(i)}+\mu\right)} D_N \ ,$$
$$D_N = \epsilon_{i_1\dots i_N}\epsilon_{j_1\dots j_N}\Gamma_{\gamma_1}[j_\mu]_{i_1 j_1}\dots\Gamma_{\gamma_N}[j_\mu]_{i_N j_N} \ . \tag{3.44}$$

Next, notice that eqs. (3.38) and (3.39) imply, for each line,

$$\int dL\, du\, du_0 \int [Dx]^L_{x_0,u_0;x,u} e^{-\int_0^L ds(\dot{u}_\mu\dot{u}_\mu+\mu)}\Gamma[j_\mu] = \int_0^\infty dL\, du\, du_0\, Q(x,u;x_0,u_0;L)$$
$$\propto G(x,x_0) \ , \tag{3.45}$$

where $OG(x,x_0) = \delta(x-x_0)I_N$. This means that

$$C_N \propto \int d^3x\, d^3x_0\, \epsilon_{i_1\dots i_N}\epsilon_{j_1\dots j_N} G(x,x_0)_{i_1 j_1}\dots G(x,x_0)_{i_N j_N} \ , \tag{3.46}$$

which strongly suggests that this correlation may be generated by the introduction of an appropriate interaction in the effective field theory. Even though the contribution of the loops may be written in terms of a single complex field φ (see Eq. (3.41)), this is not possible for the N lines correlation, due to the presence of the Levi-Civita tensors. Instead, we need to consider N fundamental fields (the same number of weights of $SU(N)$) and an appropriate interaction among them in order to reproduce (3.44). Specifically, the following partition function is able to reproduce them

$$\int [D\Phi^\dagger][D\Phi] e^{-\int d^3x \left(\frac{1}{3\kappa}\mathrm{Tr}((D_\mu\Phi)^\dagger D_\mu\Phi)+\mu\mathrm{Tr}(\Phi^\dagger\Phi)-\xi_0(\det\Phi+\det\Phi^\dagger)\right)} , \tag{3.47}$$

where the covariant derivative is defined by $D_\mu\Phi = (\partial_\mu - ij_\mu)\Phi$, and the components of Φ by $\Phi_{ij} = \phi^j|_i$, where each ϕ^j, $j = 1,\ldots,N$ is a complex field in the fundamental representation. That is, lines and columns of Φ are associated to color and flavor indices of the fundamental fields, respectively. A perturbative expansion of this interaction of to second order gives

$$\int [D\Phi^\dagger][D\Phi] \big(1 + \xi_0^2 \int d^3x \int d^3x_0$$
$$\epsilon_{i_1\ldots i_N}\epsilon_{i'_1\ldots i'_N}\,\varphi^{i'_1}|_{i_1}\ldots\varphi^{i'_N}|_{i_N}\epsilon_{j_1\ldots j_N}\epsilon_{j'_1\ldots j'_N}\,\bar{\varphi}^{j'_1}|_{j_1}\ldots\bar{\varphi}^{j'_N}|_{j_N} + \ldots\big)\, e^{-\int d^3x\,\bar{\varphi}^j|_i O^{jj'}_{ii'}\varphi^{j'}|_{i'}} . \tag{3.48}$$

The first term is simply Z_l^N, which is the contribution of N uncorrelated loop types. Then, by multiplying and diving the second order term by

$$(\det O)^{-N} = \int [D\Phi^\dagger][D\Phi]\, e^{-\int d^3x\,\bar{\varphi}^j|_i O^{jj'}_{ii'}\varphi^{j'}|_{i'}} \quad , \quad O^{jj'}_{ii'} \equiv \delta^{jj'}O_{i'i} ,$$

it is possible to use Wick's theorem to obtain

$$\begin{aligned} &\tfrac{\xi_0^2}{(N!)^2}\int [D\Phi^\dagger][D\Phi]\,\int d^3x\int d^3x_0\,\epsilon_{i_1\ldots i_N}\epsilon_{i'_1\ldots i'_N}\varphi^{i'_1}|_{i_1}(x)\ldots\varphi^{i'_N}|_{i_N}(x)\\ &\quad\times\epsilon_{j_1\ldots j_N}\epsilon_{j'_1\ldots j'_N}\bar{\varphi}^{j'_1}|_{j_1}(x_0)\ldots\bar{\varphi}^{j'_N}|_{j_N}(x_0)e^{-\int d^3x\,\bar{\varphi}^j|_i O^{jj'}_{ii'}\varphi^{j'}|_{i'}}\\ &= Z_l^N\xi_0^2\int d^3x\int d^3x_0\,\epsilon_{i_1\ldots i_N}\epsilon_{j_1\ldots j_N}G_{i_1j_1}(x,x_0)\ldots G_{i_Nj_N}(x,x_0) . \end{aligned} \tag{3.49}$$

Using Eqs. (3.45), (3.41), we obtain that the full contribution of the ξ_0 interaction, up to second order, is

$$\begin{aligned} &\xi_0^2\prod_{l=1}^{N}\int d^3x\,d^3x_0\,\epsilon_{i_1\ldots i_N}\epsilon_{k_1\ldots k_N}\int dL_l du^l du_0^l\int [Dx^{(l)}]^{L_l}_{x_0^l,u_0^l;x_l,u^l}\, e^{-\int_0^{L_l}ds_l\left[\frac{1}{2\kappa}\dot{u}^{(l)}_\mu\dot{u}^{(l)}_\mu+\mu\right]}\Gamma^{(l)}_{i_lk_l}[j_\mu]\\ &\qquad\left(\sum_v\frac{1}{v!}\prod_{k=1}^{v}\int_0^\infty\frac{dL_k}{L_k}\int\, dv_k\int [dx^{(k)}]^{L_k}_{x_k,u_k;x_k,u_k}\, e^{-\int_0^{L_k}ds_k\left(\frac{1}{2\kappa}\dot{u}^{(k)}_\mu\dot{u}^{(k)}_\mu+\mu\right)}W_{l_k}[j_\mu]\right)^N . \end{aligned} \tag{3.50}$$

This not only reproduces the contribution originated from the N vortex lines interaction, but also their mixing with the uncorrelated loops.

Further interaction terms compatible with the symmetries of the action in Eq. (3.47) may be added. Two of its important symmetries are local color transformations

$$\Phi \to S_c(x)\Phi \quad , \quad j_\mu \to S_c(x) j_\mu S_c^{-1}(x) + i S_c(x)\partial_\mu S_c^{-1}(x) \quad , \quad S_c(x) \in SU(N) \; , \tag{3.51}$$

and global flavor transformations

$$\Phi \to \Phi S_f \quad , \quad S_f \in SU(N) \; . \tag{3.52}$$

In our case, we will take the vortex contribution to the Wilson Loop to be

$$Z_v[j_\mu] = \int [D\Phi^\dagger][D\Phi]\, e^{-\int d^3x\, \mathcal{L}_v} \; ,$$
$$\mathcal{L}_v = \frac{1}{3\kappa}\mathrm{Tr}((D_\mu\Phi)^\dagger D_\mu\Phi) + \mu\,\mathrm{Tr}(\Phi^\dagger\Phi) + \lambda_0 \mathrm{Tr}(\Phi^\dagger\Phi)^2 - \xi_0(\det\Phi + \det\Phi^\dagger) \; , \tag{3.53}$$

which contains not only the loop and N-lines contribution, but also a quartic interaction (compatible with color and flavor symmetries) with parameter λ_0.

3.4.2 Further correlations

Up to this point, our effective model is able to describe uncorrelated loops and correlations between N vortex lines in a non-Abelian setting. A well-known description for these correlations within the Abelian framework is that of t' Hooft, who proposed the effective model

$$L = \partial^\mu \bar{V}\, \partial_\mu V + m^2\, \bar{V} V + \frac{\lambda}{2}\, (\bar{V}V)^2 + \xi\, (V^N + \bar{V}^N) \; . \tag{3.54}$$

This theory may also be obtained as an effective description of an Abelian center vortex ensemble [92]. It has a $Z(N)$ discrete vacuum, which implies the existence of one dimensional domain-walls. These walls represent the confining string, as they carry a finite energy per unit length. A good effective theory for the confining string should therefore have a discrete vacuum. This implies that the Lagrangian in Eq. (3.53) is not yet complete. First, the vacuum of this model is not discrete. Second, as the Spontaneous Symmetry Breaking is not complete, there will be Goldstone modes, and the path-integral will contain large fluctuations. As we will see, these issues will be solved after the introduction of the contribution of instantons in the ensemble.

An important observation in the lattice is that most (97 %) center vortices configurations contain lower-dimensional defects, forming nonoriented vortices or chains [93]. It is therefore natural that an ensemble of vortices should include the contributions of these objects in order to describe the confining string more efficiently. In the continuum, these nonoriented vortices have different Lie algebra orientations [94]. Similarly to the pure vortex (see Eq. (3.19)), these chains may be written locally in terms of singular gauge transformations of the type

$$S = e^{i\chi\beta\cdot T} W(x) \ . \tag{3.55}$$

The Cartan sector creates the thin vortices, while $W(x)$ is a different Weyl transformation on each vortex guiding-center, creating lower dimensional defects [96]. In the 2+1 dimensional case, these defects are one-dimensional (instantons), and change the Lie Algebra orientation of the gauge field from a weight to another.

The physical properties of the defects are encoded in the gauge-invariant field strength $f_\mu(A) = \epsilon_{\mu\alpha\beta} S^{-1} F_{\alpha\beta}(A) S$. More general defects may be created by considering the singular phase $S\tilde{U}^{-1}$, with $\tilde{U}^{-1}$ being a regular gauge transformation. Then, the gauge-invariant field strengths for a single vortex and the $N-$vortex lines configurations are given respectively by

$$f_\mu(A) = f_\mu(l, g(s), \beta) \ , f_\mu(A) = \sum_{j=1}^{N} f_\mu(l_j, g_j(s_j), \beta_j) \ , \tag{3.56}$$

where we defined the dual field strength

$$f_\mu(\gamma, g(s), \beta) = \int_\gamma ds \frac{dx_\mu}{ds}\, \delta(x - x(s))\, g(s)\beta\cdot T g^{-1}(s) \quad , \quad g(s) \equiv \tilde{U}(x(s)) \ . \tag{3.57}$$

For a chain with a pair of instantons,

$$f_\mu(A) = f_\mu(\gamma, g(s), \beta) + f_\mu(\gamma', g'(s'), \beta') \ . \tag{3.58}$$

For $N \geq 3$, there is also the three instanton configuration, characterized by

$$f_\mu(A) = f_\mu(\gamma, g(s), \beta) + f_\mu(\gamma', g'(s'), \beta') + f_\mu(\gamma'', g''(s''), \beta'') \ . \tag{3.59}$$

3.4.3 Introducing chains in the ensemble

Previously, we were able to obtain an effective theory for the loops, together with the $N-$vortex configurations, from their respective Wilson Loop contributions

$$W_l[j_\mu^{\mathcal{C}_\mathrm{e}}] \quad , \quad \frac{1}{N!}\epsilon_{i_1\dots i_N}\epsilon_{i'_1\dots i'_N}\Gamma_{\gamma_1}[j_\mu^{\mathcal{C}_\mathrm{e}}]_{i_1 i'_1}\dots\Gamma_{\gamma_N}[j_\mu^{\mathcal{C}_\mathrm{e}}]_{i_N i'_N} \ . \tag{3.60}$$

In order to identify the proper contribution of the chains, it is illuminating to express the above-mentioned contributions in terms of Gilmore-Peleremov group coherent states (see Refs [95, 97] for a complete introduction to these objects). Given a weight vector

$|\omega\rangle$ of a given representation, we define the state $|g, w\rangle = g|w\rangle$. Then, we may write

$$W_l[j_\mu] = \int d\mu(g) \langle g, w|\Gamma_l[j_\mu]|g, w\rangle \ . \tag{3.61}$$

In order to find the corresponding representation for the $N-$ vortex configuration, we need the formula [98],

$$\int d\mu(g) g_{i_1 j_1} \dots g_{i_N j_N} = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} \ , \tag{3.62}$$

which implies

$$\int d\mu(g)\, |g, w_1\rangle|_{i_1} \dots |g, w_N\rangle|_{i_N} = \int d\mu(g)\, g_{i_1 j_1} \dots g_{i_N j_N} |w_1\rangle|_{j_1} \dots |w_N\rangle|_{j_N} = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \ . \tag{3.63}$$

Thus, the N-line contribution may be written as

$$D_N[b_\mu] = (N!)^2 \int d\mu(g) d\mu(g_0) \langle g, w_1|\Gamma_{\gamma_1}[j_\mu]|g_0, w_1\rangle \dots \langle g, w_N|\Gamma_{\gamma_N}[j_\mu]|g_0, w_N\rangle \ . \tag{3.64}$$

These alternative representations imply that each loop is associated with a fundamental weight and each vortex line (on the matched configuration) corresponds to a different weight.

As in the chain contribution the weight must change at the location of the zero-dimensional defects (instantons), the contribution of a chain with n instantons is proposed to be

$$\begin{aligned} &\langle g_1, w'|\Gamma_{\gamma_n}[j_\mu]|g_n, w\rangle \dots \langle g_3, w'|\Gamma_{\gamma_2}[j_\mu]|g_2, w\rangle \langle g_2, w'|\Gamma_{\gamma_1}[j_\mu]|g_1, w\rangle \\ &= \mathrm{Tr} \left(|\Gamma_{\gamma_n}[j_\mu]|g_n, w\rangle \langle g_n, w'| \dots |\Gamma_{\gamma_2}[j_\mu]|g_2, w\rangle \langle g_2, w'|\Gamma_{\gamma_1}[j_\mu]|g_1, w\rangle \langle g_1, w'| \right. . \end{aligned} \tag{3.65}$$

However, the integrals of $g_j|w\rangle\langle w'|g_j^\dagger$ vanish, as implied by the formula

$$\int d\mu(g)\, R^{(i)}(g)|_{ab}\, R^{(j)}(g^{-1})|_{\beta\alpha} = \delta_{ij} \delta_{a\alpha} \delta_{b\beta} \ , \tag{3.66}$$

where $R^{(i)}$ and $R^{(j)}$ are unitary irreducible representations (irreps) of SU(N) [91], in the case of the trivial and adjoint irreps. Additionaly, chains also contribute a center element to the Wilson Loop [89]. This can be seen in Eq. (3.55), as the $W(x)$ factor is single valued in any closed path around the chain, and therefore the only contribution comes from the singular phase S. The proposal of Eq. (3.65) contains additional phase factors that must be canceled, leaving only the center element contribution. Thus, the proper

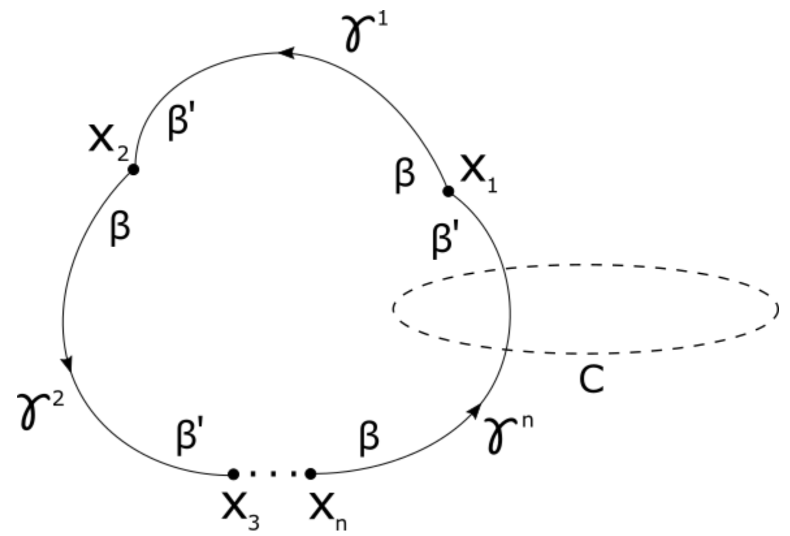


Figure 3.2: A configuration corresponding to a chain, with n instantons. The Wilson Loop along C gives a center element when it links the chain configuration.

chain contribution is

$$\int d\mu(g_1)\dots d\mu(g_n)\,\langle g_1,w|g_2,w'\rangle\langle g_2,w|g_3,w'\rangle\dots\langle g_n,w|g_1,w'\rangle$$
$$\times\langle g_1,w'|\Gamma_{\gamma_n}[j_\mu]|g_n,w\rangle\dots\langle g_3,w'|\Gamma_{\gamma_2}[j_\mu]|g_2,w\rangle\langle g_2,w'|\Gamma_{\gamma_1}[j_\mu]|g_1,w\rangle \tag{3.67}$$
$$=\int d\mu(g_1)\dots d\mu(g_n)\,\mathrm{Tr}\big(|g_n,w'\rangle\langle g_n,w|\dots|g_2,w'\rangle\langle g_2,w|\,|g_1,w'\rangle\langle g_1,w|\big)$$
$$\times\mathrm{Tr}\,\big(|\Gamma_{\gamma_n}[j_\mu]|g_n,w\rangle\langle g_n,w'|\dots|\Gamma_{\gamma_2}[j_\mu]|g_2,w\rangle\langle g_2,w'|\Gamma_{\gamma_1}[j_\mu]|g_1,w\rangle\langle g_1,w'|\;.$$

For $j_\mu = j_\mu^C$, the center-vortex line of the chain that links C (see Fig. 3.2) will contribute a center element times the real and positive factor

$$\left(e^{i2\pi k/N}\right)^{L(\mathcal{C},l)}\int d\mu(g_1)\dots d\mu(g_n)\;\left|\mathrm{Tr}\big(|g_n,w'\rangle\langle g_n,w|\dots|g_2,w'\rangle\langle g_2,w|\,|g_1,w'\rangle\langle g_1,w|\big)\right|^2\;, \tag{3.68}$$

thus coinciding with the Wilson Loop of the chain.

It is possible to obtain an alternative representation for the chain and other defect contributions by performing appropriate Weyl transformations. For the $n = 2$ case, we replace $g_2 \to g_2 W$, where W is an odd Weyl reflection, to get

$$\int d\mu(g_1)d\mu(g_2)\,\langle g_1,w|g_2,w\rangle\langle g_2,w'|g_1,w'\rangle\times\langle g_1,w'|\Gamma_{\gamma_2}[j_\mu]|g_2,w'\rangle\langle g_2,w|\Gamma_{\gamma_1}[j_\mu]|g_1,w\rangle\;. \tag{3.69}$$

For $n = 3$, $N > 2$ we can perform an even Weyl transformation that takes $g_2 \to g_2 P_A$, where P_A changes w, w', w'' to w'', w, w', and then $g_3 \to g_3 P_B$, where P_B changes w,

w', w'' to w', w'', w, and the variable becomes

$$\int d\mu(g_1)d\mu(g_2)d\mu(g_3)\,\langle g_1, w|g_2, w\rangle\langle g_2, w''|g_3, w''\rangle\langle g_3, w'|g_1, w'\rangle$$
$$\times\langle g_1, w'|\Gamma_{\gamma_3}[j_\mu]|g_3, w'\rangle\langle g_3, w''|\Gamma_{\gamma_2}[j_\mu]|g_2, w''\rangle\langle g_2, w|\Gamma_{\gamma_1}[j_\mu]|g_1, w\rangle\ . \tag{3.70}$$

The next step is to use the Gilmore-Perelemov representation

$$\langle g, w|\Gamma_\gamma[j_\mu]|g_0, w\rangle = \int [dg(s)]\, e^{i\int ds \mathrm{Tr}\left(\left(g(s)^\dagger j(s)g(s)+ig^\dagger(s)\dot{g}(s)\right)w\cdot T\right)} \quad , \quad j(s) = j_\mu(x(s))\frac{dx_\mu}{ds}\ , \tag{3.71}$$

where $g(s)$ satisfies $g(0) = g_0, g(L) = g$. In principle, this formula is valid when $|w\rangle$ is the state associated to the highest weight of the representation. However, as all the weights of the fundamental representation may be connected by Weyl transformations, this formula holds for any fundamental weight. With this representation, it is possible to show that the contributions of all the configurations (noninteracting loops, N-vortex lines matching, chains) may be written as

$$e^{i\int d^3x \mathrm{Tr}(j_\mu f_\mu(A))}\ , \tag{3.72}$$

where A is the gauge field of the corresponding configuration.

Moreover, the chain contribution (3.70) may be generated by the vertex

$$V_{\text{chain}} \propto \int d\mu(g)\langle g, w'|\Phi^\dagger|g, w'\rangle\langle g, w|\Phi|g, w\rangle, \tag{3.73}$$

which is the same as

$$\begin{aligned} V_{\text{chain}} &\propto \int d\mu(g)\mathrm{Tr}\,\left(|g, w\rangle\langle g, w'|\Phi^\dagger|g, w'\rangle\langle g, w|\Phi\right) \\ &= \int d\mu(g)\mathrm{Tr}\,\left(g|w\rangle\langle w'|g^\dagger\Phi^\dagger g|w'\rangle\langle w|g^\dagger\Phi\right). \end{aligned} \tag{3.74}$$

Notice that $|w'\rangle\langle w| = E_\alpha$, with $\alpha = w' - w$. This is an element of the root sector in the Cartan decomposition of $SU(N)$ (see Appendix A), which may be written in terms of the hermitian generators T_α, $T_{\bar{\alpha}}$,

$$E_\alpha = \frac{T_\alpha + iT_{\bar{\alpha}}}{\sqrt{2}}\ . \tag{3.75}$$

Using this and that $gT_Ag^\dagger = R_{AB}(g)T_B$, $R(g)$ being the adjoint representation of g, we obtain

$$V_{chain} \propto \frac{1}{2} \int d\mu(g)\,\text{Tr}\left((R_{\alpha B}(g) + iR_{\bar{\alpha} B}(g))T_B \Phi^\dagger (R_{\alpha C}(g) - iR_{\bar{\alpha} C}(g))T_C \Phi\right) \ . \quad (3.76)$$

Using the orthogonality formula

$$\int d\mu(g) D^{(i)}(g)|_{ab} D^{(j)}(g^{-1})|_{cd} = \delta_{ij}\delta_{ad}\delta_{bc} \quad (3.77)$$

when i and j are the adjoint representation, we obtain

$$\int d\mu(g)\, R_{AB}(g)\, R_{A'B'}(g) = \delta_{AA'}\delta_{BB'} \ . \quad (3.78)$$

Therefore,

$$V_{chain} \propto \text{Tr}(\Phi^\dagger T_A \Phi T_A) \ . \quad (3.79)$$

We now have all the necessary elements to compute the average of the Wilson Loop in our ensemble. It is given by the formula

$$\langle W_C^R \rangle = \frac{Z[j_\mu^C]}{Z[0]} \ , Z[j_\mu] = \int [D\Phi][D\Phi^\dagger] e^{-S_{eff}(\Phi, j_\mu)} \ , \quad (3.80)$$

where the effective action contains the contributions of the loops, of the $N-$ vortex lines matching (as well as a quartic vortex interaction), and of chains with different numbers of instantons attached

$$\begin{aligned} S_{eff}(\Phi, j_\mu) &= \int d^3x \left(\text{Tr}\,(D_\mu\Phi)^\dagger D^\mu \Phi + V(\Phi)\right) \quad , \quad D_\mu = \partial_\mu - ij_\mu \ , \\ V(\Phi) &= \frac{\lambda}{2}\text{Tr}\,(\Phi^\dagger\Phi - a^2 I_N)^2 - \xi\big(\det\Phi + \det\Phi^\dagger\big) - \vartheta \text{Tr}\,(\Phi^\dagger T_A \Phi T_A) + c \ . \end{aligned} \quad (3.81)$$

The constant c is included so as to assure that the vacua have energy equal to zero. To write this formula, we considered a negative tension μ and a positive stiffness $\frac{1}{\kappa}$ for the vortex lines, which corresponds to a phase where they are very large (percolating phase). Indeed, this is the phase observed in lattice calculations [81, 82]. Notice that the $SU(N)$ color and flavor symmetries of the vortex sector are broken by the chain term. The unbroken symmetries are global color-flavor transformations ($S_c = S_f^{-1}$) and a local discrete $Z(N)$ symmetry $\Phi \to e^{if(x)\beta\cdot T}$, where $f(x)$ is equal to 2π inside a volume V, and 0 outside. This kind of center transformation localized in a volume may be used to change the location of $S(C)$, the unobservable surface whose border is the Wilson Loop C.

3.4.4 Physical consequences

In this section we shall explore the physical consequences of the effective model (3.81). Let us begin by analyzing the vacuum of this model. For this purpose, it is useful to perform a polar decomposition on Φ, by writing it as the product of a positive semidefinite hermitian matrix P and a phase $U \in U(N)$, $\Phi = PU$. The potential reads

$$V(P,U) = \frac{\lambda}{2}\text{Tr}\,\left((P^2 - a^2 I_N)^2\right) - \xi\,\det P\,(\det U + \det U^\dagger) - \vartheta \text{Tr}\,\left(PT_B PUT_B U^\dagger\right)\ . \quad (3.82)$$

If only the vortex sector were present, vacuum configurations would have P proportional to the identity due to the λ term, and $U \in SU(N)$ due to the ξ term, thus forming a continuum. Such a model would not accommodate stable domain walls, and its partition function would contain large Goldstone fluctuations. In Ref [92], an Abelian ensemble of center vortices was studied. In that case, the large fluctuations of the partition function were mapped into those of an XY-model with frustration, and allowed the authors to obtain an area law for the Wilson Loop. However, the Abelian effective model is not able to accommodate the asymptotic Casimir Law.

As the ϑ term is present in our model, and is positive, the matrix U of the vacuum configurations should maximize the overlap of T_B and $n_B = UT_BU^{-1}$. This implies that U should belong to the center $Z(N)$ of $SU(N)$. Our vacuum configurations are therefore of the type

$$P = vI_N\ ,\ U \in Z_N = \left\{ e^{i\frac{2\pi n}{N}} I_N \,\middle|\, n = 0, 1, 2, ..., N-1 \right\}\ ,$$
$$2\lambda N(v^2 - a^2) - 2\xi N v^{N-2} - \frac{\vartheta}{N}\left(N^2 - 1\right) = 0\ . \quad (3.83)$$

Therefore, the presence of the instanton contribution in our model allow for the existence of stable domain walls. In the following paragraphs we shall study how these walls describe the confining string within our approach.

As our partition function does not have associated Goldstone modes, it may be well approximated by means of a saddle-point expansion, the leading contribution being that of the classical solution for Φ, which satisfies

$$D^2\Phi = \lambda\Phi(\Phi^\dagger\Phi - a^2) - \xi\, C[\Phi^*] - \vartheta T_B \Phi T_B \quad , \quad D_\mu = \partial_\mu - ij^{\mathcal{C}}_\mu\ , \quad (3.84)$$

where $C[]$ stands for the cofactor matrix, which arises upon variation of the determinant. Let us study the solution to this problem when C is a circle contained in the $x_1 = 0$ plane. we shall choose $S(C)$ not to be the minimal area, but its (unbounded) complement in the $x_1 = 0$ plane. In this case, the source in the covariant derivative implies that Φ must "jump" by a factor of $e^{i2\pi\beta_e\cdot T}$ on nearby points that are on the opposite sides of

$S(C)$. For an asymptotic Wilson Loop, the solution will be independent of (x_2, x_3), and the presence of the source will imply the boundary conditions

$$\lim_{x_1 \to -\infty} \Phi(x_1, x_2, x_3) = v I_N \quad , \quad \lim_{x_1 \to +\infty} \Phi(x_1, x_2, x_3) = v\, e^{i 2\pi \beta_e \cdot T} \ . \tag{3.85}$$

Then, the leading contribution to the saddle-point expansion will be

$$S_{\text{eff}} \approx \epsilon A \ , \tag{3.86}$$

A being the area of the disk, and the string tension shall be obtained from the one-dimensional soliton that minimizes the action given by

$$s = \int dx \, \left(\text{Tr}\, (\partial_x \Phi)^\dagger \partial_x \Phi + V(\Phi, \Phi^\dagger) \right) \ , \tag{3.87}$$

where $\Phi(-\infty) = v I_N$, $\Phi(+\infty) = v\, e^{i 2\pi \beta_e \cdot T}$. Clearly, $\Phi(x)$ satisfies (3.84), replacing D^2 by ∂_x^2, as the effect of the source was already taken into account.

Due to Eq. (3.86), it is clear that this model is compatible with a linear confining potential between static sources. The next question is whether the effective string tension is compatible with the observed Casimir Law or not. More precisely, we need to understand how ϵ depends on the $N-$ality of the representation with weight β_e. For definiteness, we will restrict our analysis to the k-Antisymmetric representations, with weights $\beta_e = 2N\omega_k^A$, k being the $N-$ality. In this case, an Ansatz of the form

$$\Phi = (h_1 P_1 + h_2 P_2)\, S \quad , \quad S = e^{i\theta_1 \frac{N-k}{N} P_1 - i\theta_2 \frac{k}{N} P_2} \ , \tag{3.88}$$

with

$$\begin{aligned} P_1 &= \begin{pmatrix} \frac{N-k}{N} I_k & 0 \\ 0 & 0 \end{pmatrix} \ , \\ P_2 &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{k}{N} I_{N-k} \end{pmatrix} \ , \end{aligned} \tag{3.89}$$

closes the equations of motion [88], in the sense that the complete matrix equations are equivalent to scalar equations for the profiles $h_1, h_2, \theta_1, \theta_2$. The obtained equations are

$$
\begin{aligned}
\partial_x^2 h_1 =& \left(\frac{N-k}{N}\right)^2 (\partial_x\theta_1)^2 h_1 + \lambda h_1(h_1^2-a^2) - \xi h_1^{k-1}h_2^{N-k}\,\mathsf{cos}\left(\frac{k(N-k)(\theta_1-\theta_2)}{N}\right) \\
&- \vartheta\frac{Nk-1}{2N^2}h_1 - \vartheta\frac{N-k}{2N}h_2\,\mathsf{cos}\left(\frac{k}{N}\theta_2+\frac{N-k}{N}\theta_1\right)\ , \qquad \text{(3.90a)} \\
\partial_x^2 h_2 =& \left(\frac{k}{N}\right)^2 (\partial_x\theta_2)^2 h_2 + \lambda h_2(h_2^2-a^2) - \xi h_1^{k}h_2^{N-k-1}\,\mathsf{cos}\left(\frac{k(N-k)(\theta_1-\theta_2)}{N}\right) \\
&- \vartheta\frac{N(N-k)-1}{2N^2}h_2 - \vartheta\frac{k}{2N}h_1\,\mathsf{cos}\left(\frac{k}{N}\theta_2+\frac{N-k}{N}\theta_1\right)\ , \qquad \text{(3.90b)} \\
\partial_x^2 \theta_1 =& -2\partial_x \ln h_1\,\partial_x\theta_1 + \xi\frac{N}{N-k}h_1^{k-2}h_2^{N-k}\,\mathsf{sin}\left(\frac{k(N-k)(\theta_1-\theta_2)}{N}\right) \\
&+ \frac{\vartheta}{2}\frac{h_2}{h_1}\,\mathsf{sin}\left(\frac{k}{N}\theta_2+\frac{N-k}{N}\theta_1\right)\ , \qquad \text{(3.90c)} \\
\partial_x^2 \theta_2 =& -2\partial_x \ln h_2\,\partial_x\theta_2 - \xi\frac{N}{k}h_1^{k}h_2^{N-k-2}\,\mathsf{sin}\left(\frac{k(N-k)(\theta_1-\theta_2)}{N}\right) \\
&+ \frac{\vartheta}{2}\frac{h_1}{h_2}\,\mathsf{sin}\left(\frac{k}{N}\theta_2+\frac{N-k}{N}\theta_1\right)\ . \qquad \text{(3.90d)}
\end{aligned}
$$

The phase can be factored in $U(1)$ and $SU(N)$ sectors

$$
S = e^{i\alpha}e^{i\theta\beta\cdot T} \quad , \quad \theta = \frac{N-k}{N}\theta_1 + \frac{k}{N}\theta_2 \quad , \quad \alpha = \frac{k(N-k)(\theta_1-\theta_2)}{N^2}\ . \qquad \text{(3.91)}
$$

In principle, as $e^{i2\pi\beta_e\cdot T} = e^{-i\frac{2k\pi}{N}}$, there are two ways to impose the boundary conditions (3.85): one where α (resp. θ) performs the transition and leaves the possibility of θ (resp. α) to remain constant. The first possibility gives rise to a model closely related with the ‘t Hooft’s model (see Eq. (3.54)), which is not consistent with a Casimir Law, while the second, corresponding to

$$
\begin{aligned}
h_1(-\infty) = h_2(-\infty) = h_0 \qquad &, \qquad h_1(\infty) = h_2(\infty) = h_0\ , \qquad \text{(3.92a)} \\
\theta_1(-\infty) = \theta_2(-\infty) = 0 \qquad &, \qquad \theta_1(\infty) = \theta_2(\infty) = 2\pi\ , \qquad \text{(3.92b)} \\
\theta(-\infty) = 0\ ,\ \theta(\infty) = 2\pi \qquad &, \qquad \alpha(-\infty) = 0\ ,\ \alpha(\infty) = 0\ , \qquad \text{(3.92c)}
\end{aligned}
$$

is the option we shall explore. To understand the conditions on the parameters of the model that favor this possibility, it is useful to look at small perturbations of the profiles around their vacuum value and keep up to linear terms. In this regard, it is convenient to write the Ansatz in terms of the variables $\eta, \eta_0, \alpha, \theta$

$$
\Phi = (\eta I_N + \eta_0\beta\cdot T)\,e^{i\theta\beta\cdot T}e^{i\alpha} \quad , \quad \eta = \frac{k}{N}h_1 + \frac{N-k}{N}h_2\ ,\ \eta_0 = h_1 - h_2\ . \qquad \text{(3.93)}
$$

The perturbations then satisfy

$$\partial_x^2\,\delta\eta = M_\eta^2\,\delta\eta \quad , \quad M_\eta^2 = \lambda(3v^2-a^2) - \xi(N-1)v^{N-2} - \vartheta\frac{N^2-1}{2N^2} \, , \tag{3.94a}$$

$$\partial_x^2\,\delta\eta_0 = M_{\eta_0}^2\,\delta\eta_0 \quad , \quad M_{\eta_0}^2 = \lambda(3v^2-a^2) + \xi v^{N-2} + \frac{\vartheta}{2N^2} \, , \tag{3.94b}$$

$$\partial_x^2\,\delta\alpha = M_\alpha^2\,\delta\alpha \quad , \quad M_\alpha^2 = N\xi v^{N-2} \, , \tag{3.94c}$$

$$\partial_x^2\,\delta\theta = M_\theta^2\,\delta\theta \quad , \quad M_\theta^2 = \frac{\vartheta}{2} \, . \tag{3.94d}$$

Therefore, we will consider the limit $\xi v^{N-2} >> \vartheta$, as this implies a very high energy cost for α to leave its trivial value $\alpha = 0$. This, together with our previous requirements that λa^2, $\xi v^{N-2} >> \vartheta$, imply that the profiles η, η_0, α will remain approximately constant, while θ will satisfy the Sine-Gordon equation

$$\partial_x^2\,\theta = \frac{\vartheta}{2}\sin\theta \, . \tag{3.95}$$

Then, as analyzed in [88], after using Derrick's theorem, the energy of the solution in this case may be written as

$$\varepsilon_k = \frac{k(N-k)}{N-1}\left(2v^2\frac{N-1}{N}\int(\partial_x\theta)^2dx\right) = \frac{k(N-k)}{N-1}\varepsilon_1 \, , \tag{3.96}$$

which is consistent with the Casimir Law for the string tension.

In this section, we have discussed ensembles of center vortices in $2+1$ dimensions. Starting from a very simple well-known example, we showed how these ensembles are able to accommodate an area law for the Wilson Loop. More complicated ensembles require more sophisticated methods to extract the physical consequences, often through the obtention of effective theories that describe them. Then, we presented an ensemble of center vortices and chains containing non-Abelian degrees of freedom in $2+1$ dimensions and showed that it is well described by a non-Abelian effective theory whose partition function may be approximated by a saddle-point expansion, due to the absence of Goldstone modes. Then, we showed that the leading order contribution to the Wilson Loop is compatible with a Casimir Law for the string tension. Moreover, the results presented predict that the chromoelectric field profile of the $2+1$d flux must be of the Sine-Gordon type. This is an important step towards the understanding of the role of these configurations to the confining properties of YM theory.

3.5 Center vortex ensembles in 4 dimensions

In $3+1$ dimensions, the center vortices are configurations localized in closed world-surfaces. Therefore, ensembles of these objects, as defined by Eq. (3.17), will involve sums over all possible closed surfaces. This is in sharp contrast with the 3 dimensional case which involved the sum over closed lines, and was thus directly representable by a field theory. In the present case, the natural effective description would be in terms of a string field theory, or a matrix model, which is considerably more complicated. The derivation of a diffusion equation for an ensemble of surfaces, and a subsequent interpretation in terms of an effective theory, is still lacking. However, some important ideas were put forward in Ref. [89], which we will describe in what follows.

3.5.1 Ensemble of surfaces

Following the general setup presented in Section 3.3, the simplest realization of a center-vortex ensemble in 4 spacetime dimensions in the continuum would be given by (see Eq. (3.17))

$$\langle W^{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})\rangle \approx \mathcal{N}\sum_{\Omega} e^{-S_{\mu}(\Omega)-S_{\kappa}(\Omega)}\frac{1}{d_{\mathsf{R}}}\mathrm{Tr}\left[R(e^{i\frac{2\pi}{N}}I)\right]^{L(\Omega,\mathcal{C}_{\mathsf{e}})} , \tag{3.97}$$

where S_{μ}, S_{κ} are contributions containing tension and stiffness terms, respectively. The tension term could be chosen to be the Nambu-Goto action

$$S_{\mu}^{\mathsf{NG}}(\Omega) = \mu\int d\sigma_1 d\sigma_2\sqrt{g(\sigma_1,\sigma_2)}\;, \tag{3.98}$$

where $g(\sigma_1,\sigma_2)$ is the worldsheet metric. As we are considering Yang-Mills theory in flat 4 dimensional Euclidean space, the surfaces Ω must be embedded in R^4 properly, i.e., a parametrization $x^{\mu}(\sigma_1,\sigma_2)$ must be given. Then, the metric may be written as

$$g_{ab} = \frac{\partial x^{\mu}}{\partial\sigma^a}\frac{\partial x^{\mu}}{\partial\sigma^b}\;. \tag{3.99}$$

Another possibility is to consider the Polyakov action

$$S_{\mu}^{\mathsf{Poly}} = \frac{\mu}{2}\int d\sigma_1 d\sigma_2\,\sqrt{h}h^{ab}\frac{\partial x^{\mu}(\sigma_1,\sigma_2)}{\partial\sigma^a}\frac{\partial x^{\mu}(\sigma_1,\sigma_2)}{\partial\sigma^b}\;, \tag{3.100}$$

where the worldsheet metric h^{ab} is treated as an independent variable. As for the stiffness term S_{κ}, it is important to note that it must contain even powers of the *extrinsic* curvature of Ω, in order to account for the observed vortex properties in the lattice [81, 82]. In particular, terms which depend only on the intrinsic curvature would vanish for e.g. a cylinder, which is not a desirable effect. Finally, it is possible to write the center element contribution in Eq. (3.97) in a more illuminating way. For this purpose, notice that $L(\Omega, \mathcal{C}_{\mathsf{e}}) = I(\Omega, S(\mathcal{C}_{\mathsf{e}}))$, with $I(\Omega, S(\mathcal{C}_{\mathsf{e}}))$ being the intersection number between the Ω and $S(\mathcal{C}_{\mathsf{e}})$, a surface whose border is $\mathcal{C}_{\mathsf{e}}$. The quantity $I(\Omega, S(\mathcal{C}_{\mathsf{e}}))$ has the integral expression

$$
\begin{aligned}
I(\Omega, S(\mathcal{C}_{\mathsf{e}})) &= \frac{1}{2}\int d^2\tilde{\sigma}_{\mu\nu}\int d^2\sigma_{\mu\nu}\delta^{(4)}(y(s,\tau) - x(\sigma_1,\sigma_2)) \ , \\
d^2\tilde{\sigma}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, d\sigma_1\, d\sigma_2 \left(\frac{\partial x_\alpha}{\partial\sigma_1}\frac{\partial x_\beta}{\partial\sigma_2} - \frac{\partial x_\alpha}{\partial\sigma_2}\frac{\partial x_\beta}{\partial\sigma_1}\right)
\end{aligned}
\tag{3.101}
$$

where $y(s,\tau), x(\sigma_1,\sigma_2)$ are parametrizations of $S(\mathcal{C}_{\mathsf{e}})$ and Ω, respectively. This formula allows us to write the center element contribution in Eq. (3.97) in terms of the coupling of a Kalb-Ramond field, concentrated in $S(\mathcal{C}_{\mathsf{e}})$, with the surface Ω:

$$
\begin{aligned}
&\frac{1}{d_{\mathsf{R}}}\mathsf{Tr}\left[\mathsf{R}\left(e^{i\frac{2\pi}{N}}I\right)\right]^{L(\Omega,\mathcal{C}_{\mathsf{e}})} = e^{-S_B} \ , \\
&S_B = \int d\sigma_1 d\sigma_2 B_{\mu\nu}(x(\sigma_1,\sigma_2))\Sigma^{\mu\nu}(x(\sigma_1,\sigma_2)) \ , \\
&\Sigma^{\mu\nu} = \frac{\partial x^\mu}{\partial\sigma_1}\frac{\partial x^\nu}{\partial\sigma_2} - \frac{\partial x^\nu}{\partial\sigma_1}\frac{\partial x^\mu}{\partial\sigma_2} \ ,
\end{aligned}
\tag{3.102}
$$

with

$$
B_{\mu\nu}(x) = \frac{2\pi k}{N}\int_{S(\mathcal{C}_{\mathsf{e}})} d^2\tilde{\sigma}_{\mu\nu}\delta^{(4)}(x - y(\sigma_1,\sigma_2)) \ . \tag{3.103}
$$

Here, k is the $N-$ality of the representation R. With this result, Eq. (3.97) may be written as

$$
\langle W^{\mathsf{R}}(\mathcal{C}_{\mathsf{e}})\rangle \approx \mathcal{N}\sum_{\Omega} e^{-S_\mu(\Omega) - S_\kappa(\Omega) - S_B(\Omega)} \ . \tag{3.104}
$$

The next step would be to write this expression in terms of a building block (in analogy with Eq. (3.38) for the 3d case), and then derive a diffusion equation for this object. This step is still lacking.

Although the description of a sum of random surfaces is made by an effective string field theory (or a matrix model) in the general case, some simplifications may occur when a condensate is formed. In this regard, the Spontaneous Symmetry Breaking phase of an Abelian string field V was studied in Ref. [99]. The action for this field coupled with an external Kalb-Ramond field $B_{\mu\nu}$ reads, in the lattice,

$$
S(B) = -\sum_C \sum_{p\in\eta(C)} \left(\bar{V}(C+p)U_p V(C) + \bar{V}(C-p)\bar{U}_p V(C)\right) + m^2\sum_C \bar{V}(C)V(C) \ , \tag{3.105}
$$

where C is a closed loop formed by lattice links, $\eta(C)$ is the set of plaquettes that share at least one common link with C, and the plaquette variable is given by $U_p = e^{ia^2 B_{\mu\nu}(p)}$. The paths $C+p$ ($C-p$) are defined as the inclusion (exclusion) of the plaquette p in the closed path C. As the field $V(C)$ is simply a complex number that depends on the curve C, a polar decomposition of the string field may always be performed. By further assuming that the phase may be written as a product of local variables, it is possible to write

$$V(C) = w(C) \prod_{l \in C} V_l \quad , \quad V_l \in U(1) \; . \tag{3.106}$$

The author argued that, when a condensate is formed (i.e. when $m^2 < 0$), the modulus $w(C)$ will be practically constant. Using this property in Eq. (3.105), the only links whose contribution do not cancel are those belonging to p:

$$\bar{V}(C+p) U_p V(C) = w^2 \prod_{l \in C+p} \prod_{l' \in C} \bar{V}_l U_p V_{l'} = w^2 U_p \prod_{l \in p} \bar{V}_l \; . \tag{3.107}$$

Then, the action becomes

$$S(B) \approx \beta \sum_p \text{Re} \left(I - \bar{U}_p \prod_{l \in p} V_l \right) \; . \tag{3.108}$$

It is then clear that these soft modes may be described in terms of the fluctuations of the link variables V_l, which correspond to a gauge field in the continuum limit. This corresponds to the generalization of the Goldstone theorem for a string field theory. For a field theory, the Goldstone modes are scalar fields. For a string field theory, these soft modes are gauge fields. These ideas imply that an ensemble of surfaces in the condensate phase could be well described by a field theory.

3.5.2 An effective theory for the condensate of surfaces

The ideas presented in the last section led, in Ref. [89], to the proposal of the following lattice ensemble average, in terms of a field $V_\mu \in SU(N)$,

$$\begin{aligned} \langle W^{\mathsf{R}}(\mathcal{C}_{\mathsf{e}}) \rangle &\approx \frac{Z^{latt}[\alpha_{\mu\nu}]}{Z^{latt}[0]} \; , \\ Z^{latt}[\alpha_{\mu\nu}] &= \int [DV_\mu] e^{-\beta \sum_{x,\mu<\nu} \text{Re}\,\text{tr}(I - V_\mu(x) V_\nu(x+\mu) V_\mu^\dagger(x+\nu) V_\nu^\dagger(x) e^{-i\alpha_{\mu\nu}})} \; , \end{aligned} \tag{3.109}$$

where the frustration $\alpha_{\mu\nu}$ is nontrivial only on plaquettes that intersect $S(\mathcal{C}_{\mathsf{e}})$, where it satisfies $e^{i\alpha_{\mu\nu}} = e^{i2\pi\beta \cdot T}$. The $(N-1)$-tuple β_e is proportional to the highest weight of the representation R. Then, using the properties of the group integral measure [100], one may show that the nontrivial contributions to Eq. (3.109) come from plaquettes that form closed surfaces, . Moreover, the matching rules of N worldsurfaces is automatically included, as the tensor product of N fundamental representations contains a singlet. Surfaces that link $\mathcal{C}_{\mathrm{e}}$ will contribute a factor $e^{\mp i\alpha_{\mu\nu}} = e^{\pm\beta_e\cdot\omega}$, which is the appropriate center element, for each intersection point. This proposal was enhanced by including the contribution of chains, i.e. configurations of vortices joined by monopole loops. The contribution of such a configuration, containing an arbitrary number n of loops $\mathcal{C}_k^{\mathrm{latt}}$, is given by

$$
\begin{aligned}
&Z_{\mathrm{mix}}^{\mathrm{latt}}[\alpha_{\mu\nu}] \propto \int [DV_\mu] e^{-\beta\sum_{x,\mu<\nu}\mathrm{Re\,tr}(I-V_\mu(x)V_\nu(x+\mu)V_\mu^\dagger(x+\nu)V_\nu^\dagger(x)e^{-i\alpha_{\mu\nu}})}\mathcal{W}_{\mathrm{Ad}}^{(1)}\dots\mathcal{W}_{\mathrm{Ad}}^{(n)} \;, \\
&\mathcal{W}_{\mathrm{Ad}}^{(k)} = \frac{1}{N^2-1}\mathrm{tr}\left(\prod_{(x,\mu)\in\mathcal{C}_k^{\mathrm{latt}}}\mathrm{Ad}(V_\mu(x))\right) \;. \qquad (3.110)
\end{aligned}
$$

Notice that the integral of a single adjoint variable $\mathrm{Ad}(V_\mu(x))$ vanishes, due to the orthogonality formula of Eq. (3.77). However, as $N\otimes\bar{N}$ contains an adjoint, the combination $\mathrm{Ad}(V_\mu)V_\nu V_\rho^\dagger$ gives a nontrivial contribution. Here, we denoted by N and $\bar{N}$ the fundamental and anti-fundamental representations, respectively. Therefore, the configurations that contribute to Eq. (3.110) are those formed by pairs of open surfaces that join at the closed loops $\mathcal{C}_l^{\mathrm{latt}}$, representing chains formed by vortices and monopoles.

In Ref. [89], using polymer techniques, an effective description for this ensemble was derived in the continuum limit. Upon inclusion of matching between 3 and 4 monopole lines carrying roots δ_i, with $\sum_i \delta_i = 0$, the resulting effective theory may be described by the action

$$
\begin{aligned}
S = &\int d^4x \Big((F_{\mu\nu}(\Lambda) - 2\pi s_{\mu\nu}\beta_e\cdot T)^2 + \frac{1}{2}\langle D_\mu\psi_A, D_\mu\psi_A\rangle + \frac{\mu^2}{2}\langle\psi_A,\psi_A\rangle \\
&+\kappa f^{IJK}\langle\psi_I,\psi_J\wedge\psi_K\rangle + \lambda\langle\psi_I\wedge\psi_J,\psi_I\wedge\psi_J\rangle\Big) \;, \\
F_{\mu\nu} = &\frac{i}{g}[D_\mu, D_\nu] \quad , \quad D_\mu = \partial_\mu - ig[\Lambda_\mu, \;] = \partial_\mu + g\Lambda_\mu\wedge \;, \qquad (3.111)
\end{aligned}
$$

Λ_μ being the gauge field that emerges from the continuum limit of the dual link variable V_μ. Here we used the wedge notation $A\wedge B \equiv -i[A,B]$. The number of adjoint Higgs fields is equal to N^2-1, i.e. $I = 1,\dots,N^2-1$. The source $s_{\mu\nu}$ is concentrated on a surface $S(\mathcal{C})$ whose border is the Wilson Loop $\mathcal{C}$. Explicitly,

$$
s_{\mu\nu} = \int_{S(\mathcal{C})} d^2\tilde{\sigma}_{\mu\nu}\delta^{(4)}(x - w(s,\tau)) \;. \qquad (3.112)
$$

Then, as in the three dimensional case, the problem of evaluating the Wilson Loop amounts to the understanding of this effective theory. The model has been extensively studied in Refs. [101, 102], where it was shown that it is able not only to produce an area law, but also to be consistent with a Casimir Law for the string tension at asymptotic distances. This behaviour is, together with a Sine Law, among the possibilities which are consistent with current lattice calculations. In the following section we show how these results were obtained.

3.6 Analysis of the effective model in 4 dimensions

Let us now study the relevant classical solutions of the effective model of Eq. (3.111). This model is invariant under the $SU(N)$ gauge transformations

$$\Lambda_\mu \to U\Lambda_\mu U^{-1} + \frac{i}{g} U\partial_\mu U^{-1} \quad , \quad \psi_I \to U\psi_I U^{-1} \; . \tag{3.113a}$$

Here, T_A and f_{ABC} are, respectively, the generators and the structure constants of the Lie Algebra of $SU(N)$. In this thesis we will use the Cartan basis. Our conventions and the relevant properties of the Lie Algebra of $SU(N)$ are explained in Appendix A. This basis consists of the diagonal generators T_q, $q = 1, \ldots, N-1$, and the off-diagonal ones

$$T_\alpha = \frac{E_\alpha + E_{-\alpha}}{\sqrt{2}} \quad , \quad T_{\bar{\alpha}} = \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}i} \; , \tag{3.114}$$

where $E_{\pm\alpha}$ are the root vectors.

Typical configurations of the vacuum manifold $\mathcal{M}$ of this theory are given by

$$\Lambda_\mu = \frac{i}{g} S\partial_\mu S^{-1} \quad , \quad \psi_A = vST_A S^{-1} \; , \tag{3.115}$$

where the mass parameter v is such that the potential $V_H(\psi)$ is minimized, i.e.

$$\mu^2 v + \kappa v^2 + \lambda v^3 = 0 \; . \tag{3.116}$$

This condition has the trivial solution $v = 0$, but also admits the nontrivial ones

$$v = -\frac{\kappa}{2\lambda} \pm \sqrt{\left(\frac{\kappa}{2\lambda}\right)^2 - \frac{\mu^2}{\lambda}} \; . \tag{3.117}$$

Therefore, for an appropriate choice of the parameters [96], this model admits a nontrivial vacuum configuration, and thus contains Spontaneous Symmetry Breaking (SSB). As the only transformation that leaves the vacuum invariant is given by $U \in Z(N)$, the SSB is $SU(N) \to Z(N)$. Due to the nontrivial first homotopy group of the vacuum manifold $\mathcal{M} = SU(N)/Z(N)$, the model admits static string-like solutions. These satisfy the equations of motion of the model, given by

$$D_j F_{ij} = g D_i \psi_A \wedge \psi_A \ , \tag{3.118}$$

$$D_i D_i \psi_A = \frac{\delta V_H}{\delta \psi_A} \ . \tag{3.119}$$

For the solution containing only one infinite straight string along the z axis, which corresponds to the confining string between a pair of heavy quarks infinitely separated, the gauge field must tend to the pure gauge $\frac{i}{g} S_0 \partial_i S_0^{-1}$ asymptotically, where $S_0 = e^{i\varphi\beta\cdot T}$, and $\beta = 2N\lambda^R$, λ^R being the highest weight of the representation R of the Wilson Loop. The Ansatz which was used to solve the above equations was

$$\Lambda_0 = 0 \quad , \quad \Lambda_i = S\mathcal{A}_i S^{-1} + \frac{i}{g} S\partial_i S^{-1} \quad , \quad \psi_A = h_{AB} S T_A S^{-1} \quad , \quad S = e^{i\varphi\beta\cdot T} \ . \tag{3.120}$$

This Ansatz may be written in a simpler form using the Cartan basis

$$\psi_\alpha = h_\alpha S T_\alpha S^{-1} \ , \ \psi_{\bar{\alpha}} = h_\alpha S T_{\bar{\alpha}} S^{-1} \ , \ \psi_q = h_{qp} S T_p S^{-1} \ . \tag{3.121}$$

In order for the gauge and $\psi_\alpha, \psi_{\bar{\alpha}}$ fields, with $\alpha \cdot \beta \neq 0$, to be well-defined along the z axis, the regularity conditions

$$a(0) = 0 \ , \tag{3.122a}$$

$$h_\alpha(0) = 0 \quad \text{when} \quad \alpha\cdot\beta \neq 0 \ , \tag{3.122b}$$

must be imposed. In this regard, note that

$$S T_\alpha S^{-1} = \cos(\varphi\beta\cdot\alpha)\, T_\alpha + \sin(\varphi\beta\cdot\alpha)\, T_{\bar{\alpha}} \ , \tag{3.123a}$$

$$S T_{\bar{\alpha}} S^{-1} = \cos(\varphi\beta\cdot\alpha)\, T_{\bar{\alpha}} - \sin(\varphi\beta\cdot\alpha)\, T_\alpha \ . \tag{3.123b}$$

This solution was initially studied in Ref. [101] for the $k-$Antisymmetric (k-A) and $k-$Symmetric ($k-$S) representations. In this case, it was sufficient to consider $\mathcal{A}_i = (a/g)\partial_i\varphi\beta\cdot T$. The goal of our work [102] was to find a solution for a general irrep R. For this purpose, it was important to study the existence of a BPS point in parameter space, which we shall discuss in the next section.

3.6.1 The BPS equations

To motivate the discussion of the BPS point of the non-Abelian model, let us initially consider the case of the Nielsen-Olesen model

$$S_{\text{Abe}} = \int d^4x \Big(-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}D_\mu\phi D_\mu\phi - \frac{\lambda}{8}(\phi\phi^* - v^2)^2 \Big) , \tag{3.124}$$

where $D_\mu = \partial_\mu - ig\Lambda_\mu$, $\phi \in \mathbb{C}$. The vacuum manifold of this model is given by $U(1)$, which has a nontrivial first homotopy group, and thus also admits string-like solutions which are topologically stable. For the case of an infinite string, it is possible to show that, for $\lambda = g^2$, the second order Euler-Lagrange equations of this model reduce to the first order ones

$$D_+\phi = 0 \ , \ B_3 = \frac{g}{2}(v^2 - \phi\phi^*) \ , \ B_1 = B_2 = 0 \ , \tag{3.125}$$

where $D_\pm = D_1 \pm iD_2$. For an in-depth discussion of this topic, see e.g. Ref. [103]. Moreover, it is well-known that some non-Abelian models containing $SU(N) \to Z(N)$ Spontaneous Symmetry Breaking also admit BPS points [104, 105, 106]. This motivated the search for a similar simplification in the model of Eq. (3.111), which turned out to be successful. In order to present the BPS equations of our model, it will be convenient to define the fields

$$\zeta_\alpha = \frac{\psi_\alpha + i\psi_{\bar\alpha}}{\sqrt{2}} \ , \tag{3.126}$$

which are in the complexified $\mathfrak{su}(N)$ Lie algebra (α is a positive root). Because we are considering an infinite static string, the first requirements are

$$B_1 = B_2 = 0 \quad , \quad D_3\psi_A = 0 \ . \tag{3.127}$$

Then, motivated by the non-Abelian generalizations of Eq. (3.125) considered in Refs. [104, 105, 106], we proposed the BPS equations

$$D_+\zeta_\alpha = 0 \quad \Leftrightarrow \quad D_-\zeta_\alpha^\dagger = 0 \quad , \quad D_1\psi_q = D_2\psi_q = 0 \ , \tag{3.128a}$$

$$B_3 = g\sum_{\alpha>0}\big(v\alpha|_q\psi_q - [\zeta_\alpha, \zeta_\alpha^\dagger]\big) \ . \tag{3.128b}$$

These equations can also be written in terms of the original fields as

$$D_\pm\psi_\alpha = \mp iD_\pm\psi_{\bar\alpha} \ , \tag{3.129}$$

$$B_3 = g\sum_{\alpha>0}(v\alpha|_q\psi_q - \psi_\alpha \wedge \psi_{\bar\alpha}) \ . \tag{3.130}$$

In section 3.6.4 we will show that these equations are indeed equivalent to the full equations of the model (3.118), (3.119) for the Ansatz that we considered.

3.6.2 The Ansatz for a general irrep R

As for the Ansatz, we will again use Eqs. (3.120), (3.121). For the $k-$A and $k-$S representations, it was sufficient, in Ref. [101], to consider $\mathcal{A}_i$ along a fixed direction in the Cartan subalgebra$\mathfrak{C}$. However, for a general R, it is necessary to consider a more general one

$$\mathcal{A}_i = \sum_{l=1}^{N-1} \frac{a_l - d_l}{g} \partial_i \varphi \beta^{l-\mathsf{A}} \cdot T \ , \tag{3.131}$$

where $\beta^{(l)} = 2N\lambda^{l-\mathsf{A}}$ and $\lambda^{l-\mathsf{A}}$, $l = 1, \ldots, N-1$ are the antisymmetric (fundamental) weights, which provide a basis $\beta^{(l)} \cdot T$ for $\mathfrak{C}$. The Dynkin numbers d_l are the positive integer coefficients obtained when expressing β as a linear combination of $\beta^{l-\mathsf{A}}$. The profiles a_l must obey the boundary conditions

$$a_l(0) = 0 \quad , \quad a_l(\infty) = d_l \ . \tag{3.132}$$

The first guarantees a finite action density and a well-defined strength field along the vortex core located at $\rho = 0$, while the second ensures that the gauge field is a pure gauge asymptotically, i.e. (3.120),

$$\Lambda_i \to \frac{\partial_i \varphi}{g} \beta \cdot T \ , \quad \text{when } \rho \to \infty \ . \tag{3.133}$$

- For this Ansatz, it holds that $D_i \psi_q = \partial_i \psi_q$ and, from Eqs. (3.127), (3.128a), that the fields ψ_q must be constant. We shall take $\psi_q \equiv vT_q$. Moreover, notice that Eq. (3.128a) implies

$$D_+ [\zeta_\alpha, \zeta_{\alpha'}] = [D_+ \zeta_\alpha, \zeta_{\alpha'}] + [\zeta_\alpha, D_+ \zeta_{\alpha'}] = 0 \ , \tag{3.134}$$

if both α and α' are positive roots. This suggests that $[\zeta_\alpha, \zeta_{\alpha'}]$ is proportional to another $\zeta_{\alpha''}$. In addition, notice that the boundary conditions imply that $\zeta_\alpha \to vE_\alpha$, for $\rho \to \infty$. Therefore,

$$[\zeta_\alpha, \zeta_{\alpha'}] \to v^2 \mathcal{N}_{\alpha,\alpha'} [E_\alpha, E_{\alpha'}] = v^2 \mathcal{N}_{\alpha,\alpha'} E_{\alpha+\alpha'}; \ , \quad \text{when } \rho \to \infty \ . \tag{3.135}$$

Then, it is natural to assume

$$[\zeta_\alpha, \zeta_{\alpha'}] = v \mathcal{N}_{\alpha,\alpha'} \zeta_{\alpha+\alpha'} \ . \tag{3.136}$$

In the following paragraph we show that this proposal is consistent with the regularity conditions at $\rho = 0$.

If $\alpha \cdot \beta \neq 0$, because of the Ansatz (3.121) and Eq. (3.123), we must impose $\zeta_\alpha(\rho \to 0) = 0$ to ensure regularity of the solution. Now, consider the case when $\beta \cdot \alpha \neq 0$ or $\beta \cdot \alpha' \neq 0$. Using the results of Appendix A, it follows that $\beta \cdot \gamma > 0$, $\forall \gamma > 0$. Then

$\beta \cdot (\alpha + \alpha') \neq 0$. In this case, to avoid the possible multivaluedness problem in Eq. (3.123), $\zeta_{\alpha+\alpha'}$ must be zero at $\rho = 0$, in accordance with the regularity condition of at least one of the factors in the left-hand side of Eq. (3.136). Moreover, when both $\beta \cdot \alpha = 0$ and $\beta \cdot \alpha' = 0$, the associated basis elements $E_\alpha, E_{\alpha'}$ do not rotate, so that ψ_α, $\psi_{\bar{\alpha}}$, $\psi_{\alpha'}$, $\psi_{\bar{\alpha}'}$ are not determined at the origin. In this case, in similarity with ψ_q, it holds that $D_i\psi_\alpha = \partial_i\psi_\alpha$, which suggests that we set $\psi_\alpha = vT_\alpha$, $\psi_{\bar{\alpha}} = vT_{\bar{\alpha}}$. This is consistent with the equations of motion and also with Eq. (3.136).

3.6.3 Reduced scalar BPS equations

Let us analyze the proposed BPS equations in more detail. Notice that

$$D_+(\Lambda)\zeta_\alpha = SD_+(\mathcal{A})(h_\alpha E_\alpha)S^{-1} = \Big(\partial_+ h_\alpha - i\partial_+\varphi h_\alpha \sum_{l=1}^{N-1}(a_l - d_l)\alpha \cdot \beta^{l-\mathsf{A}}\Big) S E_\alpha S^{-1} \ , \tag{3.137}$$

$$B^3 = \sum_{l=1}^{N-1} \frac{1}{g\rho}\frac{\partial a_l}{\partial \rho}\beta^{l-\mathsf{A}} \cdot T = g\sum_{\alpha>0} v^2\alpha \cdot T - \psi_\alpha \wedge \psi_{\bar{\alpha}} = g\sum_{\alpha>0}(v^2 - h_\alpha^2)S\alpha \cdot TS^{-1} \ . \tag{3.138}$$

These relations imply the BPS equations for the the gauge and Higgs scalar profiles

$$\partial_+ \ln h_\alpha = i\partial_+\varphi \sum_{l=1}^{N-1}(a_l - d_l)\alpha \cdot \beta^{l-\mathsf{A}} \ , \tag{3.139a}$$

$$\frac{1}{\rho}\frac{\partial a_l}{\partial \rho} = g^2 \sum_{\alpha>0}(v^2 - h_\alpha^2)\alpha \cdot \alpha^{(l)} \ . \tag{3.139b}$$

To obtain these equations, we used the well-known formula involving the fundamental weights and the simple roots $\alpha^{(p)} = \omega_p - \omega_{p+1}$:

$$\alpha^{(p)} \cdot \beta^{l-\mathsf{A}} = \delta^{pq} \ . \tag{3.140}$$

We have already discussed the property $[\zeta_\alpha, \zeta_{\alpha'}] = v\zeta_{\alpha+\alpha'}$. Naturally, this leads to $h_\alpha h_{\alpha'} = vh_{\alpha+\alpha'}$, which is consistent with Eq. (3.139a). Furthermore, as a general root can be written as a linear combination of simple roots, the scalar profiles $h_{\alpha^{(p)}}$ associated with simple roots, which satisfy

$$\partial_+ \ln h_{\alpha^{(p)}} = i\partial_+\varphi(a_p - d_p) \ , \tag{3.141}$$

can be used to generate all the others.

3.6.4 Making contact with the $SU(N) \to Z(N)$ model

The gauge-field equations

The BPS Eqs. (3.127), (3.128b), recalling that

$$B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk} \quad , \quad F_{ij} = \varepsilon_{ijk}B_k \; , \tag{3.142}$$

imply that

$$D_jF_{ij} \;\; = \;\; \varepsilon_{ijk}D_jB_k = -g\varepsilon_{ij3}D_j(\psi_\alpha \wedge \psi_{\bar{\alpha}}) \; . \tag{3.143}$$

Considering $i = 1$ and using the BPS equation for ψ_α, $\psi_{\bar{\alpha}}$, we get

$$\begin{aligned} D_jF_{1j} &= -gD_2(\psi_\alpha \wedge \psi_{\bar{\alpha}}) = -gD_2\psi_\alpha \wedge \psi_{\bar{\alpha}} - g\psi_\alpha \wedge D_2\psi_{\bar{\alpha}} \\ &= \frac{ig}{2}\left(D_+\psi_\alpha \wedge \psi_{\bar{\alpha}} - D_-\psi_\alpha \wedge \psi_{\bar{\alpha}} + \psi_\alpha \wedge D_+\psi_{\bar{\alpha}} - \psi_\alpha \wedge D_-\psi_{\bar{\alpha}}\right) \\ &= \frac{ig}{2}\left(-iD_+\psi_{\bar{\alpha}} \wedge \psi_{\bar{\alpha}} - iD_-\psi_{\bar{\alpha}} \wedge \psi_{\bar{\alpha}} + i\psi_\alpha \wedge D_+\psi_\alpha + i\psi_\alpha \wedge D_-\psi_\alpha\right) \\ &= -g\left(\psi_\alpha \wedge \frac{D_+ + D_-}{2}\psi_\alpha + \psi_{\bar{\alpha}} \wedge \frac{D_+ + D_-}{2}\psi_{\bar{\alpha}}\right) = gD_1\psi_A \wedge \psi_A \; . \end{aligned} \tag{3.144}$$

This is nothing but the component $i = 1$ of the full equation of motion of the model (3.118). A similar calculation can be done for $i = 2$, while $i = 3$ is trivially satisfied.

The Higgs-field equations

Cartan sector

Now, to make contact with the full equation of motion for the Higgs-field (3.119), we have to look for a Higgs potential V_{H} that is compatible with the BPS equations. In particular, Eqs. (3.127), (3.128a) imply $D_iD^i\psi_q = 0$, so that V_{H} must imply

$$\frac{\delta V_H}{\delta \psi_q} = 0 \tag{3.145}$$

on the ansatz given in Eqs. (3.120), (3.121) and (3.131), which closes the BPS equations. In what follows, we will see that this happens when it is given by

$$V_{\mathsf{H}}(\psi) = c + \frac{\mu^2}{2}\langle \psi_A, \psi_A \rangle + \frac{\kappa}{3} f_{ABC}\langle \psi_A \wedge \psi_B, \psi_C \rangle + \frac{\lambda}{4}\langle \psi_A \wedge \psi_B \rangle^2 \; , \tag{3.146}$$

with $\mu^2 = 0$ and $\lambda = g^2$. Of course, this is precisely the potential of Eq. (3.111). In this case,

$$\frac{\delta V_H}{\delta \psi_A} = \lambda \psi_B \wedge (\psi_A \wedge \psi_B - v f_{ABC}\psi_C) \; , \tag{3.147}$$

where $v = -\frac{\kappa}{\lambda}$. Indeed, using the proposed Ansatz, we get

$$\begin{aligned}\frac{\delta V_H}{\delta \psi_q} &= \lambda \sum_{\alpha>0} \psi_\alpha \wedge (\psi_q \wedge \psi_\alpha - v f_{q\alpha\bar{\alpha}} \psi_{\bar{\alpha}}) + \psi_{\bar{\alpha}} \wedge (\psi_q \wedge \psi_{\bar{\alpha}} - v f_{q\bar{\alpha}\alpha} \psi_\alpha) \\ &= \lambda v \sum_{\alpha>0} \left(h_\alpha S T_\alpha S^{-1}\right) \wedge \left(\alpha|_q h_\alpha S T_{\bar{\alpha}} S^{-1} - \alpha|_q h_\alpha S T_{\bar{\alpha}} S^{-1}\right) = 0 \; . \end{aligned} \tag{3.148}$$

Off-diagonal sector

Let us now analyze the equations for the fields ζ_α labeled by roots. The BPS equations imply

$$D^2 \zeta_\alpha = D_- D_+ \zeta_\alpha - g[B_3, \zeta_\alpha] = g^2 \sum_{\alpha'>0} \left[[\zeta_{\alpha'}, \zeta^\dagger_{\alpha'}] - v^2 \alpha' \cdot T, \zeta_\alpha \right] \; . \tag{3.149}$$

The sum over α' involves all positive roots, including α. On the other hand, according to the full equations of the model, we have

$$D^2 \zeta_\alpha = F_\alpha \quad , \quad F_\alpha = \frac{1}{\sqrt{2}} \left(\frac{\delta V}{\delta \psi_\alpha} + i \frac{\delta V}{\delta \psi_{\bar{\alpha}}} \right) \; . \tag{3.150}$$

According to Eq. (3.147), F_α receives contributions from the index types $B = q, \alpha, \bar{\alpha}, \gamma, \bar{\gamma}$ where $\gamma > 0$ is a root different from α. The partial contribution originated from the Cartan labels $B = q$ is given by

$$F_\alpha^{(B=q)} = \frac{\lambda}{\sqrt{2}} \psi_q \wedge (\psi_\alpha \wedge \psi_q - v f_{\alpha q \bar{\alpha}} \psi_{\bar{\alpha}} + i \psi_{\bar{\alpha}} \wedge \psi_q - i v f_{\bar{\alpha} q \alpha} \psi_\alpha) \; . \tag{3.151}$$

Using the Ansatz equations (3.120), (3.121), and also $\psi_q = v T_q$, we have

$$\psi_\alpha \wedge \psi_q = v f_{\alpha q \bar{\alpha}} \psi_{\bar{\alpha}} \; , \tag{3.152a}$$

$$\psi_{\bar{\alpha}} \wedge \psi_q = v f_{\bar{\alpha} q \alpha} \psi_\alpha \; , \tag{3.152b}$$

which imply $F_\alpha^{(B=q)} = 0$. Next, there are the contributions originated from $B = \alpha, \bar{\alpha}$

$$\begin{aligned} F_\alpha^{(B=\alpha,\bar{\alpha})} &= \frac{\lambda}{\sqrt{2}} \left(\psi_{\bar{\alpha}} \wedge (\psi_\alpha \wedge \psi_{\bar{\alpha}} - v f_{\alpha\bar{\alpha}q} \psi_q) + i \psi_\alpha \wedge (\psi_{\bar{\alpha}} \wedge \psi_\alpha - v f_{\bar{\alpha}\alpha q} \psi_q) \right) \\ &= \lambda \frac{\psi_{\bar{\alpha}} - i \psi_\alpha}{\sqrt{2}} \wedge (\psi_\alpha \wedge \psi_{\bar{\alpha}} - v f_{\alpha\bar{\alpha}q} \psi_q) \\ &= \lambda \left[[\zeta_\alpha, \zeta^\dagger_\alpha] - v \alpha \cdot \psi, \zeta_\alpha \right] \; , \end{aligned} \tag{3.153}$$

where we used the property

$$\psi_\alpha \wedge \psi_{\bar{\alpha}} = \left[\zeta_\alpha, \zeta^\dagger_\alpha \right] \; . \tag{3.154}$$

Finally, we evaluate $F_\alpha^{(B=\gamma,\bar\gamma)} = P_\alpha + Q_\alpha$, where P_α (Q_α) is the part without (with) explicit dependence on the structure constants. They are given by a sum over positive roots $\gamma \neq \alpha$

$$P_\alpha = \lambda \sum_{\gamma\neq\alpha} \left(\psi_\gamma \wedge (\zeta_\alpha \wedge \psi_\gamma) + \psi_{\bar\gamma} \wedge (\zeta_\alpha \wedge \psi_{\bar\gamma})\right) \tag{3.155a}$$

$$Q_\alpha = \frac{\lambda v}{\sqrt{2}} \sum_{\gamma\neq\alpha} \left(f_{\alpha\gamma\bar\delta}\psi_\gamma \wedge \psi_{\bar\delta} - f_{\alpha\bar\gamma\delta}\psi_{\bar\gamma} \wedge \psi_\delta - i f_{\bar\alpha\gamma\delta}\psi_\gamma \wedge \psi_\delta - i f_{\bar\alpha\bar\gamma\bar\delta}\psi_{\bar\gamma} \wedge \psi_{\bar\delta}\right). \tag{3.155b}$$

Using Eq. (3.136), we arrive at

$$\begin{aligned} P_\alpha = \lambda \sum_{\gamma\neq\alpha} \left(\zeta_\gamma \wedge (\zeta_\alpha \wedge \zeta_\gamma^\dagger) + \zeta_\gamma^\dagger \wedge (\zeta_\alpha \wedge \zeta_\gamma)\right) = \\ \lambda \sum_{\gamma\neq\alpha} \left([[\zeta_\gamma, \zeta_\gamma^\dagger], \zeta_\alpha] - 2v\mathcal{N}_{\alpha,\gamma}[\zeta_\gamma^\dagger, \zeta_{\alpha+\gamma}]\right) \ . \end{aligned} \tag{3.156}$$

On the other hand, by using Eqs. (A.19) and (3.126) it is possible to write Q_α as follows

$$Q_\alpha \;=\; \lambda v \sum_{\gamma\neq\alpha} \left(\mathcal{N}_{\alpha,\gamma}[\zeta_\gamma^\dagger, \zeta_{\alpha+\gamma}] + \mathcal{N}_{\alpha,-\gamma}[\zeta_\gamma, \zeta_{\alpha-\gamma}]\right) \ . \tag{3.157}$$

Let us analyze the term with label $\alpha - \gamma$. Because γ is a positive root, $\alpha - \gamma$ is not necessarily positive, so we cannot use Eq. (3.136) right away. Instead, we shall split the sum over γ as follows:

$$\sum_{\gamma\neq\alpha} = \sum_{\gamma^+\neq\alpha} + \sum_{\gamma^-\neq\alpha} \ , \tag{3.158}$$

where $\gamma = \gamma^+$ ($\gamma = \gamma^-$) is such that $\alpha - \gamma^+$ ($\alpha - \gamma^-$) is a positive (negative) root. As for the γ^- contribution,

$$\lambda v \mathcal{N}_{\alpha,-\gamma^-}[\zeta_{\gamma^-}, \zeta_{\alpha-\gamma^-}] = \lambda v \mathcal{N}_{\alpha,-\sigma-\alpha}[\zeta_{\sigma+\alpha}, \zeta_{-\sigma}] = \lambda v \mathcal{N}_{\alpha,\sigma}\left[\zeta_\sigma^\dagger, \zeta_{\sigma+\alpha}\right] \ , \tag{3.159}$$

where on the second equality we have defined $\sigma = -(\alpha - \gamma^-)$ which is, by definition, a positive root. Moreover, $\alpha + \sigma$ yields another positive root. This is equal to the first term of Eq. (3.157). Therefore,

$$Q_\alpha = \lambda v \sum_{\gamma\neq\alpha} 2\mathcal{N}_{\alpha,\gamma}[\zeta_\gamma^\dagger, \zeta_{\alpha+\gamma}] + \lambda v \sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+}[\zeta_{\gamma^+}, \zeta_{\alpha-\gamma^+}] \ , \tag{3.160}$$

which together with the result for P_α yields

$$F_\alpha^{(B=\gamma,\bar\gamma)} \;=\; \lambda \sum_{\gamma\neq\alpha} \left[[\zeta_\gamma, \zeta_\gamma^\dagger], \zeta_\alpha\right] + \lambda v \sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+}[\zeta_{\gamma^+}, \zeta_{\alpha-\gamma^+}] \ . \tag{3.161}$$

By the definition of γ^+, $\alpha - \gamma^+$ is positive so we can use Eq. (3.136) to write

$$\begin{aligned} F_\alpha^{(B=\gamma,\bar{\gamma})} = & \quad \lambda \sum_{\gamma \neq \alpha} \left[[\zeta_\gamma, \zeta_\gamma^\dagger], \zeta_\alpha \right] + \lambda v^2 \sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+} \mathcal{N}_{\gamma^+,\alpha-\gamma^+} \zeta_\alpha \\ = & \quad \lambda \sum_{\gamma \neq \alpha} \left[[\zeta_\gamma, \zeta_\gamma^\dagger], \zeta_\alpha \right] - \lambda v^2 \sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+}^2 \zeta_\alpha \ . \end{aligned} \tag{3.162}$$

To evaluate the sum over γ^+, we must count how many roots are consistent with the condition $\alpha - \gamma^+ > 0$. For this purpose, we can use that $\alpha = \omega_I - \omega_J$ for some $I < J$. Then, there are two cases

$$\begin{aligned} \gamma^+ &= \omega_I - \omega_l \ , I < l < J \Rightarrow J - I - 1 \quad \text{possibilities,} \\ \gamma^+ &= \omega_l - \omega_J \ , I < l < J \Rightarrow J - I - 1 \quad \text{possibilities.} \end{aligned}$$

As $\mathcal{N}_{\alpha,-\gamma^+}^2 = \frac{1}{2N}$ in both of these cases, we have

$$\sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+}^2 = \frac{J - I - 1}{N} \ . \tag{3.163}$$

The sum of the $\mathcal{N}^2$-factors in Eq. (3.162) can be rewritten as a sum of $(\alpha \cdot \gamma)$-factors:

$$\sum_{\gamma \neq \alpha} \alpha \cdot \gamma = \frac{N + J - I - 3}{2N} - \frac{N - J + I - 1}{2N} = \sum_{\gamma^+} \mathcal{N}_{\alpha,-\gamma^+}^2 \ , \tag{3.164}$$

where we used a similar counting to determine how many positive roots γ different from α have $\alpha \cdot \gamma = \pm \frac{1}{2N}$. In addition, using that, for our Ansatz,

$$\alpha \cdot \gamma \zeta_\alpha = [\gamma \cdot T, \zeta_\alpha] \ , \tag{3.165}$$

it follows that

$$F_\alpha^{(B=\gamma,\bar{\gamma})} = \lambda \sum_{\gamma \neq \alpha} \left[[\zeta_\gamma, \zeta_\gamma^\dagger] - v^2 \gamma \cdot T, \zeta_\alpha \right] \ . \tag{3.166}$$

Finally, using this result, together with $F_\alpha^{(B=q)} = 0$ and Eq. (3.153), we get

$$D^2 \zeta_\alpha = \lambda \left[[\zeta_\alpha, \zeta_\alpha^\dagger] - v^2 \alpha \cdot T, \zeta_\alpha \right] + \lambda \sum_{\gamma \neq \alpha} \left[[\zeta_\gamma^\dagger, \zeta_\gamma] - v^2 \gamma \cdot T, \zeta_\alpha \right] = \lambda \sum_{\alpha' > 0} \left[v^2 \alpha' \cdot T - [\zeta_{\alpha'}, \zeta_{\alpha'}^\dagger], \zeta_\alpha \right] \ , \tag{3.167}$$

which equals Eq. (3.149) for $\lambda = g^2$. We have thus shown that the proposed BPS equations imply the full equations of motion of the model.

3.6.5 Physical analysis

Stability of the asymptotic Casimir scaling law

In the previous sections, for each quark representation, we reviewed that at $\mu^2 = 0$, $\lambda = g^2$ the proposed vortex Ansatz that satisfies the BPS equations provide a static vortex solution for the $SU(N) \to Z(N)$ YMH model defined in Eq. (3.111). From Eqs. (3.126)-(3.128), the corresponding energy per unit-length is

$$\epsilon \;=\; \int d^2x \left(\frac{1}{2}\langle B_3, B_3\rangle + \sum_{\alpha>0}\langle D_i\zeta_\alpha^\dagger, D_i\zeta_\alpha\rangle + V_{\mathsf{H}}(\psi)\right) , \tag{3.168}$$

where d^2x integrates over the transverse directions to the infinite string. Using Derrick's theorem in two dimensions, we can equate the potential energy of the Higgs field to that of the gauge field, thus obtaining

$$\begin{aligned}
\epsilon &= \int d^2x \,\langle B_3, B_3\rangle - \langle \zeta_\alpha^\dagger, D^2\zeta_\alpha\rangle \\
&= \int d^2x \,\langle B_3, B_3\rangle - \langle \zeta_\alpha^\dagger, D_- D_+\zeta_\alpha\rangle + g\langle \zeta_\alpha^\dagger, [B_3, \zeta_\alpha]\rangle = \int d^2x \,\langle B^3, B^3 + g[\zeta_\alpha, \zeta_\alpha^\dagger]\rangle \\
&= \int d^2x \, gv^2 \langle B^3, 2\delta\cdot T\rangle = gv^2 \oint \langle \Lambda_i, 2\delta\cdot T\rangle \, dx_i \ ,
\end{aligned} \tag{3.169}$$

where δ is the sum of all positive roots and the last integral must be taken along a cicle with infinite radius. Recalling Eq. (3.133), this implies that

$$\epsilon = 2\pi g v^2 \beta\cdot 2\delta \ . \tag{3.170}$$

at the BPS point. In particular, note that the k-A string tension scales with the quadratic Casimir, as $\beta\cdot 2\delta = \frac{N}{N+1} C_2(k\text{-A})$ in this case. This is the result obtained in Ref. [101]. The important physical consequence that we will derive from Eq. (3.170) is that for a general representation $\mathsf{D}(\cdot)$ with N-ality k, the asymptotic string tension satisfies

$$\frac{\sigma(\mathsf{D})}{\sigma(\mathsf{F})} = \frac{C_2(k\text{-A})}{C_2(\mathsf{F})} \ , \tag{3.171}$$

which is among the possibilities which are consistent with lattice simulations.

In what follows, we shall see that the smallest $\beta\cdot 2\delta$ factor, and thus that the lowest energy, is given by the k-A weight. This was the result obtained in our work [102]. To prove this, some Young Tableaux technology, useful to study the properties of the irreducible representations, was required. In this discussion, we closely followed the ideas in Ref. [107]. A Young Tableau consists of a set of adjacent boxes organized according to the following rules:

1. The maximum allowed number of boxes (n_i) in a given column is $N-1$.

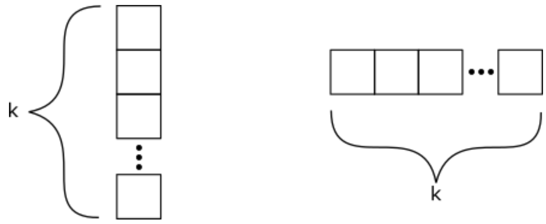


Figure 3.3: Young tableaux for the k-Symmetric (right) and k-Antisymmetric (left) representations.

2. n_i should be lower or equal than the number of boxes n_j in any column to the left. That is, $i > j \to n_i \leq n_j$.

3. The number of boxes in a given row (m_i) should be lower or equal than the number of boxes in any row above. That is, $i > j \to m_i \leq m_j$.

Every diagram drawn according to these rules corresponds to an irreducible representation of $SU(N)$. Many properties of these irreps can be easily identified in this language [107]. The $N-$ality of a representation is given by the number of boxes of the Young Tableau, modulo N. The Dynkin indices d_k of the highest weight λ^D satisfy [107][1]

$$\lambda^D = \sum_{l=1}^{N-1} d_l \lambda^{l\text{-A}} \quad , \quad d_i = m_i - m_{i+1} \; . \tag{3.172}$$

When a box is moved from an upper to a lower row, an irrep. with more antisymmetries is obtained. For example, the Young tableau for the k-A (k-S) irrep. has one column (row) with k boxes, as shown in Fig. 3.3. For an irrep. with N-ality k, that is, a Young tableau with a total number of boxes of the form $k + nN$, the factor $\beta \cdot 2\delta$ can be written as

$$\beta \cdot 2\delta \;\; = \;\; \frac{N}{N+1} \sum_{l=1}^{N-1} d_l \, l(N-l) = N(k+nN) - \frac{2N}{N+1} \sum_{l=1}^{N-1} m_l \, l \; . \tag{3.173}$$

Then, if a pair of irreps. D and D′ with magnetic weights β and β', respectively, have the same N-ality k, it follows that

$$\Delta\beta \cdot 2\delta = \beta' \cdot 2\delta - \beta \cdot 2\delta = N^2 \Delta n - \frac{2N}{N+1} \sum_{l=1}^{N-1} \Delta m_l \, l \; , \tag{3.174}$$

$\Delta m_l = m_l' - m_l$, $\Delta n = n' - n$, where the primed variables refer to D′. Let us initially consider a pair of Young tableaux with the same number of boxes, so that $\Delta n = 0$. If a box is moved from an upper row I to a lower row J (see Fig. 3.4 for an example), we have $I < J$ and $\Delta m_J = -\Delta m_I = 1$; consequently, $\Delta\beta \cdot 2\delta = \frac{2N}{N+1}(I - J) < 0$.

[1] When $i = N - 1$, we take $m_N = 0$.

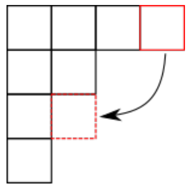
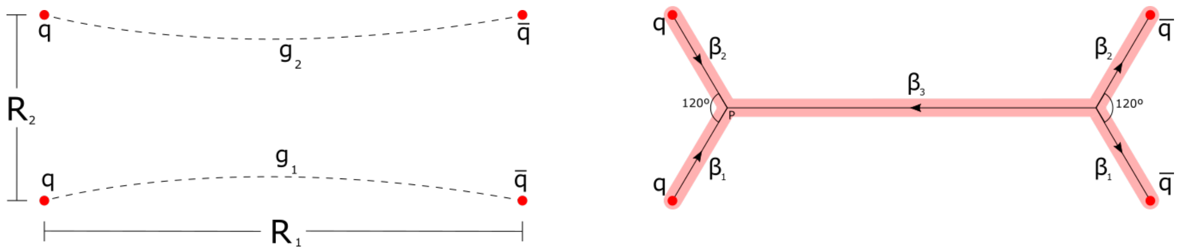


Figure 3.4: A transformation on a tableau that decreases the factor $\beta \cdot \delta$.

This means that, for a given number of boxes $k + nN$, the tableau with smallest $\beta \cdot 2\delta$ is that in which the boxes are as lowered as possible. Now, let us consider a general representation D with $N-$ality k. Its Young tableau contains $k + nN$ boxes, where n is a natural number. Clearly, by moving the boxes from the upper to the lower rows, it will be possible to obtain a Young tableau whose first n columns are composed of N boxes, and the last column will contain the last k boxes. The first N columns can then be erased, as they correspond to singlets, and the resulting Young tableau is that of the $k-$A irrep. As the product $\beta \cdot 2\delta$ decreased in each step of this process, it follows that the lowest factor $\beta \cdot 2\delta$ is that of the $k-$A representation, which is what we wanted to show.

Now, to complete the analysis of the asymptotic scaling, we need to discuss how the Wilson loop would be assessed in the effective model in Eq. (3.111), as this is the observable used in the lattice to compute string tensions. Indeed, as discussed in section 3.5.2, this model emerges as an effective description of center-element averages, which depend on the linking number between center vortices and the Wilson loop $\mathcal{C}$. Recall that β is a magnetic weight associated with the quark representation, and $s_{\mu\nu}$ is concentrated on any surface $S(\mathcal{C})$, parametrized by $w(s,\tau)$, whose border is $\mathcal{C}$. More precisely, $J_{\mu\nu} = 2\pi s_{\mu\nu}\beta \cdot T$ was introduced to compute intersection numbers in the initial ensemble, which are equivalent to the linking numbers between $\mathcal{C}$ and the vortex worldsurfaces. As usual, the confining state in the presence of a static quark-antiquark pair is obtained from a rectangular Wilson loop with one side along the Euclidean time with length $T \to \infty$. In the energy functional, $J_{\mu\nu}$ gives place to unobservable Dirac strings with endpoints at the (physical) quark and antiquark locations. These strings need to be taken into account for vortices of finite length. In this case, solutions of the form (3.120), with modified regularity conditions so as to cancel the Dirac strings, can be obtained. They correspond to smooth finite strings, which in the limit of large quark-antiquark separations make contact with the BPS solutions studied in this work. However, most of these solutions are in fact local minima or metastable states. Other finite energy solutions where the Dirac strings are also canceled may involve dynamical adjoint monopoles (also known as valence gluons) created around the sources [108]. As the adjoint representation has trivial N-ality, the favored asymptotic confining string will be the one with the lowest energy among those with the same N-ality (k) of D$(\cdot)$. From the previous discussion, this corresponds to the k-A string, which completes the



(a) Two quark-antiquark pairs (b) Double Y-shaped flux tube in $SU(3)$

Figure 3.5: $qq\bar{q}\bar{q}$ probes: a) The stable flux configuration includes the energy minimization over all possible guiding-centers g_1, g_2. b) For $R_1 > \sqrt{3}R_2$, the coalescence of g_1, g_2 is favored, as the sum of the fundamental $\mathfrak{su}(3)$ weights β_1, β_2 is the antifundamental weight $-\beta_3$ (N-ality).

proof of the asymptotic Casimir scaling in Eq. (3.171).

Configurations induced by a pair of external quark-antiquark sources

In Monte Carlo simulations, when studying an observable that creates static sources during a large time interval T, the leading behavior is dominated by the lowest energy state, consistent with the quantum numbers of the observable, that can be created. In the effective model, this state corresponds to the lowest energy configuration compatible with the conditions imposed by the sources. For example, it is clear that the lattice simulation of the Wilson loop in the k-A irrep. corresponds to a straight string (with cylindrical symmetry), running from the quark to the antiquark. This will be the global minimum, as the introduction of dynamical monopoles or wiggles in the string will certainly increase the energy. Indeed, at asymptotic distances, where the effective model is expected to be valid, this will make contact with the translationally symmetric BPS k-A string solution studied in the previous sections.

Now, at $\mu^2 = 0$, the nontrivial profiles for translationally symmetric configurations with any number of k-A strings, given by the ansatz in Eq. (3.120), were shown to obey Nielsen-Olesen equations [101]. At the critical coupling, this implies that they do not interact. However, this is not necessarily related with the behavior of flux tubes in Yang-Mills theories. For example, to analyze a situation with a pair of sources and sinks (see Fig. 3.5a), an observable that creates a tetraquark must be considered. Again, the lattice result has to be compared with the global minimization of the effective energy functional in the presence of the external sources induced by this observable, without any further restrictions on the fields. Moreover, the multivortex critical solutions do not contemplate the minimization with respect to translationally nonsymmetric configurations. That is, when the sources and sinks are far apart from each other, the noninteracting translationally invariant configuration could be a metastable state associated with a local minimum. To settle this discussion, let us take a closer look to the case of $SU(3)$ with fundamental quarks. As pointed out in Refs. [109, 110, 111], the flux

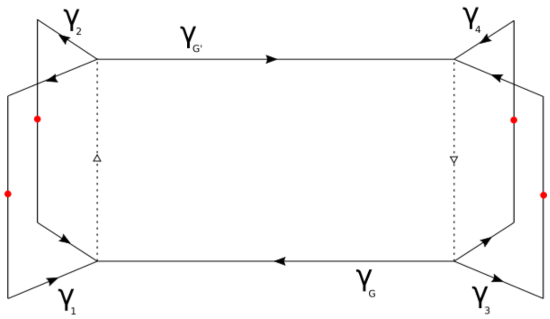


Figure 3.6: Visualization of the tetraquark observable W_{4q}. The dashed lines are optional holonomies that can be included without changing the observable.

distribution of this tetraquark configuration strongly depends on the distance between the quark-antiquark pairs. For $R_1 > \sqrt{3}R_2$ (with asymptotic values for both R_1 and R_2), the energy distribution is given by a double Y-shaped configuration, as depicted in Fig. 3.5b. This behavior was computed in the lattice, by considering the tetraquark observable [109]

$$W_{4q}[A_\mu] = \frac{1}{12}\epsilon^{abc}\epsilon^{def}\epsilon^{a'b'c'}\epsilon^{d'e'f'}\Gamma_1|^{aa'}\Gamma_2|^{bb'}\Gamma_G|^{cf}\Gamma_3|^{d'd}\Gamma_4|^{e'e}\Gamma_{G'}|^{f'c'} \; , \tag{3.175}$$

where A_μ is the fundamental field of pure Yang-Mills theory and the different holonomies Γ are evaluated along the paths $\gamma_1, \ldots, \gamma_4$, γ_G, $\gamma_{G'}$ (see Fig. 3.6).

In the center-vortex ensemble picture, the tetraquark observable is related with the average of

$$W_{4q} = \prod_{i=1}^{4} z^{\sum_w L(\gamma_i^c, w)} z^{\sum_w 2L(\gamma_5^c, w)} \tag{3.176}$$

over closed worldsurfaces w, as this is the contribution to the tetraquark variable W_{4q} when evaluated on thin center-vortices. Here, $z = e^{i2\pi/3}$ is a center element, and the closed paths γ_1^c, γ_2^c (resp. γ_3^c, γ_4^c) are the composition of γ_1, γ_2 (resp. γ_3, γ_4) with the adjacent dotted line γ_L (resp. γ_R). In addition, the closed path γ_5^c is given by the composition of $\gamma_G, \gamma_L, \gamma_{G'}$ and γ_R. $L(\gamma_k^c, w)$ is the linking number between w and the closed paths γ_k^c, while the factor 2 is because γ_5^c has opposite orientation compared with $\gamma_1^c, \ldots, \gamma_4^c$, and $z^{-1} = z^2$. Due to Eq. (3.176), a possibility is given by

$$J_{\mu\nu} = 2\pi \sum_{k=1}^{5} \beta(\gamma_k^c) \cdot T s_{\mu\nu}^k \tag{3.177}$$

where $s_{\mu\nu}^k$ is localized on a surface $S(\gamma_k^c)$ whose border is γ_k^c and

$$\beta(\gamma_1^c) = \beta(\gamma_3^c) = \beta_1 \quad , \quad \beta(\gamma_2^c) = \beta(\gamma_4^c) = \beta_2 \quad , \quad \beta(\gamma_5^c) = \beta_3 = -\beta_1 - \beta_2 \; , \tag{3.178}$$

where $\beta_k = 2N\omega_k$, and ω_1, ω_2, ω_3 are the three (ordered) weights of the fundamental representation of $SU(3)$. Indeed, in the lattice, this introduces a frustration factor in the Wilson action

$$e^{-i\alpha_{\mu\nu}} \quad , \quad \alpha_{\mu\nu} = \alpha^1_{\mu\nu} + \cdots + \alpha^1_{\mu\nu} - \alpha^5_{\mu\nu} \quad , \quad \alpha^k_{\mu\nu} = \begin{cases} 2\pi\beta(\gamma^c_k) \cdot T & \text{if } \langle\mu\nu\rangle \text{ intersects } S(\gamma^c_k) \\ 0 & \text{otherwise,} \end{cases}$$

defined on the lattice plaquettes $\langle\mu\nu\rangle$. In the expansion of the Wilson action of the ensemble defined by Eq. (3.109), the nontrivial contribution is originated from plaquettes distributed on closed worldsurfaces w. When γ^c_k links w, then $S(\gamma^c_k)$ is intersected. This gives a factor $e^{i2\pi\beta_1 \cdot T} = e^{i2\pi\beta_2 \cdot T} = zI$ or $e^{-i2\pi\beta_3 \cdot T} = e^{i2\pi(\beta_1+\beta_2)\cdot T} = z^2 I$, thus reproducing Eq. (3.176). Similarly to the case of a single Wilson loop, at fixed time the external source in Eq. (3.177) will give rise to unobservable Dirac lines, which can be chosen as entering the lower (upper) antiquark and leaving the lower (upper) quark with β_1 (β_2). In this case, in order for the energy to be finite, a field configuration based on a phase $S = e^{i(\beta_1\chi_1 + i\beta_2\chi_2)\cdot T}$ is required, where χ_1 (χ_2) is multivalued when going around a closed path designed to cancel the Dirac string of type β_1 (β_2). This leaves the effect of a pair of guiding centers g_1, g_2 (Fig. 3.5a) where the fields must be in a false vacuum, so that the energy will be mainly concentrated around them. It is clear that for $R_1 > \sqrt{3}R_2$ (with asymptotic R_1, R_2), the energy minimization, which includes the variation of g_1 and g_2, will favor a Y-shaped global minimum as shown in Fig. 3.5b. This is due to the fact that, in the common part, the sum of fundamental magnetic weights β_1 and β_2 will combine to $-\beta_3$, which implies the same energy cost of a single fundamental string. In other words, the observed Y-shaped configuration is nothing but the implications of N-ality stated in the language of weights.

Chapter 4

The Yang-Mills ensemble

The center vortex scenario, together with its effective descriptions discussed in this thesis, are powerful tools for the description of the phenomenology of confinement. However, up to this point, it is not clear at all how such an ensemble could emerge from a first principles calculation. In fact, it is not even clear how to perform a first principles calculation in continuum Yang-Mills theory in the infrared regime, as any global gauge is doomed by the existence of Gribov Copies [13]. We start this section by briefly reviewing global gauge fixing procedures and their limitations. Then, we present an alternative solution, initially introduced in Ref. [112], and then further studied in Ref [113], where the configuration space of YM theory is divided into sectors labeled by defects, and the gauge is fixed by a local, sector-dependent gauge condition. We show that this approach is not only a promising candidate to deal with the Gribov problem, but also is closely connected with a center-vortex ensemble.

4.1 Yang-Mills (global) gauge-fixings

In this section, we provide a brief discussion of some of the global gauge-fixings commonly used for continuum and lattice nonabelian gauge theories. These gauges are global, in the sense that a unique condition is imposed on the whole configuration space $\{A_\mu\}$. Then, in the next sections, we will introduce our local procedure and discuss how it could avoid the limitations of the global procedures.

Let us initially consider gauge theories in the continuum. In this case, globally defined gauge-fixing conditions,

$$f(A) = 0 \quad , \quad A_\mu \in \{A_\mu\} \; , \tag{4.1}$$

were extensively studied. For example, the Landau gauge corresponds to $f(A) = \partial_\mu A_\mu$. Due to Singer's theorem [13], it is impossible to find a continuous condition on the whole configuration space $\{A_\mu\}$ that avoids Gribov copies, i.e. such that $f(A^U) = 0 \Rightarrow U =$

$\mathbb{I}$. Then, in this framework, to continue working with the traditional methods, which are based on a single global $f(A)$, one possibility is to restrict the path integral to the first Gribov region, which is a subset of $\{A_\mu\}$. This region is defined as the smallest connected set, containing the trivial configuration $A_\mu = 0$, such that the Fadeev-Popov (FP) operator is positive definite [114]. This region generally contains, however, finite copies. The restriction of the path integral to the first Gribov region is done via the introduction of nonlocal extra terms in the action, which are then localized with the aid of auxiliary fields [115]. This approach was initially plagued by infrared instabilities which were then solved with the use of the Refined Gribov-Zwanziger action [116, 117]. Finally, we point out the existence of a Becchi-Rouet-Stora-Tyutin (BRST) invariant formulation of the path integral restriction, with a local and renormalizable action, that was implemented in Refs. [118, 119, 120]. The computation of an observable regarding the confining flux tube in this framework is, however, still lacking. It should be mentioned that the first Gribov region is a gauge-dependent concept, as it is defined in terms of the gauge-dependent FP operator. In the infrared regime, it is believed that the YM path-integral in Landau gauge is dominated by configurations on the Gribov horizon [121, 122], which is the boundary of the first Gribov region. The corresponding FP operators were extensively studied in the continuum and in the lattice for the Landau and Coulomb gauges [122, 123, 124]. For example, in the Landau gauge, where the FP operator is given by

$$M^{ab}_{\text{Landau}} = -\partial_\mu D^{ab}_\mu \delta^{(4)}(x-y) \;, \tag{4.2}$$

it was shown that typical center vortices and instantons belong to the corresponding Gribov horizon [75, 125, 126].

In the lattice, as mentioned in chapter 3, center vortices have been extensively studied in the infrared regime. In this case, although a gauge-fixing is not necessary for the computation of observables, it is important for identifying the dominant configurations in the confining regime. These studies were initially carried out in the Maximal Center Gauge (MCG) [64, 65, 66], which brings each link element as close as possible to an element of the center $Z(N)$ of $SU(N)$. Given an initial link configuration $U_\mu(x) \in SU(N)$, the gauge is defined by means of the following maximization over gauge transformations $g(x)$

$$\text{max}_g \sum_{x,\mu} \big(\text{tr}\,\text{Ad}\big(U^g_\mu(x)\big)\big) \quad , \quad \text{Ad}\big(U^g_\mu(x)\big) = R^{\mathsf{T}}(x)\,\text{Ad}\big(U_\mu(x)\big)R(x+\mu) \;, \tag{4.3}$$

with $R = \text{Ad}\big(g\big)$, $\text{Ad}(\cdot)$ denoting the adjoint representation of $SU(N)$. In Ref. [66], this gauge was generalized to the continuum by means of the condition

$$\text{min}_\Sigma\, \text{min}_g \int d^Dx\, \left(\text{tr}\,(A^g - a_\Sigma)^2\right) \;, \tag{4.4}$$

where a_Σ is the gauge field of a thin vortex whose guiding center is localized on the closed surface $\partial\Sigma$. A condition for local extrema can be obtained by first considering the extremization with respect to $g = e^{i\theta}$, with infinitesimal θ, and fixed Σ, which leads to

$$[\partial_\mu + a_\mu^\Sigma, A_\mu] - \partial_\mu a_\mu^\Sigma = 0 \ . \tag{4.5}$$

If this condition was free from Gribov copies, a well-defined map $\Sigma \to A_\mu[\Sigma]$, with $A[\Sigma]$ satisfying Eq. (4.5), would exist. Then, the continuum Maximal Center Gauge would be achieved by minimizing over Σ:

$$\mathrm{min}_\Sigma \int d^D x \, \mathrm{tr}(A[\Sigma] - a_\Sigma)^2 \ . \tag{4.6}$$

This is a conceptually interesting procedure, which tries to bring A_μ as close as possible to a thin center vortex field $a_\Sigma|_\mu$. However, as noticed in Ref. [66], this idea would require further developments, as there is a large mismatch between a smooth A_μ and a thin center-vortex field $a_\Sigma|_\mu$ at points that are close to any $\partial\Sigma$, where the difference $A - a_\Sigma$ diverges. Thus, the condition (4.4) is always achieved for a trivial a_Σ, even for vortex-like smooth configurations A_μ. Some possibilities to avoid this problem were considered in Ref. [66]: a smoothed a_Σ or the replacement $\mathrm{tr}(\cdot) \to s(\mathrm{tr}(\cdot))$ in Eq. (4.4), with $s(t)$ a monotonically increasing function. A problem pointed in that work is that, to avoid the divergence at $\partial\Sigma$, $s(t)$ cannot diverge as $t \to \infty$. However, in this case, large deviations between A_μ and $a_\Sigma|_\mu$ in other regions would not be penalized. Additionally, for certain functions as $s(t) = -\tanh(R^4 t^2)$, it was noted that the best $\partial\Sigma$ does not coincide with the spatial location of the guiding-center of a smooth center-vortex A_μ, even for the simplest example.

Another important class of gauges in the lattice considers a set of eigenvectors $\phi^{(j)}$ corresponding to the lowest eigenvalues of the lattice covariant adjoint Laplacian,

$$\Delta_{xy}^{ab}(U)\phi_b^{(j)}(y) = \mu_j \phi_a^{(j)}(x) \ . \tag{4.7}$$

The gauge can then be fixed by imposing different conditions on the eigenfunctions associated with the lowest eigenvalues. For instance, in the Laplacian Center Gauge (LCG) [127], the gauge-fixed configuration is achieved by the composition of a pair of gauge transformations on the link variables. The first one orients the lowest eigenfunction $\phi^{(1)}$ along the Cartan subalgebra. Then, a second transformation is performed to ensure that the color components of the second lowest eigenfunction $\phi^{(2)}$ satisfy some conventional conditions. The second transformation must, additionally, keep $\phi^{(1)}$ fixed. The possibility of extending this gauge to the continuum was first pointed out in Ref. [128]. For this purpose, it was suggested that the pair of Laplacian eigenfunctions should be replaced by other adjoint fields. However, a specific realization for these fields was not presented. Additionally, the use of such a global gauge-fixing condition on these adjoint fields would, in general, lead to singular gauge-fixed fields. This would happen, in particular, for vortex-like configurations, due to the topological character of their phases. In the lattice, we would also like to mention the Direct Laplacian Center Gauge (DLCG), introduced in Ref. [129], motivated by the above mentioned discrepancy between smooth and thin configurations in the MCG. For $SU(2)$, instead of introducing the function $s(t)$, the MCG was smoothed out by promoting $R(x) \in SO(3)$ to a new degree of freedom $M(x)$, given by a 3×3 real matrix. The initial step to achieve this gauge is to perform the constrained maximization

$$\max_M \sum_{x,\mu} \mathrm{tr}\,\left(M^{\mathsf{T}}(x)\,\mathrm{Ad}\big(U_\mu(x)\big)M(x+\mu)\right) \quad , \quad \frac{1}{\mathcal{V}}\sum_x M^{\mathsf{T}}(x)M(x) = I_{3\times 3}\ , \tag{4.8}$$

with $\mathcal{V}$ being the lattice volume. Then, it was shown that the solution to this maximization can be written as $M_{ab}(x) = \phi_b^{(a)}(x)$. In the next step, an $SO(3)$-field is extracted from $M(x)$ through a polar decomposition. This field is then mapped to $SU(2)$ and the link-variables are gauge transformed to satisfy the adjoint version of the lattice Laplacian Landau Gauge (LLG) introduced in Ref. [130]. Finally, the DLCG is achieved by bringing these link-variables to the closest configuration that satisfies the MCG. In Ref. [129], it was argued that the DLCG is preferable to the LLG, as it avoids the presence of small scale fluctuations in the P-vortex surfaces of projected configurations.

4.2 The local gauge-fixing in continuum YM theory

In the lattice, the use of global gauge-fixing conditions, in the various center gauges discussed in Sec. 4.1, is always possible because there is no concept of singular phase field $S(x)$, where x represents the discrete lattice sites. On the other hand, in the continuum, any attempt of defining a global condition, in a procedure that detects nonabelian topological phases $S(x)$, $x \in \mathbb{R}^4$, would lead to singular gauge-fixed fields. For example, this occurs in the global gauge of Ref. [131]. In that case, among the natural large phases there are those corresponding to monopoles. Then, a gauge-fixing based on a global orientation of the auxiliary fields, where $S(x)$ is set to the identity, leads to gauge fields A_μ containing singularities (Dirac strings). A similar situation would occur in gauge fixings in the continuum based on a set of adjoint auxiliary fields $\psi_I \in \mathfrak{su}(N)$, $I = 1,\dots,N_{\mathsf{f}}$. This time, the topological phases $S(x) \in SU(N)$ will certainly include center-vortex defects. In addition, monopole-like phases will generally be attached to a pair of (physical) center-vortex defects.[1] Again, there will be an obstruction to implement a global ψ_I orientation, for every $A_\mu \in \{A_\mu\}$. By enforcing such a condition,

[1]These configurations are known as nonoriented center vortices (see Ref. [66]).

singular gauge fixed fields A_μ^{gf} would be produced. On the other hand, in the continuum, it is precisely the clear distinction between regular and singular $SU(N)$-mappings that enables the introduction of the equivalence relation

$$S(x) \sim S'(x) \qquad \text{if } \exists \text{ regular } U(x) \;/\qquad S'(x) = U(x)S(x)\;. \tag{4.9}$$

Such distinction and equivalence relation have no meaning for fields defined on the lattice. In the continuum, it enables us to think of generating, a priori, different equivalence classes $[S_0]$, where $S_0(x)$ is a class representative. For example, in gauges based on adjoint auxiliary fields, a possible reference would be $S_0 = e^{i\chi\beta\cdot T}$, where χ is a multivalued harmonic function and β is a fundamental magnetic weight, such that S_0 changes by a center element when going around a closed surface $\partial\Sigma$. Of course, there is also a defect-free sector that can be labeled by the identity. There are also more general topological phases representing center-vortices that are nonoriented in the Lie algebra (see Refs. [66, 89]). Here, we will not discuss the general classification of the topological sectors. Instead, we shall analyze some examples. However, it is important to underline that, as discussed in Ref. [89], multiplying a label S_0 by a regular mapping on the right generally leads to a physically inequivalent label. The identification of these nonabelian degrees of freedom is an important property in the continuum which has no clear counterpart in the lattice. A possibility to achieve a local gauge-fixing is to use a mechanism that maps A_μ to a phase S in a gauge covariant way, and look for the previously defined reference label S_0 that is equivalent to S. Then, instead of a global condition on $\{A_\mu\}$, we can require the gauge-fixed A_μ^{gf} to be mapped into S_0, which is attained by a regular gauge transformation.

The simplest known example where local gauge-fixings are used is in the context of the Abelian Higgs Model [132]. In the unitary gauge, the phase of the Higgs field is required to be trivial. However, this condition cannot be applied to the Nielsen-Olesen vortex. For a straight infinite vortex, the best that can be done is to fix the gauge by requiring that $\phi = h\, e^{i\varphi}$, where φ is the polar angle ($\partial^2\varphi = 0$). This is one of the motivations that led to the gauge-fixing proposal for pure YM theories in Ref. [112], which we will consider in this thesis. There, the mapping $A_\mu \to S(A_\mu)$ was done by introducing a set of adjoint auxiliary fields that minimize an auxiliary action

$$S_{\text{aux}} = \int d^4x\, \left((D_\mu\psi_I, D_\mu\psi_I) + V_{\text{aux}}\right)\;. \tag{4.10}$$

The Killing product is defined between elements of the Lie Algebra according to

$$(X,Y) = \text{Tr}(\text{Ad}(X)\text{Ad}(Y))\;. \tag{4.11}$$

The consideration of $\psi(A) = (\psi_1, \dots, \psi_{N_{\text{f}}})$, solution to this minimization problem, has the advantage that, unlike the lowest eigenfunctions of the covariant adjoint Laplacian, it is a a well-posed problem in the continuum. At the quantum level, as reviewed in the next section, these fields were introduced by means of an identity, keeping the pure Yang-Mills dynamics unchanged. Regarding the field content and auxiliary potential, they were chosen such that the components ψ_I of the classical solution $\psi(A)$ enable a simple concept of "modulus" tuple and the extraction of a phase. For this aim, we proposed S_{aux} to display $SU(N) \to Z(N)$ SSB, which requires $N_{\mathrm{f}} \geq N$. Among the many possible sets of auxiliary fields, we preferred the choice $N_{\mathrm{f}} = N^2 - 1$, as a simple auxiliary action and procedure to extract the phase S can be given for general $SU(N)$. For example, V_{aux} can be chosen such that it is minimized by the nontrivial solutions to

$$-i[\psi_I, \psi_J] = v\, f_{IJK}\, \psi_K \ , \tag{4.12}$$

namely, $\psi_I = vST_IS^{-1}$, where $I = 1, \ldots, N^2 - 1$. In regions where A_μ is close to a pure gauge, the solution will be close to this rotated frame. This "dynamical tendency" can be thought of as playing a similar role to the orthonormality property of the Laplacian eigenvector fields in the DLCG (see Eq. (4.8)).

The polar decomposition of a tuple ψ was done by defining a modulus tuple q as the transformed ψ that minimizes the average square distance

$$\sum_I \langle q_I - vT_I \rangle^2 \ . \tag{4.13}$$

This implies that q_I is "aligned" with the Lie basis T_I on average,

$$[q_I, T_I] = 0 \ . \tag{4.14}$$

Then, this procedure allows for the identification of the phase $S(A)$ of the solution $\psi(A)$ and identify the corresponding sector $\mathcal{V}(S_0)$ where A_μ is. Finally, the gauge can be fixed by the sector-dependent condition

$$f_{S_0}(\psi) = [S_0^{-1} \psi_I(A) S_0, T_I] = 0 \ . \tag{4.15}$$

This procedure, proposed in Ref. [112], has many points of contact with Laplacian center gauges used in the lattice. As discussed in section 4.1, the possibility of using adjoint fields other than the Laplacian eigenfunctions in the continuum was first pointed out in Ref. [128]. In the procedure to be analyzed here, various adjoint field were considered. This field content simplified the extraction of a covariant phase out of ψ. Indeed, the above-mentioned concept of polar decomposition generalizes to $SU(N)$ the usual decomposition of the 3×3 real matrix, formed with the three lowest eigenvectors, used in the lattice adjoint LLG in $SU(2)$. In addition, as already explained, by considering local gauge-fixing conditions on $\mathcal{V}(S_0) \subset \{A_\mu\}$, singular gauge-fixed fields are avoided in this procedure.

On the other hand, for oriented center vortices, our procedure differs from the continuum global MCG, as it is not based on comparing A_μ with the singular configurations a_Σ. The closed manifold $\partial\Sigma$ is not obtained after a best fit of $a|_\Sigma$ to A_μ, but by reading the defects in $S(A)$. It is also very different from the traditional global gauge-fixings. For instance, in the Landau gauge, the Gribov copies associated with smooth center vortex or instanton configurations (cf. Eq. (4.2)) are related with zero mode solutions to a Schrödinger-like differential equation. It should be emphasized that the FP operator for this type of global gauges is completely different from the FP operator J_{S_0} in any local sector $\mathcal{V}(S_0)$, which is related with the algebraic condition in Eq. (4.15). Therefore, there is no a priori reason to expect J_{S_0} to contain zero modes. In order to study the existence of copies, the analysis must be completely reformulated. Instead of considering a general $A_\mu \in \{A_\mu\}$, it should be done separately for $A_\mu \in \mathcal{V}(S_0)$, for every possible label S_0.

4.3 A different quantization scheme: the Yang-Mills ensemble

In this section we will present the necessary techniques to perform the local gauge-fixing procedure discussed in the previous section. As proposed by Singer, the approach should be based on a locally finite open covering $\{\vartheta_\alpha\}$ of the total space of gauge field configurations $\{A_\mu\}$, i.e.

$$\{A_\mu\} = \cup_\alpha \vartheta_\alpha \; , \tag{4.16}$$

together with a subordinate partition of unity [133, 134]

$$\sum_\alpha \rho_\alpha(A_\mu) = 1 \quad , \quad \forall A_\mu \in \{A_\mu\} \; , \tag{4.17}$$

where the function ρ_α is nonzero only in ϑ_α. Using this identity, it is possible to write the Yang-Mills partition function as

$$Z_{\mathrm{YM}} = \sum_\alpha Z_{(\alpha)} \quad , \quad Z_{(\alpha)} = \int_{\vartheta_\alpha} [DA]\, \rho_\alpha(A)\, e^{-S_{\mathrm{YM}}[A]} \; . \tag{4.18}$$

Note that, in each term, the path-integral can be done on the support of $\rho_\alpha(A)$. Now, notice that it is always possible to choose the open sets ϑ_α so as to assure that they admit local cross sections

$$f_\alpha(A) = 0 \ , \tag{4.19}$$

without copies. This follows from the fact that it is always possible to define local cross-sections in a principal fiber bundle, even when the bundle is not trivial. Then, the Faddeev-Popov procedure can be safely implemented on each $Z_{(\alpha)}$

$$Z_{\text{YM}} = \sum_\alpha \int_{\vartheta_\alpha} [DA]\, \rho_\alpha(A)\, e^{-S_{\text{YM}}}\, \delta(f_\alpha(A))\, \text{Det} \frac{\delta f_\alpha(A^U)}{\delta U}\bigg|_{U=I} \ . \tag{4.20}$$

4.3.1 Center vortices as labels for the different sectors

If each of the terms in the sum on the rhs of eq. (4.20) could be well-approximated by a dominant contribution of the sector ϑ_α, this equation would allow us to express the YM partition function in terms of an ensemble of these relevant configurations. As discussed in chapter 3, the most promising candidates for describing the YM degrees of freedom in the infrared are center-vortex ensembles. Therefore, it would be very interesting if these labels α were related to these field configurations. This is precisely achieved by implementing the local gauge-fixing procedure on the sets $\mathcal{V}(S_0)$ labeled by center vortices discussed in section 4.2. This was proposed in Ref. [112], and the YM partition function written as

$$Z_{\text{YM}} = \sum_{S_0} Z_{\text{YM}}^{S_0} \ . \tag{4.21}$$

Concretely, the procedure is achieved by means of the introduction of two identities in Z_{YM}. The first, given by

$$1 = \int [D\Psi]\, \det \frac{\delta^2 S_{aux}}{\delta\psi_I \delta\psi_J} \delta\left(\frac{\delta S_{aux}}{\delta \psi_I}\right) \ , \tag{4.22}$$

correlates the gauge fields with the tuple of adjoint scalar fields that minimize S_{aux} in their presence. For this identity to be well-defined, it is sufficient to assume that the minimization equations for S_{aux} admits an unique solution for a given A_μ. Then, a second identity is introduced

$$1 = \textstyle\sum_{S_0} 1_{S_0} \ , \tag{4.23}$$

$$1_{S_0} = \textstyle\int [DU] \delta(f_{S_0} U^{-1}\psi_I U)\, \det J(\psi) \tag{4.24}$$

where $f_{S_0}(\psi) = [S_0 u_I S_0^{-1}, \psi_I]$ is a sector-dependent gauge condition. To define the operator $J_{AB}(\psi)$, we proceed similarly to the usual FP procedure (eq. (2.17)), by considering an infinitesimal gauge transformation $G = e^{i\alpha^a T^a}$, and set

$$J^{S_0}_{ab} = \frac{\delta f^a_{S_0}(\psi)}{\delta \alpha^b} \ . \tag{4.25}$$

For this identity to be consistent, it is necessary, for each ψ, that: a) There are no elements $U \in SU(N)$ which leave ψ invariant. b) Only one element of the orbit $S^{-1}\psi S$ ($S = US_0^{-1}$) satisfies the condition $f_{S_0}(\psi) = 0$. The first condition may be posed alternatively in terms of the injectivity of the functional $\psi(A)$, obtained by associating to A_μ the adjoint scalar fields ψ_I that minimize S_{aux}. Condition b) is the usual requirement that a good gauge-fixing should satisfy, with the difference that in our case the gauge condition is imposed in the auxiliary fields ψ. Additionaly, as the second identity is introduced together with the first one (eq. (4.22)), we only need to worry about conditions a) and b) for fields ψ that are in the image of the functional $\psi(A)$. In Ref. [113], we showed that this is true for typical configurations in the vortex-free sector, in the limit of large v, and for a specific configuration in the one-vortex sector, for $N = 2$. In section 4.4, we will review how this result was obtained.

Then, the partial contribution of the sector S_0 is given by

$$Z^{S_0}_{YM} = \int_{V(S_0)} [DA][D\psi] \det \left(\frac{\delta^2 S_{aux}}{\delta\psi_I \delta\psi_J} \right) \delta \left(\frac{\delta S_{aux}}{\delta \psi_I} \right) \delta(f_{S_0}(U^{-1}\psi_I U) \det J^{S_0}(\psi) \ . \tag{4.26}$$

The determinants and deltas may be exponentiated in the usual way. For now, we exponentiate only the contributions arising from the first identity, thus

$$Z^{S_0}_{YM} = \textstyle\int_{V(S_0)} [D\mathcal{A}][D\psi][Dw][D\bar{w}][D\xi] e^{-S} \delta(f_{S_0}(U^{-1}\psi_I U) \det J^{S_0}(\psi) \ , \tag{4.27}$$

$$\begin{aligned} S = S_{YM} &+ \textstyle\int d^4x \left(\langle D_\mu \bar{\omega}_I, D_\mu \omega_I \rangle + \langle \bar{\omega}_I, \frac{\delta^2 V}{\delta\psi_J \psi_I} \omega_J \rangle \right) + \\ &\textstyle\int d^4x \left(\langle D_\mu \xi_I, D_\mu \psi_I \rangle + \langle \xi_I, \frac{\delta V}{\delta \psi_I} \rangle \right) \ , \end{aligned} \tag{4.28}$$

where w_I, $\bar{w}_I$ and ξ_I are adjoint fermionic and bosonic fields, respectively. Next, we perform the change of variables

$$\mathcal{A}_\mu \to A_\mu = \mathcal{A}^U_\mu \ , \psi_I \to \zeta_I = \psi^U_I \ , \xi_I \to b_I = \xi^U_I \ , \tag{4.29}$$

$$w_I \to c_I = w^U_I \ , \bar{w}_I \to \bar{c}_I = \bar{w}^U_I \ , \tag{4.30}$$

and use the gauge invariance of S and of the FP determinant to arrive at the gauge-fixed partition function for the sector S_0

$$Z_{YM}^{S_0} = \left(\int [DU]\right) \int_{V(S_0)} [DA_\mu][D\zeta_I][Dc_I][D\bar{c}_I][Db_I][Dc][D\bar{c}][Db] e^{-S_{gf}} \ , \quad (4.31)$$

$$\begin{aligned} S_{gf} = S_{YM} + \int d^4x \left(\langle D_\mu \bar{c}_I, D_\mu c_I \rangle + \langle \bar{c}_I, \tfrac{\delta^2 V}{\delta \psi_J \psi_I}|_{\psi=\zeta} c_J \rangle \right) + \\ \int d^4x \left(\langle D_\mu b_I, D_\mu \zeta_I \rangle + \langle b_I, \tfrac{\delta V}{\delta \psi_I}{}_{\psi=\zeta} \rangle \right) + \\ \int d^4x \left(\langle b, [u_I^{S_0}, \zeta_I] \rangle + \langle \bar{c}, [u_I^{S_0}, [\zeta_I, c]] + \langle \bar{c}, [u_I^{S_0}, c_I] \rangle \right) \ . \end{aligned} \quad (4.32)$$

The presence of S_0 in S_{gf}^0 is expected, as it may not be eliminated by a regular gauge transformation, and is responsible for distinguishing the different sectors. The term $\langle \bar{c}, [u_I^{S_0}, c_I] \rangle$ was included for later convenience: it is possible to show that it does not contribute to the partition function [112].

We have thus shown that the YM partition function may be written as a sum over sectors labeled by center vortices, where the gauge is fixed in each partial contribution through a sector-dependent condition. This is a very promising representation for non-perturbative purposes, as it allows us to study the contributions of sectors with vorticity in a more controlled way. In particular, we see that thin and thick center vortices with the same guiding center should belong to the same sector. In the best case scenario, it could be possible to describe each sector with only a few dominant vortex configurations, thus arriving at an ensemble of center vortices. This provides a glimpse of a path from pure Yang-Mills theory to the phenomenological ensembles of center vortices discussed in Chapter 3.

4.3.2 A class of nontrivial sectors

In this section we discuss some examples of gauge field configurations that belong to nontrivial sectors. Eq. (4.24) implies that a configuration A_μ belongs to a given sector $\mathcal{V}(S_0)$ if and only if there is a regular transformation $U(x) \in$ SU(N) such that

$$[S_0^{-1} U^{-1}(x) \psi_I(x)[A] U(x) S_0, u_I] = 0 \ , \forall x \ . \quad (4.33)$$

We will choose, for convenience, $u_I = v T_I$ as the element of $\mathcal{M}$. Let us now consider a class of configurations with cylindrical symmetry

$$A_\mu^{(k)} = a_{(k)}(\rho) \partial_\mu \varphi \beta^{(k)} \cdot T \ , \quad (4.34)$$

with ρ, φ being the radial and angular coordinates. We are using the convention that indices between parenthesis are not summed. The $N-1$ dimensional vectors $\beta^{(k)}$ are the highest weights of the antisymmetric representation with $N-$ality k, given explicitly by

$$\beta^k = 2N \sum_{i=1}^{k} \omega_i \ . \tag{4.35}$$

The scalar profile $a_k(\rho)$ satisfies the conditions

$$a_k(\rho \to \infty) = 1 \quad , \quad a_k(0) = 0 \ . \tag{4.36}$$

The first implies that the configurations contributes a center element to the k power to the Wilson Loop, the definining property of a center vortex. The second assures that the gauge field $A_\mu^{(k)}$ is smooth and regular. In Ref. [101] $\psi_I(A)$ was obtained for the special case in which the gauge field minimizes the auxiliary action, and is given by

$$\psi_q^{(k)} = h_{qp}^{(k)} V_{(k)} T_p V_{(k)}^{-1} \ , \tag{4.37}$$

$$\psi_\alpha^{(k)} = \psi_{\bar{\alpha}}^{(k)} = h_\alpha^{(k)} V_{(k)} T_\alpha V_{(k)}^{-1} \ , \tag{4.38}$$

$$V_{(k)} = e^{i\varphi\beta^k \cdot T} \ , \tag{4.39}$$

with the profiles $h_\alpha(\rho)$ satisfying the regularity condition $h_\alpha(0) = 0$ if $\alpha \cdot \beta \neq 0$. The profiles $a(\rho), h_{qp}(\rho), h_\alpha(\rho)$ satisfy scalar equations, which were solved numerically for SU(N) in Refs. [135, 101]. This solution satisfies the condition (4.33) for $S_0 = V_{(k)} = e^{i\varphi\beta^k \cdot T}$, $U = I$. This implies that the smooth thick center vortex configuration of Eq. (4.34) belong to the sector $V_{(k)}$. Moreover, for each k, the set of roots that satisfy $\alpha \cdot \beta^k$ is different, which implies that, for different k, the singular phases $V_{(k)}$ label inequivalent sectors of Yang-Mills theory.

4.3.3 Nonabelian degrees of freedom

The study of the labels $V_{(k)}$ discussed in the previous section implies the existence of a discrete set of physically inequivalent sectors with defects located at the same space-time points. Similar internal degrees of freedom were observed in the context of effective Yang-Mills-Higgs models [136, 137, 138, 139, 140, 141], but here they appear naturally in pure YM theory. As discussed in [89, 113], a continuum of such degrees of freedom may also be generated from a defect S_0 by multiplicating it on the right by a regular phase $\tilde{U}(x)$. For example, when $S_0 = e^{i\varphi\beta \cdot T}$, the generated configuration is

$$A_\mu = aiS\partial_\mu S^{-1} = S\mathcal{A}_\mu S^{-1} + iS\partial_\mu S^{-1} \ , \tag{4.40}$$

$$\mathcal{A}_\mu = (1-a)iS^{-1}\partial_\mu S \ , \tag{4.41}$$

$$S = \tilde{U} e^{i\varphi\beta \cdot T} \tilde{U}^{-1} \ , \tag{4.42}$$

and the associated solution $\psi(A)$ being of the form $\psi_I(A) = S\bar{\psi}_I(A)S^{-1}$, with $\bar{\psi}_I(A)$ regular. The equations of motion for the auxiliary action imply

$$S^{-1}(D_\mu(A)D^\mu(A)\psi_I)S = \partial^2\bar{\psi}_I + 2\mathcal{A}\wedge\partial_\mu\bar{\psi}_I + \partial_\mu\mathcal{A}_\mu\wedge\bar{\psi}_I + \mathcal{A}_\mu\wedge(\mathcal{A}_\mu\wedge\bar{\psi}_I)\ . \quad (4.43)$$

Next, notice that the regularity conditions of $\bar{\psi}_I(x)$ and $a(x)$ (see Eq. (4.36)) imply that they may be expanded as

$$\bar{\psi}_I = \bar{\psi}_I^{(0)} + \bar{\psi}_I^{(1)}\rho + \dots\ , \quad (4.44)$$

$$a(x) = a^{(1)}\rho + a^{(2)}\rho^2 + \dots \quad (4.45)$$

Substituting these in Eq. (4.43), the term proportional to ρ^{-2} is

$$\frac{\partial^2\bar{\psi}_I^{(0)}}{\partial\varphi^2} - 2\bar{X}\wedge\frac{\partial\bar{\psi}_I^{(0)}}{\partial\varphi} + \bar{X}\wedge(\bar{X}\wedge\bar{\psi}_I^{(0)})\ , \quad (4.46)$$

$$\bar{X} = \tilde{U}\beta\cdot T\tilde{U}^{-1}\ . \quad (4.47)$$

As $\bar{\psi}_A$ is single-valued and regular, the zeroth order term $\bar{\psi}_A^{(0)}$ in the ρ expansion can't depend on φ. Therefore, in the guiding center it must hold that

$$\bar{X}\wedge(\bar{X}\wedge\bar{\psi}_A^{(0)}) = 0\ . \quad (4.48)$$

Evaluating the scalar product of this equation with $\bar{\psi}_A^{(0)}$ yields

$$\bar{X}\wedge\bar{\psi}_A^{(0)} = 0\ . \quad (4.49)$$

That is, the fields must satisfy $\bar{U}-$ dependent regularity conditions. This implies the existence of a continuum of physically inequivalent sectors S for the Yang-Mills ensemble, which are localized in the same spacetime points as the original defect S_0.

4.4 Study of Gribov copies in the Yang-Mills ensemble

As discussed in section 4.3.1, for our local gauge-fixing to be well defined, there are two main requirements: a) The mapping $A\to\psi(A)$ must be injective. b) The sector-dependent gauge-fixing condition $f_{S_0}(\psi) = 0$ must be free of copies. We will study these conditions in the following sections.

4.4.1 Infinitesimal injectivity of $\psi(A)$

In this section we will show that injectivity is related to the positivity of the operator introduced in the identity of Eq. (4.22), and to the absence of nontrivial gauge transformations that leave the auxiliary fields invariant. Then, we show that the functional $\psi(A)$ is injective for typical configurations of the vortex-free sector. A particular example in the vortex labeled by one vortex is also provided.

Conditions for injectivity

The equations of motion originated from the auxiliary action (4.10) is a functional of ψ and A_μ, $\Sigma \equiv= \delta S_{\text{aux}}/\delta\psi = \Sigma(\psi, A_\mu)$, and it is invariant under an infinitesimal gauge transformation, i.e. $\delta\Sigma = \delta_A\Sigma + \delta_\psi\Sigma = 0$, with

$$\delta_A \equiv \int \delta A^a_\mu \frac{\delta}{\delta A^a_\mu} \quad , \quad \delta_\psi \equiv \int \delta\psi^a_I \frac{\delta}{\delta\psi^a_I} \; . \tag{4.50}$$

To address the infinitesimal injectivity of $\psi(A)$, we should study if the equation

$$\delta_A \frac{\delta S}{\delta\psi^a_I(x)} = -\delta_\psi \frac{\delta S}{\delta\psi^a_I(x)} = -\int dy \frac{\delta^2 S}{\delta\psi^a_I(x)\psi^b_J(y)} f^{bmn}\xi^m(y)\psi^n_J(y) = 0 \tag{4.51}$$

has nontrivial solutions. We may multiply this equation by $f^{am'n'}\xi^{m'}(x)\psi^{n'}_I(x)$ and integrate over x to arrive at

$$\int dx\, dy\, \frac{\delta^2 S}{\delta\psi^a_I(x)\psi^b_J(y)} v^a_I(x)v^b_J(y) = 0 \; , \quad , \quad v^a_I(x) = f^{amn}\xi^m(x)\psi^n_I(x) = (\xi(x)\wedge\psi_I(x))|_a \; . \tag{4.52}$$

For ψ^a_I to be a minimum of S, all the eigenvalues of $\frac{\delta^2 S}{\delta\psi^a_I(x)\psi^b_J(y)}$ must be positive, as was already required for the identity in Eq. (4.22) to be well-defined. Therefore, non-trivial solutions for (4.4.3), if they exist, are given by

$$v^a_I = \delta\psi^a_I = 0 \; . \tag{4.53}$$

We see that the lack of infinitesimal injectivity is associated to the existence of nontrivial gauge transformations that leave ψ_I invariant. By using the definitions $\Psi \equiv \psi^B_A$, $X \equiv \xi^A M^A$, $M^A|_{BC} \equiv if^{ABC}$, we can rewrite condition (4.53) as

$$\Psi X = 0 \; . \tag{4.54}$$

Therefore, we conclude that a lack of infinitesimal injectivity would be associated to configurations of auxiliary fields that satisfy det $\Psi = 0$.

Vortex-free sector

For this sector, in the limit of large v, we expect that the adjoint scalar fields will be close to the vacuum configuration, i.e. $\Psi = v\mathbb{I} + \epsilon$, where ϵ is a small matrix. We must then show that $b(\epsilon) = \det(v\mathbb{I} + \epsilon) \neq 0$ for small ϵ. By expanding $b(\epsilon)$, we may write

$$b(\epsilon) \approx b(0) + \frac{\partial g}{\partial \epsilon^a}\epsilon^a \, . \tag{4.55}$$

Since $b(0) = \det v\mathbb{I} = v^{N^2-1}$ is a finite (and large) value, we may conclude that the only solution to Eq. (4.54) is $X = 0$. Therefore, in the vortex-free sector, injectivity is ensured.

4.4.2 Sectors with center-vortices

The argument of the vortex-free sector cannot be extended to sectors labeled by vortices, as Ψ will necessarily be far from the vacuum near their guiding-centers. Therefore, we shall consider a particular example for $SU(2)$. The simplest case is the sector labeled by an antisymmetric vortex with charge k=1.Then, as $\beta = \sqrt{2}$, the relevant singular phase is $S_0 = e^{i\varphi\sqrt{2}T_1}$, where φ is the angle of cylindrical coordinates. The solution for $\psi(A)$, when A is a minimum of the action as well, is known to be [101]

$$\begin{aligned}
&\psi_1 = h_1(\rho)T_1 \, , \\
&\psi_{\alpha_1} = h(\rho)S_0T_{\alpha_1}S_0^{-1} \, , \\
&\psi_{\bar{\alpha}_1} = h(\rho)S_0T_{\bar{\alpha}_1}S_0^{-1} \, .
\end{aligned} \tag{4.56}$$

For $SU(2)$, there is only one root $\alpha_1 = \frac{1}{\sqrt{2}}$, satisfying $\alpha_1 \cdot \beta = 1$, and the following relations hold

$$\begin{aligned}
S_0T_{\alpha_1}S_0^{-1} &= \cos(\varphi)T_{\alpha_1} - \sin(\varphi)T_{\bar{\alpha}_1} \, , \\
S_0T_{\bar{\alpha}_1}S_0^{-1} &= \cos(\varphi)T_{\alpha_1} + \sin(\varphi)T_{\bar{\alpha}_1} \, .
\end{aligned} \tag{4.57}$$

This implies the following matrix Ψ of auxiliary fields:

$$\begin{pmatrix} h_1(\rho) & 0 & 0 \\ 0 & h(\rho)\cos(\varphi) & -h(\rho)\sin(\varphi) \\ 0 & h(\rho)\cos(\varphi) & h(\rho)\sin(\varphi) \end{pmatrix} \, . \tag{4.58}$$

Now, the condition (4.54) implies that, if injectivity was violated, the following equation would hold

$$\begin{pmatrix} 0 & h_1(\rho)\xi_3 & h_1(\rho)\xi_2 \\ -\xi_3 h(\rho)\cos(\varphi) - \xi_2 h(\rho)\sin(\varphi) & \xi_1 h(\rho)\sin(\varphi) & \xi_1 h(\rho)\cos(\varphi) \\ -\xi_3 h(\rho)\cos(\varphi) + \xi_2 h(\rho)\sin(\varphi) & -\xi_1 h(\rho)\sin(\varphi) & \xi_1 h(\rho)\cos(\varphi) \end{pmatrix} = 0 \,. \tag{4.59}$$

For $\rho \neq 0$, this gives $\xi_1 = \xi_2 = \xi_3 = 0$, i.e. there are no problems of injectivity in this case. The only region where problems could arise is the plane $\rho = 0$, which is a set of null measure in R^4. The gauge transformations that would leave Ψ invariant, thus leading to the lack of injectivity, should be nontrivial only in this plane. Such transformations are not continuous, so they can be disregarded. The functional $\psi(A)$ is therefore infinitesimally injective for this particular configuration belonging to the one-vortex sector.

4.4.3 A polar decomposition without infinitesimal copies

In this section we will study the existence of copies in requirement that, for all sectors S_0,

$$f_{S_0}(\psi(A)) = f_{S_0}(\psi(A^U)) = 0 \to U = \mathbb{I} \,. \tag{4.60}$$

We shall see that this condition, as expected, is related to the absence of zero modes of the operator introduced in the identity of Eq. (4.24). For instance, to analyze Eq. (4.60) in the vortex-free sector, we must show that if a given set of auxiliary fields q_I satisfies

$$(q_I \wedge T_I)|_\gamma = f^{aI\gamma} q_I^a = 0 \,, \tag{4.61}$$

then there is no nontrivial gauge transformation with parameters ξ^a, such that

$$f^{aI\gamma} f^{anm} q_I^n \xi^m = 0 \,. \tag{4.62}$$

It should be emphasized that these are just necessary conditions that a problematic tuple q_I should satisfy, as it should also minimize the auxiliary action ((4.10)) i.e. it should belong to the image of the functional $\psi(A)$. These algebraic conditions for the existence of Gribov copies (4.61),(4.62) can also be written by using the generators in the adjoint representation:

$$\mathrm{Ad}(T_A)|_{BC} \equiv M_A|_{BC} = i f^{ABC} \,, \tag{4.63}$$

and the matrix

$$Q|_{Ia} = q_I^a \,. \tag{4.64}$$

Then, equations (4.61) and (4.62) become, respectively,

$$Tr(M_b Q) = 0 \;, \tag{4.65}$$

$$Tr(M_\gamma M_b Q)\xi^\gamma = 0 \;. \tag{4.66}$$

These conditions may also be written as

$$J^{AB}\xi^B = 0 \qquad , \qquad J^{AB} \equiv Tr(M^A M^B Q) \;.$$

Therefore, copies are associated with configurations having $\det J = 0$. In fact, this is precisely the operator J introduced in the Yang-Mills partition function by means of the Fadeev-Popov procedure (see eq. (4.24)). It is therefore expected that copies are related to zeros of this determinant.

Let us begin by analyzing the above equations for $SU(2)$. In this case, $f^{ABC} = \frac{\epsilon^{ABC}}{\sqrt{2}}$, and the matrices M and X are given by

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix} \quad , \quad M_3 = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;, \tag{4.67}$$

$$X = \xi^A M^A = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}}\xi_3 & -\frac{i}{\sqrt{2}}\xi_2 \\ -\frac{i}{\sqrt{2}}\xi_3 & 0 & \frac{i}{\sqrt{2}}\xi_1 \\ \frac{i}{\sqrt{2}}\xi_2 & -\frac{i}{\sqrt{2}}\xi_1 & 0 \end{pmatrix} \;. \tag{4.68}$$

The pure modulus condition (4.65) implies that Q is a symmetric matrix, and thus can be parametrized as

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix} \;. \tag{4.69}$$

The equation for copies (4.66) is then

$$J^{ab}\xi^b = 0 \quad , \quad J = \begin{pmatrix} Q_{22}+Q_{33} & -Q_{12} & -Q_{13} \\ -Q_{12} & Q_{11}+Q_{33} & -Q_{23} \\ -Q_{13} & -Q_{23} & Q_{11}+Q_{22} \end{pmatrix} \quad , \quad \xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} \;. \tag{4.70}$$

A necessary condition for the system (4.70) to have a nontrivial solution is that the determinant of J should be 0 . This yields

$$\det J = (Q_{22}+Q_{33})(Q_{11}+Q_{33})(Q_{11}+Q_{22}) - 2Q_{12}Q_{23}Q_{13} - Q_{12}^2(Q_{11}+Q_{22}) - Q_{23}^2(Q_{22}+Q_{33}) - Q_{13}^2(Q_{11}+Q_{33}) = 0 \; . \tag{4.71}$$

Study of copies in the vortex-free sector

In the vortex-free sector, the gauge-fixed functional $q_I(A)$ satisfies

$$q_I(A) \wedge u_I = 0 \; , \tag{4.72}$$

$$q_I(A) \to vT_I \; , x \to \infty \; . \tag{4.73}$$

If there is a Gribov copy, then there exists a gauge transformation $U(x)$ such that

$$q_I^U(A) \wedge u_I = 0 \; , \tag{4.74}$$

$$U(x) \to \mathbb{I}, x \to \infty \; . \tag{4.75}$$

For infinitesimal transformations, equation (4.75) implies

$$f^{aI\gamma} f^{anm} q_I^n \xi^m = 0 \; . \tag{4.76}$$

In the vortex-free sector, the boundary condition of Eq. (4.73) implies, in the large v limit, that the fields Q are close to the vacuum $v\,\mathbb{I}$ everywhere, i.e. $q_I^a = v\delta_I^a + \epsilon_I^a$. Eq. (4.76) may thus be written as

$$v\xi^\gamma + f^{aI\gamma} f^{anm} \xi^n \epsilon_I^m = 0 \; , \tag{4.77}$$

$$\xi^m (v\delta^{m\gamma} + f^{aI\gamma} f^{anm} \epsilon_I^n) = 0 \; . \tag{4.78}$$

This is a system of $N^2 - 1$ linear equations in the variables ξ^a, with coefficients that depend on ϵ_I^a, i.e.

$$M(\epsilon)\xi = 0 \; , \tag{4.79}$$

where M is the matrix of coefficients. For this system to have a nontrivial solution, it must hold that

$$k(\epsilon) \equiv \det M(\epsilon) = 0 \; . \tag{4.80}$$

Since $k(\epsilon)$ is polynomial on the infinitesimal parameters ϵ_I^a, we may approximate:

$$k(\epsilon) \approx k(0) + \frac{\partial k(\epsilon)}{\partial \epsilon_I^a} \epsilon_I^a \; . \tag{4.81}$$

As $M(0)$ is proportional the $(N^2-1) \times (N^2-1)$ identity matrix, we have $k(0) = v^N$, a finite value. Therefore, in the large v-limit, there are no Gribov copies in the vortex-free sector.

Study of copies in a general sector

In a general sector labeled by a defect S_0, the functional $\psi_I(A)$ satisfies, by definition,

$$\frac{\delta S_{\text{aux}}}{\delta \psi_I} = 0 \; . \tag{4.82}$$

For a general gauge field A in this sector, the corresponding ψ will be of the form $\psi_I = U S_0 q_I S_0^{-1} U^{-1}$, with U regular. The gauge-fixed A_μ will be associated to $\zeta_I \equiv S_0 q_I S_0^{-1}$, and the following properties must hold

$$\zeta_I(A) \wedge \eta_I = 0 \; , \tag{4.83}$$

$$\eta_I \equiv v S_0 T_I S_0^{-1} \; , \tag{4.84}$$

$$\zeta_I(A) \to v S_0 T_I S_0^{-1} \quad , \quad x \to \infty \; . \tag{4.85}$$

If there is a Gribov copy, then there exists a gauge transformation $U(x)$ such that

$$\zeta_I^U(A) \wedge \eta_I = (U S_0 q_I S_0^{-1} U^{-1}) \wedge S_0 T_I S_0^{-1} = 0 \; , \tag{4.86}$$

$$U(x) \to \mathbb{I} \quad , \quad x \to \infty \; . \tag{4.87}$$

Condition (4.86) may be written in terms of q_I:

$$(S_0^{-1} U S_0 q_I (S_0^{-1} U S_0)^{-1}) \wedge u_I = 0 \; , \tag{4.88}$$

and in terms of the matrix Q

$$R(S_0^{-1} U S_0) Q = Q' \; , \tag{4.89}$$

with Q, Q' being pure modulus matrices. Therefore, for a Gribov copy to exist, it is necessary that

$$R(\tilde{U}) Q = Q' \; , \tag{4.90}$$

$$\tilde{U}(x) \to \mathbb{I} \quad , \quad x \to \infty \; , \tag{4.91}$$

where we defined $\tilde{U} \equiv S_0^{-1} U S_0$. Clearly, if U is infinitesimal, so is $\tilde{U}$. The algebraic equation for infinitesimal copies is therefore the same in all sectors. However, in a general sector there is no reason to expect that q_I will be close to vT_I everywhere, since some of its components must go to zero at the guiding centers of the vortices to assure regularity. Gauge transformations which are nontrivial only in the regions surrounding the guiding-centers of the vortices could, in principle, yield copies. However, as v grows, these regions become increasingly small.

An example of field configuration that could yield copies is when $A_\mu = a(\rho)\partial_\mu \varphi \beta \cdot T$, belonging to the sector labeled by a vortex along the z axis. As reviewed in (4.56), for $SU(2)$, the solution for $\psi(A)$ is known. It is of the form

$$\psi_I = h_{IJ} S_0 T_I S_0^{-1} \;, \tag{4.92}$$

which implies

$$q_I = h_{IJ} T_J \;. \tag{4.93}$$

The associated $Q-$matrix is symmetric, as required by the gauge fixing. For this configuration to admit infinitesimal copies, eq (4.71) should be satisfied in some finite region. This condition corresponds to

$$2h(h_1 + h)^2 = 0 \;. \tag{4.94}$$

Since the scalar profiles $h_1(\rho)$ and $h(\rho)$ are positive for all $\rho > 0$ (see Ref. [101]), it is easy to see that this condition is only satisfied at $\rho = 0$, which is a set of null measure in R^4. The transformations that could lead to copies are not continuous, as they should be nontrivial only in the plane $\rho = 0$. This configuration, therefore, does not admit Gribov copies.

Chapter 5

Studying the perturbative renormalizability of the Yang-Mills ensemble

5.1 The vortex-free sector

A first interesting point to investigate in the Yang-Mills ensemble is the contribution of the vortex-free sector. As the expectation is that it will be essentially perturbative, it is important to study its renormalizability, which is the subject of this section. For definiteness, we will consider the following potential for the auxiliary action

$$V_{aux} = \kappa f^{IJK} f^{abc} \psi_I^a \psi_J^b \psi_K^c + \lambda \gamma_{IJKL}^{abcd} \psi_I^a \psi_J^b \psi_K^c \psi_L^d \;, \tag{5.1}$$

where γ_{IJKL}^{abcd} is the most general tensor invariant under color and flavor transformations, i.e.,

$$R^{aa'} R^{bb'} R^{cc'} R^{dd'} \gamma_{IJKL}^{a'b'c'd'} = \gamma_{IJKL}^{abcd} \;, \tag{5.2}$$

$$R^{II'} R^{JJ'} R^{KK'} R^{LL'} \gamma_{I'J'K'L'}^{abcd} = \gamma_{IJKL}^{abcd} \;, \tag{5.3}$$

where R belongs to the adjoint representation of $SU(N)$. Additionally, in this sector, we will denote the field ζ_I by q_I. In this case, it was shown in Ref. [112] that the action S_{gf} has the following nilpotent BRST symmetry

$$sA_\mu = \frac{i}{g} D_\mu^{ab} c^b \;, sc = -\frac{i}{2} f^{abc} c^b c^c \;, \tag{5.4}$$

$$s\bar{c}^a = -b^a \;, sb^a = 0 \;, \tag{5.5}$$

$$sq_I^a = i f^{abc} q_I^b c^c + c_I^a \;, sb_I^a = i f^{abc} b_I^b c^c \;, \tag{5.6}$$

$$s\bar{c}_I^a = -i f^{abc} \bar{c}_I^b c^c - b_I^a \;, sc_I^a = -i f^{abc} c_I^b c^c - b_I^a \;. \tag{5.7}$$

The BRST operator is useful for the construction of the physical space, which are non-trivial elements of its cohomology, i.e., observables $\mathcal{O}$ of the theory must satisfy $s\mathcal{O} = 0$, $\mathcal{O} \neq s\tilde{O}$, for some $\tilde{O}$ [142, 143]. The elements E of the cohomology that may be written as $s\tilde{E}$, for some $\tilde{E}$, are said to be trivial elements, as they may not appear in the spectrum. The expectation is that the gauge-fixing terms belong to this trivial cohomology. In fact, this is true: S_{gf} may be written as a trivial variation

$$
\begin{aligned}
S_{gf} = \quad & s \int d^4x \left(D^{ab}_{\mu}\bar{c}^b_I D^{ac}_{\mu} c^c_I + \bar{c}^a_I(\mu^2 q_I + \kappa f^{IJK} f^{abc} q^b_J q^c_K) + \gamma^{abcd}_{IJKL} \lambda \bar{c}^a_I q^b_J q^c_K q^d_L + \right. \\
& \left. i f^{abc} \bar{c}^a u^b_I q^c_I \right) \ . \qquad (5.8)
\end{aligned}
$$

The BRST symmetry is also a powerful tool for studying the algebraic renormalizability of the theory, which will be done in the following sections.

5.2 Controlling the dependence of observables on the gauge-fixing parameters

The auxiliary action (Eq. (5.1)) contains the parameters μ, κ, λ which, in the classical theory, are associated to the interactions among the fields ψ_I. However, in our setup, these are simply gauge-fixing parameters from which the observables should be independent. In order to make this evident, it is necessary to extend the BRST transformations to these parameters as well:

$$ s\mu^2 = U^2 \ , sU^2 = 0 \ , \qquad (5.9) $$

$$ s\kappa = \mathcal{K} \ , s\mathcal{K} = 0 \ , \qquad (5.10) $$

$$ s\lambda = \Lambda \ , s\Lambda = 0 \ , \qquad (5.11) $$

where U^2, $\mathcal{K}$, Λ are constant Grassmann variables with ghost number 1 and mass dimension 2,1,0, respectively. With this extension, the pairs (μ^2, U), $(\kappa, \mathcal{K})$ and (λ, Λ) are now BRST doublets from which physical quantities must be independent [144, 145, 146, 147]. However, with this modification, the flavor sector of the full action (eq. (4.32)) is given by

$$
\begin{aligned}
\tilde{S}_f = \int d^4x \Big(& D^{ab}_{\mu}\bar{c}^b_I D^{ac}_{\mu} c^c_I + \mu^2 \Big(\bar{c}^a_I c^a_I + b^a_I q^a_I \Big) + \kappa f_{IJK} f^{abc} \Big(b^a_I q^b_J q^c_K - 2\bar{c}^a_I q^b_K c^c_J \Big) + \\
& + \lambda \gamma^{abcd}_{IJKL} (b^a_I q^b_J q^c_K q^d_L + 3\bar{c}^a_I c^b_J q^c_K q^d_L) + D^{ab}_{\mu} b^b_I D^{ac}_{\mu} q^c_I \Big) \ . \qquad (5.12)
\end{aligned}
$$

Accordingly, we define the new flavor action

$$ S_f = \tilde{S}_f - \int d^4x \Big(U^2 \bar{c}^a_I q^a_I + \mathcal{K} f^{IJK} f^{abc} \bar{c}^a_I q^b_J q^c_K + \Lambda \gamma^{abcd}_{IJKL} \bar{c}^a_I q^b_J q^c_K q^d_L \Big) \qquad (5.13) $$

which may also be written as a trivial variation. The full action of the flavor sector thus becomes

$$\begin{aligned}
\Sigma_f &= \int d^4x \Big(\left(D_\mu^{ab}\bar{c}_I^b\right) D_\mu^{ac} c_I^c + \left(D_\mu^{ab} b_I^b\right) D_\mu^{ac} q_I^c + \mu^2 \Big(\bar{c}_I^a c_I^a + b_I^a q_I^b\Big) + \\
&+ \kappa f_{IJK} f^{abc} \Big(b_I^a q_J^b q_K^c - 2\bar{c}_I^a q_K^b c_J^c\Big) + \lambda \gamma_{IJKL}^{abcd} (b_I^a q_J^b q_K^c q_L^d + 3\bar{c}_I^a c_J^b q_K^c q_L^d) + \\
&- U^2 \bar{c}_I^a q_I^a - \mathcal{K} f^{IJK} f^{abc} \bar{c}_I^a q_J^b q_K^c - \Lambda \gamma_{IJKL}^{abcd} \bar{c}_I^a q_J^b q_I^d q_J^e \Big) \, . \qquad (5.14)
\end{aligned}$$

5.3 Useful symmetries for the renormalization procedure

For the algebraic characterization of the counter-terms, it is important to seek for symmetries of the classical action Σ [148]. These are transformations of the fields of the form

$$\delta\phi_i = P_i(\phi) \, , \qquad (5.15)$$

where ϕ_i denotes, in an abstract way, fields of the theory, together with their indices, which in our case may be of color and or/ flavor type. $P_i(\phi)$ stands for a local polynomial in the fields and derivatives. A specially important case is that of linear symmetries, for which

$$P_i(\phi) = a_{ij}\phi_j \, , \qquad (5.16)$$

a_{ij} being independent of the fields. In this case, the invariance of the action may be expressed in the functional form

$$W\Sigma = 0 \, , W = \int d^4x \, P_i(\phi(x)) \frac{\delta}{\delta\phi_i(x)} \, . \qquad (5.17)$$

It should be noted that this will be an integrated Ward Identity for the case of global symmetries, and a local, stronger one for gauge symmetries.

Another important class of symmetries are those associated to non-linear transformations of the fields. For instance,

$$P_i(\phi) = c_{ijk}\phi_j(x)\phi_k(x) \, , \qquad (5.18)$$

where c_{ijk} are independent of the fields. The variation of correlation functions with respect to this type of symmetry ends up being dependent on Green's functions of the composite operator $P_i(\phi)$ [149]. Therefore, it is necessary to generalize the starting action in this case

$$\Sigma' = \Sigma + S_J \ , \tag{5.19}$$

$$S_J = \int d^4x \, \rho_i P_i(\phi) \ , \tag{5.20}$$

ρ_i being classical external sources. Of course, as the action is modified in this case, one should check whether Σ' satisfies the same Ward Identities of Σ or not. Suppose that $\delta P_i(\phi) = 0$. Then, the nonlinear symmetry (5.18) may be written in the form

$$W\Sigma' = \int d^4x \, \frac{\delta \Sigma'}{\delta \rho_i} \frac{\delta \Sigma'}{\delta \phi_i} = 0 \ . \tag{5.21}$$

Finally, there is the case of linearly broken symmetries

$$W\Sigma = \Delta_{lin} \ , \tag{5.22}$$

where W is a functional operator implementing a symmetry of a general kind, and Δ_{lin} is linear in the quantum fields.

5.4 The Quantum Action Principle (QAP)

A natural question is whether the symmetries obeyed by the classical action may be extended to the quantum level or not. For this purpose, the Quantum Action Principle (QAP) is a really useful result. Here we merely state it, and refer the reader to [150, 151, 152, 153] for proofs.

Consider a local, Lorentz invariant and power-counting renormalizable theory whose propagators have the following behaviour on the UV

$$\lim_{k \to \infty} \langle \phi_i(k) \phi_j(-k) \rangle \propto \frac{P(k)}{k^2} \ , \tag{5.23}$$

$P(k)$ being a polynomial in k. Additionaly, consider the loop expansion of the quantum action Γ,

$$\Gamma = \sum_{l=0}^{\infty} \hbar^l \Gamma^{(l)} \ . \tag{5.24}$$

The QAP states that, if a symmetry of the kind (5.15), implemented in the functional language by the operator W, is known to hold (or to be linearly broken by Δ^{lin}) to order $\hbar^{l-1}$, then, in the next order,

$$W\Gamma = \Delta^{lin} + \hbar^l \Delta + O(\hbar^{l+1}) \ , \tag{5.25}$$

where Δ is an integrated (or not, depending on the nature of Γ) local polynomial in the fields, with dimension bounded by $4 - d_i + d_P$, d_i and d_P being the mass dimensions of ϕ_i and P_i, respectively. Additionaly, it has the same quantum numbers as W. Equation (5.25) is a very strong one, and allows (unless there is an anomaly, see [148]) for the introduction of non-invariant terms in the action Γ^l to assure that the Ward Identity will also hold to order $\hbar^l$. By using an induction argument, the QAP allows for the extension of linearly broken symmetries of the classical action to symmetries of the full quantum action Γ. With the set of Ward Identities satisfied by the quantum action, it is possible to restrict the possible counterterms and study the perturbative stability of the theory to all orders, also with an induction argument. One considers that the theory has been multiplicatively renormalized to some order $l - 1$, and that

$$\Gamma^{(l)} = \Gamma^{(l-1)} + \hbar^l \Sigma^{c.t.} \ , \tag{5.26}$$

where $\Sigma^{c.t.}$ is the most general local polynomial in the fields with dimension bounded by 4. We will consider $l = 1$ for simplicity. Then, the set of Ward Identities W^a are imposed on the next order, giving the constraint

$$W^a(\Gamma^{(0)} + \hbar^l \Sigma^{c.t.}) = \Delta^a_{lin} \ . \tag{5.27}$$

For a symmetry W linear in the fields, this implies

$$W\Sigma^{c.t.} = 0 \ , \tag{5.28}$$

i.e. the counter-term must be invariant. The situation is different for non-linear Ward Identities, as

$$W\Gamma^{(l)} = \int d^4x \frac{\delta(\Gamma^{(0)} + \hbar\Sigma^{c.t.})}{\delta\rho_i} \frac{\delta(\Gamma^{(0)} + \hbar\Sigma^{c.t.})}{\delta\phi_i} = 0 \ . \tag{5.29}$$

To order $\hbar$, this implies

$$\int d^4x \left(\frac{\delta\Gamma^{(0)}}{\delta\rho_i} \frac{\delta\Sigma^{c.t.}}{\delta\phi_i} + \frac{\delta\Gamma^{(0)}}{\delta\phi_i} \frac{\delta\Sigma^{c.t.}}{\delta\rho_i} \right) = 0 \ . \tag{5.30}$$

For symmetries containing mixed linear and non-linear parts, the corresponding mixture of eqs (5.29), (5.30) will hold. Finally, after all the symmetries have been imposed on $\Sigma^{c.t.}$, one proceeds to study the multiplicative renormalizability of the theory.

5.5 The non-linear symmetries of our action

As many of our BRST transformations are non-linear, it is necessary to introduce external sources coupled to these variations [149]. We do this in a BRST-invariant way by introducing

$$\begin{aligned}
\Sigma^{(1)}_{sources} &= \int d^4x \left(K^a_\mu(sA^a_\mu) + \bar{C}^a(sc^a) + Q^a_I(sq^a_I) + \bar{L}^a_I(sc^a_I) + L^a_I(s\bar{c}^a_I) + B^a_I(sb^a_I)\right) \\
&= \int d^4x \left(\frac{i}{g} K^a_\mu D^{ab}_\mu c^b - \frac{1}{2} i\bar{C}^a f^{abc} c^b c^c + Q^a_I(if^{abc}q^b_I c^c + c^a_I) - if^{abc}\bar{L}^a_I c^b_I c^c + \right. \\
&- \left. L^a_I(if^{abc}\bar{c}^b_I c^c + b^a_I) + if^{abc}B^a_I b^b_I c^c \right) \ , \qquad (5.31)
\end{aligned}$$

with $s(\bar{C}^a, K^a_\mu, \bar{L}^a_I, L^a_I, Q^a_I, B^a_I) = 0$. Clearly, $\Sigma^{(1)}_{sources}$ belongs to the trivial cohomology of s. We also introduce two additional external sources M^{ab}_I, N^{ab}_I which will be important to ensure that the ghost and anti-ghost Ward identities are satisfied by the full action. They are naturally introduced through a trivial variation:

$$\Sigma^{(2)}_{sources} = s\int_x M^{ab}_I \bar{c}^b q^b_I = \int d^4x \left(N^{ab}_I \bar{c}^a q^b_I - M^{ab}_I b^a q^b_I - M^{ab}_I \bar{c}^a \frac{\delta\Sigma}{\delta Q^b_I} \right) \ . \qquad (5.32)$$

Then, we finally arrive at our full classical action Σ in the vortex-free sector

$$\begin{aligned}
\Sigma &= S_{YM} + \int d^4x \left(\left(D^{ab}_\mu \bar{c}^b_I\right) D^{ac}_\mu c^c_I + \left(D^{ab}_\mu b^b_I\right) D^{ac}_\mu q^c_I + \kappa f_{IJK} f^{abc} \left(b^a_I q^b_J q^c_K - 2\bar{c}^a_I q^b_K c^c_J \right) + \right. \\
&+ \mu^2 \left(\bar{c}^a_I c^a_I + b^a_I q^b_I \right) + \lambda \gamma^{abcd}_{IJKL} (b^a_I q^b_J q^c_K q^d_L + 3\bar{c}^a_I c^b_J q^c_K q^d_L) + \\
&- U^2 \bar{c}^a_I q^a_I - \mathcal{K} f^{IJK} f^{abc} \bar{c}^a_I q^b_J q^c_K - \Lambda f^{abc} f^{cde} \bar{c}^a_I q^b_J q^d_I q^e_J + if^{abc}\left(b^a u^b_I q^c_I + \bar{c}^a u^b_I c^c_I\right) + \\
&+ f^{ecd} f^{eba} \bar{c}^a u^b_I q^c_I c^d + \frac{i}{g} K^a_\mu (D^{ab}_\mu c^b) - \frac{1}{2} i\bar{C}^a f^{abc} c^b c^c + Q^a_I(if^{abc} q^b_I c^c + c^a_I) - if^{abc}\bar{L}^a_I c^b_I c^c + \\
&- \left. L^a_I(if^{abc}\bar{c}^b_I c^c + b^a_I) + if^{abc} B^a_I b^b_I c^c + N^{ab}_I \bar{c}^a q^b_I - M^{ab}_I b^a q^b_I - M^{ab}_I \bar{c}^a \frac{\delta\Sigma}{\delta Q^b_I} \right) , \qquad (5.33)
\end{aligned}$$

which is invariant under the full extended BRST transformations

$$\begin{aligned}
sA^a_\mu &= \frac{i}{g} D^{ab}_\mu c^b \ , & sc &= -\frac{i}{2} f^{abc} c^b c^c \ , \\
s\bar{c}^a &= -b^a \ , & sb^a &= 0 \ , \\
sq^a_I &= if^{abc} q^b_I c^c + c^a_I \ , & s\bar{c}^a_I &= -if^{abc}\bar{c}^b_I c^c - b^a_I \ , \\
sb_I &= if^{abc} b^b_I c^c \ , & sc^a_I &= -if^{abc} c^b_I c^c \ , \\
s\mu^2 &= U^2 \ , & sU^2 &= 0 \ , \\
s\kappa &= \mathcal{K} \ , & s\mathcal{K} &= 0 \ , \\
s\lambda &= \Lambda \ , & s\Lambda &= 0 \ , \\
sM^{ab}_I &= N^{ab}_I \ , & sN^{ab}_I &= 0 \ , \\
s\bar{C}^a &= sK^a_\mu = sL^a_I = s\bar{L}^a_I = sQ^a_I = sB^a_I = 0 \ . && \qquad (5.34)
\end{aligned}$$

5.6 Ward Identities

We now display the Ward Identities satisfied by the full action Σ (eq. (5.33)):

- The Slavnov-Taylor identity:

$$
\begin{aligned}
S(\Sigma) &= \int d^4x \left(\frac{\delta\Sigma}{\delta K^a_\mu}\frac{\delta\Sigma}{A^a_\mu} + \frac{\delta\Sigma}{\delta Q^a_I}\frac{\delta\Sigma}{\delta q^a_I} + \frac{\delta\Sigma}{\delta B^a_I}\frac{\delta\Sigma}{\delta b_I} + \frac{\delta\Sigma}{\delta \bar{L}^a_I}\frac{\delta\Sigma}{\delta c^a_I} + \frac{\delta\Sigma}{\delta L^a_I}\frac{\delta\Sigma}{\delta \bar{c}^a_I} + \frac{\delta\Sigma}{\delta \bar{C}^a}\frac{\delta\Sigma}{\delta c^a} + \right. \\
&- \left. b^a\frac{\delta\Sigma}{\delta\bar{c}^a} + N^{ab}_I\frac{\delta\Sigma}{\delta M^{ab}_I}\right) + U^2\frac{\delta\Sigma}{\delta\mu^2} + \mathcal{K}\frac{\delta\Sigma}{\delta\kappa} + \Lambda\frac{\delta\Sigma}{\delta\lambda} = 0 \ . \qquad (5.35)
\end{aligned}
$$

- Gauge fixing condition

$$
\frac{\delta\Sigma}{\delta b^a} = -M^{ab}_I q^b_I + if^{abc}u^b_I q^c_I \ . \qquad (5.36)
$$

- The antighost equation

$$
\bar{G}^a\Sigma = N^{ab}_I q^b_I \ , \qquad (5.37)
$$

where the anti-ghost operator is given by

$$
\bar{G}^a = M^{ab}_I\frac{\delta}{\delta Q^b_I} + \frac{\delta}{\delta\bar{c}^a} - if^{abc}u^b_I\frac{\delta}{\delta Q^c_I} \ . \qquad (5.38)
$$

- The ghost equation

$$
G^a\Sigma = \frac{i}{g}D^{ab}_\mu K^b_\mu + if^{abc}\left(B^b_I b^c_I + Q^b_I q^c_I + \bar{L}^b_I c^c_I + L^b_I \bar{c}^c_I + \bar{C}^b c^c\right) \ , \qquad (5.39)
$$

where the ghost operator is given by

$$
G^a = \frac{\delta}{\delta c^a} - f^{cmn} f^{abc} u^n_I \frac{\delta}{\delta N^{mb}_I} \ . \qquad (5.40)
$$

- Ghost number equation:

$$
N_{gh}\Sigma = 0 \ , \qquad (5.41)
$$

$$
\begin{aligned}
\mathcal{N}_{gh} = \int d^4x \Big(& c^a\tfrac{\delta}{\delta c^a} - \bar{c}^a\tfrac{\delta}{\delta\bar{c}^a} + c^a_I\tfrac{\delta}{\delta c^a_I} - \bar{c}^a_I\tfrac{\delta}{\delta\bar{c}^a_I} + U^2\tfrac{\delta}{\delta U^2} + \mathcal{K}\tfrac{\delta}{\delta\mathcal{K}} + \Lambda\tfrac{\delta}{\delta\Lambda} + \\
& -K^a\tfrac{\delta}{\delta K^a} - 2\bar{L}^a_I\tfrac{\delta}{\delta\bar{L}^a_I} - 2\bar{C}^a\tfrac{\delta}{\delta\bar{C}^a} - Q^a_I\tfrac{\delta}{\delta Q^a_I} - B^a_I\tfrac{\delta}{\delta B^a_I} + N^{ab}_I\tfrac{\delta}{\delta N^{ab}_I}\Big) \ . \qquad (5.42)
\end{aligned}
$$

- Global flavor symmetry:

$$F\Sigma = 0 \ , \tag{5.43}$$

where we defined the flavor charge operator

$$\begin{aligned} F \ \equiv \ & q_I^a \frac{\delta}{\delta q_I^a} - b_I^a \frac{\delta}{\delta b_I^a} + c_I^a \frac{\delta}{\delta c_I^a} - \bar{c}_I^a \frac{\delta}{\delta \bar{c}_I^a} - u_I^a \frac{\delta}{\delta u_I^a} + B_I^a \frac{\delta}{\delta B_I^a} - Q_I^a \frac{\delta}{\delta Q_I^a} + L_I^a \frac{\delta}{\delta L_I^a} + \\ - \ & \bar{L}_I^a \frac{\delta}{\delta \bar{L}_I^a} - \kappa \frac{\delta}{\delta \kappa} - 2\lambda \frac{\delta}{\delta \lambda} - \mathcal{K} \frac{\delta}{\delta \mathcal{K}} - 2\Lambda \frac{\delta}{\delta \Lambda} - M_I^{ab} \frac{\delta}{\delta M_I^{ab}} - N_I^{ab} \frac{\delta}{\delta N_I^{ab}} \ . \end{aligned} \tag{5.44}$$

This Ward Identity can be used to define a new conserved quantum number, the $\mathcal{Q}$-charge, carried by the fields with a flavor index.

- Rigid symmetry:

$$R\Sigma = L_I^a \bar{c}_I^a + \bar{L}_I^a c_I^a - q_I^a Q_I^a \ , \tag{5.45}$$

where

$$\mathcal{R} = q_I^a \frac{\delta}{\delta c_I^a} + \bar{c}_I^a \frac{\delta}{\delta b_I^a} - i f^{abc} u_I^a \frac{\delta}{\delta N_I^{bc}} + \bar{L}_I^a \frac{\delta}{\delta Q_I^a} - B_I^a \frac{\delta}{\delta \bar{L}_I^a} - \kappa \frac{\delta}{\delta \mathcal{K}} - 2\lambda \frac{\delta}{\delta \Lambda} - M_I^{ab} \frac{\delta}{\delta N_I^{ab}} \ . \tag{5.46}$$

The symmetries (5.36), (5.37), (5.39), (5.45) are not exact, but the breaking is linear in the quantum fields. They are therefore compatible with the Quantum Action Principle, as the breaking remains classical upon quantization.

As the dimensions and quantum numbers of the fields will be useful for the algebraic characterization of the counter-term, we display these in Tables 1,2, and 3.

Fields	A_μ	b_I	$\bar{c}_I$	c_I	q_I	u_I	$\bar{c}$	c	b
Dimension	1	1	1	1	1	1	2	0	2
Ghost number	0	0	-1	1	0	0	-1	1	0
$\mathcal{Q}$-charge	0	-1	-1	1	1	-1	0	0	0
Statistics	B	B	F	F	B	B	F	F	B

Table 5.1: The quantum numbers of the fields. B stands for bosons and F for fermions.

Sources	$\bar{C}$	K_μ	L_I	$\bar{L}_I$	Q_I	B_I	N_I	M_I
Dimension	4	3	3	3	3	3	1	1
Ghost number	-2	-1	0	-2	-1	-1	1	0
$\mathcal{Q}$-charge	0	0	1	-1	-1	1	-1	-1
Statistics	B	F	B	B	F	F	F	B

Table 5.2: The quantum numbers of the external sources.

Parameters	μ^2	κ	λ	U^2	$\mathcal{K}$	Λ
Dimension	2	1	0	2	1	0
Ghost number	0	0	0	1	1	1
$\mathcal{Q}$-charge	0	-1	-2	0	-1	-2
Statistics	B	B	B	F	F	F

Table 5.3: The quantum numbers of the parameters.

5.7 Renormalizability of the vortex-free sector

With all the Ward Identities at hand, we now proceed to the algebraic characterization of the most general counter-term [148]. The quantum action will be given, to the lowest nontrivial order, by

$$\Gamma = \Sigma + \hbar\Sigma^{c.t.} \,, \tag{5.47}$$

where $\Sigma^{c.t.}$ is the most general integrated local polynomial in the fields, of dimension bounded by four. Then, the QAP allows us to impose that the Ward Identites of the previous section must be satisfied by Γ, which implies, for the linear symmetries,

$$\begin{aligned} &\bar{G}^a\Sigma^{c.t.} = 0\,, \quad G^a\Sigma^{c.t.} = 0\,, \\ &N_{gh}\Sigma^{c.t.} = 0\,, F\Sigma^{c.t.} = 0\,, \quad R\Sigma^{c.t.} = 0\,. \end{aligned} \tag{5.48}$$

As for the implications of the Slavnov-Taylor identity, we need to define the nilpotent operator

$$\begin{aligned} S_\Sigma \;=\; & \int_x \left(\frac{\delta\Sigma}{\delta K^a_\mu}\frac{\delta}{A^a_\mu} + \frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta K^a_\mu} + \frac{\delta\Sigma}{\delta L^a_I}\frac{\delta}{\delta\bar{c}^a_I} + \frac{\delta\Sigma}{\delta\bar{c}^a_I}\frac{\delta}{\delta L^a_I} + \frac{\delta\Sigma}{\delta\bar{L}^a_I}\frac{\delta}{\delta c^a_I} + \frac{\delta\Sigma}{\delta c^a_I}\frac{\delta}{\delta\bar{L}^a_I} + \right. \\ & + \frac{\delta\Sigma}{\delta B^a_I}\frac{\delta}{\delta b_I} + \frac{\delta\Sigma}{\delta b^a_I}\frac{\delta}{\delta B^a_I} + \frac{\delta\Sigma}{\delta Q^a_I}\frac{\delta}{\delta q^a_I} + \frac{\delta\Sigma}{\delta q^a_I}\frac{\delta}{\delta Q^a_I} + \frac{\delta\Sigma}{\delta\bar{C}^a}\frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a}\frac{\delta}{\delta\bar{C}^a} + \\ & \left. - \; b^a\frac{\delta}{\delta\bar{c}^a} + N^{ab}_I\frac{\delta}{\delta M^{ab}_I} \right) + U^2\frac{\delta}{\delta\mu^2} + \mathcal{K}\frac{\delta}{\delta\kappa} + \Lambda\frac{\delta}{\delta\lambda}\,. \end{aligned} \tag{5.49}$$

Then, the Slavnov-Taylor symmetry implies

$$S_\Sigma\Sigma^{c.t.} = 0\,. \tag{5.50}$$

This means that $\Sigma^{c.t.}$ must belong to the cohomology of the operator S_Σ, whose domain is the space of integrated local polynomials in the fields, sources and parameters. As any element of a cohomology, it may be decomposed as the sum of a trivial and a non-trivial part

$$\Sigma^{c.t.} = \Delta + S_\Sigma\Delta^{(-1)}\,, \tag{5.51}$$

where Δ^{-1} is an integrated local polynomial of dimension four and ghost number -1, and Δ is the non-trivial part, i.e., it may not be written as a trivial variation. Then, with the help of tables 1,2,3, we may write the most general counter-term as

$$\Sigma^{c.t.} = a_0 S_{YM} + S_\Sigma \Delta^{(-1)} \ , \tag{5.52}$$

$$\begin{aligned}
\Delta^{(-1)} &= \int d^4x \Big(b_1 \bar{C}^a c^a + b_2 K^a_\mu A^a_\mu + b_3 c^a_I \bar{L}^a_I + b_4 f^{abc} \bar{L}^a_I q^b_I c^c + b^{abc}_5 B^a_I M^{bc}_I + b_6 L^a_I c^a_I \\
&+ b_7 Q^a_I q^a_I + + b_8 u^a_I B^a_I + b_9 B^a_I b^a_I + b_{10} f^{abc} (\partial_\mu A^a_\mu) \bar{c}^b_I q^c_I + b^{abcd}_{11} A^a_\mu A^b_\mu \bar{c}^c_I q^d_I \\
&+ b_{12} \partial^2 q^a_I \bar{c}^a_I + b_{13} (\partial_\mu A^a_\mu) \bar{c}^a + \\
&+ b^{abcd}_{14,IJKL} b^a_I \bar{c}^b_J q^c_K q^d_L + b_{15} f^{abc} b^a_I \bar{c}^b q^c_I + b^{acbd}_{16,IKJL} c^a_I \bar{c}^c_K \bar{c}^b_J q^d_L + b_{17} f^{abc} c^a_I \bar{c}^b_I \bar{c}^c + \\
&+ b^{abcd}_{18,IJKL} u^b_J \bar{c}^a_I q^c_K q^d_L + b_{19} f^{IJK} f^{abc} \kappa \bar{c}^a_I q^b_J q^c_K + b^{abcd}_{20,IJKL} \lambda \bar{c}^a_I q^b_J q^c_K q^d_L + \\
&+ b^{abcd}_{21} \bar{c}^a_I q^b_I \bar{c}^c c^d + b_{22} f^{abc} u^b_I q^a_I \bar{c}^c + b_{23} f^{abc} \bar{c}^a_I q^b_I b^c + b_{24} \mu^2 \bar{c}^a_I q^a_I + \\
&+ b_{25} f^{abc} \bar{c}^a A^b_\mu A^c_\mu + b_{26} f^{abc} \bar{c}^a \bar{c}^b c^c + b^{abcde}_{27,IJKL} q^c_K q^d_L \bar{c}^a_I \bar{c}^b_J c^e + b_{28} \bar{c}^a b^a + \\
&+ b^{abcde}_{29,IJKL} q^d_K q^e_L M^{ab}_I \bar{c}^c_J + b^{abcd}_{30} M^{ab}_I \bar{c}^c q^c_I \Big) \ .
\end{aligned} \tag{5.53}$$

After a long computation, we find that the following constraints are imposed by the Ward Identities [155]:

$$\begin{aligned}
& b_1 = b_2 = \cdots = b_9 = b_{13} = b_{14} = b_{16} = b_{18} = b_{22} = b_{25} = \cdots = b_{30} = 0 \ , \\
& b^{cban}_{21} = i b_{23} (f^{mbn} f^{cma} + f^{mcn} f^{mba}) \ , \\
& b_{15} = -b_{17} = b_{23} = b_{24} \ , \\
& b_{10} = g b_{12} \ , \\
& b^{abcd}_{11} = g^2 f^{ca\alpha} f^{\alpha bd} b_{12} \ , \\
& b^{cban}_{21} = i b_{23} (f^{mcn} f^{mba} + f^{mbn} f^{cma}) \ , \\
& b^{cbae}_{30} = -\delta^{be} \delta^{ca} b_7 \ , \\
& f^{mna} b^{mbcd}_{20,IJKL} + f^{mba} b^{nmcd}_{20,IJKL} + f^{mca} b^{nbmd}_{20,IJKL} + f^{mda} b^{nbcm}_{20,IJKL} = 0 \ .
\end{aligned} \tag{5.54}$$

Therefore, the counter-term is reduced to

$$\begin{aligned}
\Sigma^{c.t.} &= \int d^4x \left(\frac{a_0}{2} (\partial_\mu A^a_\nu)^2 - \frac{a_0}{2} \partial_\nu A^a_\mu \partial_\mu A^a_\nu + \frac{a_0}{2} g f^{abc} A^a_\mu A^b_\nu \partial_\mu A^c_\nu + \frac{a_0}{4} g^2 f^{abc} f^{cde} A^a_\mu A^b_\nu A^d_\mu A^e_\nu + \right. \\
&+ b_{12} (\partial_\mu \bar{c}^a_I \partial_\mu c^a_I + g f^{abc} \partial_\mu \bar{c}^a_I A^b_\mu c^c_I + g f^{abc} \bar{c}^a_I \partial_\mu c^b_I A^c_\mu + g^2 f^{abe} f^{cde} A^a_\mu \bar{c}^b_I A^c_\mu c^d_I + \\
&+ \partial_\mu b^a_I \partial_\mu q^a_I + g f^{abc} \partial_\mu b^a_I A^b_\mu q^c_I + g f^{abc} b^a_I \partial_\mu q^b_I A^c_\mu + g^2 f^{abe} f^{cde} A^a_\mu b^b_I A^c_\mu q^d_I) + \\
&+ b_{19} f^{IJK} f^{abc} (\mathcal{K} \bar{c}^a_I q^b_J q^c_K - 2\kappa \bar{c}^a_I c^b_J q^c_K - \kappa b^a_I q^b_J q^c_K) + \\
&+ b^{abcd}_{20,IJKL} (\Lambda \bar{c}^a_I q^b_J q^c_K q^d_L - 3\lambda \bar{c}^a_I c^b_J q^c_K q^d_L - \lambda b^a_I q^b_J q^c_K q^d_L) + \\
&+ \left. b_{24} (U^2 \bar{c}^a_I q^a_I - \mu^2 \bar{c}^a_I c^a_I - \mu^2 b^a_I q^a_I) \right) \ .
\end{aligned} \tag{5.55}$$

The last step is to check if this general counter-term may be absorbed in Σ by a redefinition of the p fields (F), sources and parameters (P), i.e., we must check if

$$
\begin{aligned}
\Sigma_0[F_0,P_0]+O(\hbar^2) &= \Sigma[F,P]+\hbar\,\Sigma^{c.t.}[F,P]\,,\\
F &= \{A_\mu, q_I, , b_I, \bar{c}_I, c_I, b, \bar{c}, c\}\,,\\
P &= \{K_\mu, L_I, \bar{L}_I, B_I, Q_I, M_I, N_I, \bar{C}, g, \mu^2, \kappa, \lambda, U^2, \mathcal{K}, \Lambda\}\,,
\end{aligned} \tag{5.56}
$$

where bare quantities are denoted with the subscript 0. In our conventions, the renormalization factors read

$$
\begin{aligned}
F_0 &= Z_F^{1/2}\,F = \left(1+\frac{\hbar}{2}\,z_F\right)F\,,\\
P_0 &= Z_P\,P = (1+\hbar\,z_P)\,P\,.
\end{aligned} \tag{5.57}
$$

where the quantities z_F, z_P should be numerical coefficients dependent on the parameters a_0, b_i. Indeed this is the case, as the general counter-term in eq. (5.55) only contains contributions that are already present in Σ. Then,

$$
\begin{aligned}
&z_A = a_0\,, && z_g = -\frac{a_0}{2}\,,\\
&z_{c_I} = 0\,, && z_{q_I} = 0\,,\\
&z_{b_I} = 2b_{12}\,, && z_{\bar{c}_I} = 2b_{12}\,,\\
&z_{U^2} = -b_{12}-b_{25}\,, && z_{\mu^2} = -b_{12}-b_{25}\,,\\
&z_{\mathcal{K}} = -b_{12}-b_{19}\,, && z_\kappa = -b_{12}-b_{19}\,,\\
&z_\Lambda = -b_{12}-b_{20}\,, && z_\lambda = -b_{12}-b_{20}\,,\\
&z_{\bar{c}} = 0\,, && z_c = 0\,,\\
&z_b = 0\,, && z_{\bar{C}} = 0\,,\\
&z_K = -\frac{\sigma}{2}\,, && z_L = -b_{12}\,,\\
&z_{\bar{L}} = 0\,, && z_B = -b_{12}\,,\\
&z_Q = 0\,, && z_M = 0\,,\\
&z_N = 0\,.
\end{aligned} \tag{5.58}
$$

This establishes the stability of the action Σ to order $\hbar$. Then, relying on the usual inductive structure of the algebraic renormalization procedure, the same will hold to all orders, and thus we have established the all-orders perturbative renormalizability of the vortex-free sector.

5.8 Renormalizability of sectors labeled by center vortices

Let us now consider a general sector labeled by a center vortex $S_0 = e^{i\chi\beta\cdot T}$,$\beta$ being proportional to a weight of the fundamental representation, and χ being a multivalued angle with respect to a set of closed surfaces $\Omega_1, \ldots, \Omega_n$. For convenience we shall define $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$, the guiding-center of the vortex configuration. Because of the sector-dependent gauge condition $[\zeta_I, S_0 T_I S_0^{-1}] = 0$, the gauge-fixed auxiliary fields ζ_I will be of the form $\zeta_I = S_0 q_I S_0^{-1}$, with $[q_I, T_I] = 0$. In this case, it is important to consider the behaviour of the Lie Algebra basis T_I under the S_0 transformation. The Cartan components T_q, $q = 1, \ldots, N-1$ are obviously left invariant by it. The root components satisfy

$$S_0 T_\alpha S_0^{-1} = \cos(\alpha\cdot\beta\chi)T_\alpha - \sin(\alpha\cdot\beta\chi)T_{\bar\alpha}\ , \tag{5.59}$$

$$S_0 T_{\bar\alpha} S_0^{-1} = \cos(\alpha\cdot\beta\chi)T_\alpha + \sin(\alpha\cdot\beta\chi)T_{\bar\alpha}\ . \tag{5.60}$$

Due to the multivaluedness property of the angle χ, the components labeled by roots α satisfying $\alpha\cdot\beta \neq 0$ will not be regular. This can easily be dealt with by imposing the regularity conditions

$$\zeta_I^\alpha(x) = \zeta_I^{\bar\alpha}(x) = 0 \quad , \quad x \in \Omega\ , \tag{5.61}$$

for $\alpha\cdot\beta \neq 0$. The roots satisfying this condition will be denoted by γ from now on. A practical implementation of these conditions may be accomplished by the introduction of the following δ functional in the path-integral [156]

$$\prod_\gamma \delta_\Omega(\zeta_I^\gamma)\delta_\Omega(\zeta_I^{\bar\gamma})\ , \tag{5.62}$$

which may be exponentiated by defining fields λ_I^γ localized on Ω, as follows:

$$\prod_\gamma \delta_\Omega(\zeta_I^\gamma)\delta_\Omega(\zeta_I^{\bar\gamma}) = \int [D\lambda]\ e^{i\sum_\gamma \int d\sigma_1 d\sigma_2 \sqrt{g(\sigma_1,\sigma_2)}\left(\lambda_I^\gamma(\sigma_1,\sigma_2)\zeta_I^\gamma(x(\sigma_1,\sigma_2)) + \lambda_I^{\bar\gamma}(\sigma_1,\sigma_2)\zeta_I^{\bar\gamma}(x(\sigma_1,\sigma_2))\right)}. \tag{5.63}$$

Here, $x(\sigma_1,\sigma_2)$ is a parametrization of the surface Ω. It is then possible to extend the fields λ_I to the whole spacetime, by defining a source $J_\Omega(x)$ localized on Ω

$$\prod_\gamma \delta_\Omega(\zeta_I^\gamma)\delta_\Omega(\zeta_I^{\bar\gamma}) = \int [D\lambda]\ e^{i\int dx\ J_\Omega(x)\sum_\gamma \left(\lambda_I^\gamma(x)\zeta_I^\gamma(x) + \lambda_I^{\bar\gamma}(x)\zeta_I^{\bar\gamma}(x)\right)}\ , \tag{5.64}$$

$$J_\Omega(x) = \int d\sigma_1 d\sigma_2 \sqrt{g(\sigma_1,\sigma_2)}\delta(x - x(\sigma_1,\sigma_2)) \ . \tag{5.65}$$

Next, it is necessary to ensure the BRST invariance of these boundary conditions, which imply that

$$s\zeta_I^\gamma(x) = s\zeta_I^{\bar\gamma}(x) = 0 \quad , \quad x \in \Omega \ . \tag{5.66}$$

For this purpose we introduced a set Grassmanian Lagrange multipliers $\xi_I^\gamma(x)$ and implement these conditions in a similar way, i.e. by inserting

$$\prod_\gamma \delta_\Omega(\zeta_I^\gamma)\delta_\Omega(\zeta_I^{\bar\gamma})\delta_\Omega(s\zeta_I^\gamma)\delta_\Omega(s\zeta_I^{\bar\gamma}) =$$
$$\int [D\lambda][D\xi] \ e^{i\int dx\ J_\Omega(x)\sum\limits_\gamma \left(\lambda_I^\gamma(x)\zeta_I^\gamma(x) + \lambda_I^{\bar\gamma}(x)\zeta_I^{\bar\gamma}(x) + \xi_I^\gamma(x)s\zeta_I^\gamma(x) + \xi_I^{\bar\gamma}(x)s\zeta_I^\gamma(x)\right)} \tag{5.67}$$

into the partition function. Next, in order to maximize the symmetries of the action, we used the Symanzik method [157], where we extended $J_\Omega(x)$ to be a general Lie-Algebra valued source $J(x) = J^a(x)T^a$ and considered the replacement

$$\prod_\gamma \delta_\Omega(\zeta_I^\gamma)\delta_\Omega(\zeta_I^{\bar\gamma})\delta_\Omega(s\zeta_I^\gamma)\delta_\Omega(s\zeta_I^{\bar\gamma}) \to \int [D\lambda][D\xi] \ e^{-\int dx\ f^{abc}J^a(\lambda_I^b\zeta_I^c - \xi_I^b s\zeta_I^c)} \ . \tag{5.68}$$

At this point it was also convenient to define the BRST transformations of the fields λ_I, ξ_I

$$\begin{aligned} s\xi_I^a &= \lambda_I^a \ , \\ s\lambda_I^a &= 0 \ . \end{aligned} \tag{5.69}$$

These ensure that the fields are BRST doublets which will not appear in the spectrum of the theory, as desired. Additionally, defining the source J to be a singlet, i.e. $sJ = 0$, the terms in the exponent of Eq. (5.68) may be written in terms of the action

$$S_J = s\int d^4x f^{abc} J^a \xi_I^b \zeta_I^c \ . \tag{5.70}$$

The object in Eq. (5.67) is then recovered by setting J^a to its physical values

$$\begin{aligned} J^\alpha\Big|_{\text{phys}} &= J^{\bar\alpha}\Big|_{\text{phys}} = 0 \ , \\ J^q\Big|_{\text{phys}} &= i\beta_q \int d\sigma_1 d\sigma_2 \sqrt{g(\sigma_1,\sigma_2)}\delta(x - x(\sigma_1,\sigma_2)) \ . \end{aligned} \tag{5.71}$$

In this case,

$$f^{abc}J^a\xi_I^b\zeta_I^c = f^{qbc}J^q\xi_I^b\zeta_I^c \ . \tag{5.72}$$

The structure constants f^{qbc} are nontrivial only when $b = \alpha, c = \bar{\alpha}$, or $b = \bar{\alpha}, c = \alpha$, with the values $f^{q\alpha\bar{\alpha}} = \alpha|_q$ (see Appendix A for our Lie Algebra conventions). Then, Eq. (5.72) becomes

$$\begin{gathered}\sum_{\alpha>0} J\cdot\alpha(\xi_I^\alpha\zeta_I^{\bar{\alpha}} - \xi_I^{\bar{\alpha}}\zeta_I^\alpha) = \\ \sum_{\alpha>0} i\beta\cdot\alpha\int d\sigma_1 d\sigma_2\sqrt{g(\sigma_1,\sigma_2)}\delta(x - x(\sigma_1,\sigma_2))(\xi_I^\alpha\zeta_I^{\bar{\alpha}} - \xi_I^{\bar{\alpha}}\zeta_I^\alpha)\ .\end{gathered} \tag{5.73}$$

As the scalar product $\alpha\cdot\beta$ is either $0, -1$,or 1, expression (5.67) is recovered. Therefore, the full action in a sector labeled by center vortices is given by

$$S = \Sigma + S_J\ , \tag{5.74}$$

where Σ is the action in the vortex-free sector (see Eq. (5.33)), replacing q_I by ζ_I.

One important point is if the imposition of these boundary conditions on the auxiliary fields is sufficient to restrict the path integral of the gauge fields to the sector $V(S_0)$. The boundary conditions discussed in this section ensure the regularity of the functional $\zeta(A)$. The δ functional introduced in the path integral by means of the identity (4.22) implements the correlation $A \to \zeta(A)$ discussed in section 4.3.1. At the classical level, this correlation is sufficient to ensure the regularity of the corresponding gauge field (see e.g. Eq. (4.45)). The question is if this remains true at the quantum level. We will make this assumption from now on, but this must be investigated more carefully in the future.

5.8.1 Ward identities in sectors labeled by center vortices

As discussed in Section 5.6, the vortex-free sector has a rich set of symmetries. It turns out that these can be extended to the sectors labeled by center vortices, with appropriate adaptations. For this purpose, it is necessary to introduce four additional external sources $\{M_I^{ab}, N_I^{ab}, m_{IJ}^{ab}, n_{IJ}^{ab}\}$ with the following BRST transformations

$$\begin{aligned} sM_I^{ab} &= N_I^{ab}\ , \\ sN_I^{ab} &= 0\ , \\ sm_{IJ}^{ab} &= -n_{IJ}^{ab}\ , \\ sn_{IJ}^{ab} &= 0\ . \end{aligned} \tag{5.75}$$

Then, we add to the action the term

$$S_{\mathsf{s}} = s\int d^4x\left(m_{IJ}^{ab}\zeta_I^a\xi_J^b + M_I^{ab}\bar{c}^a\zeta_I^b\right)\ , \tag{5.76}$$

thus arriving at the final action in sectors containing vortices

$$S_{\text{cv}} = \Sigma + S_J + S_s \ . \tag{5.77}$$

With the appropriate modifications, the action S_{cv} then satisfies the same Ward Identities of Σ. In particular, it satisfies the modified ghost equation

$$\begin{aligned}
\mathcal{G}^a S_{\text{c.v.}} &= \left(\frac{\delta}{\delta c^a} + (if^{abn}M_I^{mn} + f^{abc}f^{cnm}\eta_I^n)\frac{\delta}{\delta N_I^{mb}} + i(im_{IJ}^{db}f^{dma} + f^{blc}f^{cma}J^l)\frac{\delta}{\delta n_{IJ}^{mb}}\right)S_{\text{c.v.}} \\
&= \frac{i}{g}D_\mu^{ab}K_\mu^b + if^{abc}\left(\bar{C}^b c^c + Q_I^b\zeta_I^c + B_I^b b_I^c + \bar{L}_I^b c_I^c + L_I^b\bar{c}_I^c\right) \ ,
\end{aligned} \tag{5.78}$$

which is crucial for the renormalizability. The ghost number equation also requires some modifications, and becomes

$$\begin{aligned}
\mathcal{N}_{\text{gh}} S_{\text{c.v.}} &= \int d^4x \left(c^a\frac{\delta}{\delta c^a} - \bar{c}^a\frac{\delta}{\delta \bar{c}^a} + c_I^a\frac{\delta}{\delta c_I^a} - \bar{c}_I^a\frac{\delta}{\delta \bar{c}_I^a} + U^2\frac{\delta}{\delta U^2} + \mathcal{K}\frac{\delta}{\delta \mathcal{K}} + \Lambda\frac{\delta}{\delta\Lambda} + \right. \\
&- K^a\frac{\delta}{\delta K^a} - 2\bar{C}^a\frac{\delta}{\delta \bar{C}^a} - Q_I^a\frac{\delta}{\delta Q_I^a} - B_I^a\frac{\delta}{\delta B_I^a} - 2\bar{L}_I^a\frac{\delta}{\delta \bar{L}_I^a} + N_I^{ab}\frac{\delta}{\delta N_I^{ab}} + \\
&+ \left. m_{IJ}^{ab}\frac{\delta}{\delta m_{IJ}^{ab}} - \xi_I^a\frac{\delta}{\delta \xi_I^a}\right) S_{\text{c.v.}} = 0 \ .
\end{aligned} \tag{5.79}$$

The modified global flavor symmetry reads

$$\begin{aligned}
\mathcal{Q}\Sigma &= \left(\zeta_I^a\frac{\delta}{\delta \zeta_I^a} - b_I^a\frac{\delta}{\delta b_I^a} + c_I^a\frac{\delta}{\delta c_I^a} - \bar{c}_I^a\frac{\delta}{\delta \bar{c}_I^a} - u_I^a\frac{\delta}{\delta u_I^a} + B_I^a\frac{\delta}{\delta B_I^a} - Q_I^a\frac{\delta}{\delta Q_I^a} + L_I^a\frac{\delta}{\delta L_I^a} + \right. \\
&- \bar{L}_I^a\frac{\delta}{\delta \bar{L}_I^a} - \kappa\frac{\delta}{\delta\kappa} - 2\lambda\frac{\delta}{\delta\lambda} - \mathcal{K}\frac{\delta}{\delta\mathcal{K}} - 2\Lambda\frac{\delta}{\delta\Lambda} - N_I^{ab}\frac{\delta}{\delta N_I^{ab}} - M_I^{ab}\frac{\delta}{\delta M_I^{ab}} + \\
&- \left. \lambda_I^a\frac{\delta}{\delta\lambda_I^a} - \xi_I^a\frac{\delta}{\delta\xi_I^a}\right) S_{\text{c.v.}} = 0 \ .
\end{aligned} \tag{5.80}$$

The linearly broken rigid symmetry is given by

$$\begin{aligned}
\mathcal{R}S_{\text{c.v.}} = &\left(\bar{c}_I^a\tfrac{\delta}{\delta b_I^a} + \zeta_I^a\tfrac{\delta}{\delta c_I^a} - B_I^a\tfrac{\delta}{\delta \bar{L}_I^a} - if^{abc}\eta_I^a\tfrac{\delta}{\delta N_I^{bc}} + \bar{L}_I^a\tfrac{\delta}{\delta Q_I^a} - \kappa\tfrac{\delta}{\delta\mathcal{K}} - 2\lambda\tfrac{\delta}{\delta\Lambda} - M_I^{ab}\tfrac{\delta}{\delta N_I^{ab}} + \right. \\
&\left. -\xi_I^a\tfrac{\delta}{\delta\lambda_I^a}\right) S_{\text{c.v.}} = -\zeta_I^a Q_I^a + \bar{L}_I^a c_I^a + L_I^a\bar{c}_I^a \ .
\end{aligned} \tag{5.81}$$

The other Ward Identities of the vortex-free sector are also satisfied without any modification. Finally, the action S_{cv} satisfies three additional symmetries:

- The J equation,

$$\mathcal{J}^a S_{\text{c.v.}} = \frac{\delta S_{\text{c.v.}}}{\delta J^a} - f^{abc}\delta_{IJ}\frac{\delta S_{\text{c.v.}}}{\delta m^{bc}_{IJ}} = 0 \ . \tag{5.82}$$

- Global symmetry in the boundary-conditions sector,

$$\mathcal{F} S_{\text{c.v.}} = \lambda^a_I \frac{\delta S_{\text{c.v.}}}{\delta \lambda^a_I} + \xi^a_I \frac{\delta S_{\text{c.v.}}}{\delta \xi^a_I} - J^a \frac{\delta S_{\text{c.v.}}}{\delta J^a} - n^{ab}_{IJ}\frac{\delta S_{\text{c.v.}}}{\delta n^{ab}_{IJ}} - m^{ab}_{IJ}\frac{\delta S_{\text{c.v.}}}{\delta m^{ab}_{IJ}} = 0 \ . \tag{5.83}$$

- The linearly broken λ equation,

$$\Lambda^a_I S_{\text{c.v.}} = \frac{\delta S_{\text{c.v.}}}{\delta \lambda^a_I} = f^{abc}\zeta^b_I J^c \ . \tag{5.84}$$

5.8.2 Counterterm of the sectors labeled by center vortices

We now may proceed as in the vortex-free case to analyze the most general counterterm consistent with all the Ward Identities. Specifically, $\Sigma^{\text{c.t.}}$ must satisfy

$$\begin{aligned}
\mathcal{B}_{S_{\text{c.v.}}}\Sigma^{\text{c.t.}} &= 0 \ ,\\
\frac{\delta \Sigma^{\text{c.t.}}}{\delta b^a} &= 0 \ ,\\
\bar{\mathcal{G}}^a\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{N}_{gh}\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{Q}\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{R}\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{G}^a\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{J}^a\Sigma^{\text{c.t.}} &= 0 \ ,\\
\mathcal{F}\Sigma^{\text{c.t.}} &= 0 \ ,\\
\Lambda^a_I\Sigma^{\text{c.t.}} &= 0 \ .
\end{aligned} \tag{5.85}$$

where we have defined the linearized Slavnov-Taylor operator of the vortex sectors

$$\begin{aligned}
\mathcal{B}_{S_{\text{c.v.}}} = & \int_x \Bigg(\frac{\delta S_{\text{c.v.}}}{\delta K^a_\mu}\frac{\delta}{A^a_\mu} + \frac{\delta S_{\text{c.v.}}}{\delta A^a_\mu}\frac{\delta}{\delta K^a_\mu} + \frac{\delta S_{\text{c.v.}}}{\delta L^a_I}\frac{\delta}{\delta \bar{c}^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta \bar{c}^a_I}\frac{\delta}{\delta L^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta \bar{L}^a_I}\frac{\delta}{\delta c^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta c^a_I}\frac{\delta}{\delta \bar{L}^a_I} + \\
& + \frac{\delta S_{\text{c.v.}}}{\delta B^a_I}\frac{\delta}{\delta b_I} + \frac{\delta S_{\text{c.v.}}}{\delta b^a_I}\frac{\delta}{\delta B^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta Q^a_I}\frac{\delta}{\delta \zeta^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta \zeta^a_I}\frac{\delta}{\delta Q^a_I} + \frac{\delta S_{\text{c.v.}}}{\delta \bar{C}^a}\frac{\delta}{\delta c^a} + \frac{\delta S_{\text{c.v.}}}{\delta c^a}\frac{\delta}{\delta \bar{C}^a} + \\
& + N^{ab}_I\frac{\delta}{\delta M^{ab}_I} - b^a\frac{\delta}{\delta \bar{c}^a} - n^{ab}_{IJ}\frac{\delta}{\delta m^{ab}_{IJ}} + \lambda^a_I\frac{\delta}{\delta \xi^a_I}\Bigg) + U^2\frac{\delta}{\delta \mu^2} + \mathcal{K}\frac{\delta}{\delta \kappa} + \Lambda\frac{\delta}{\delta \lambda} \ .
\end{aligned} \tag{5.86}$$

As this operator is nilpotent, the full counterterm may be written as

$$\Sigma^{c.t.} = \Delta_0 + \mathcal{B}_{S_{\text{c.v.}}}\Delta^{-1} \ , \tag{5.87}$$

with Δ_0 being a nontrivial element of the cohomology of $\mathcal{B}_{S_{\text{c.v.}}}$, and Δ^{-1} the trivial one. As the additional fields in this sector are doublets, the term Δ_0 is trivial, i.e., it contains only the usual Yang-Mills action

$$\Delta_0 = a_0 S_{\text{YM}} \ . \tag{5.88}$$

The trivial part Δ^{-1} consists of the most general integrated local polynomial of dimension four, with ghost number -1. It may be written as

$$\Delta^{-1} = \bar{\Delta}^{-1}(\varphi) + D^{-1}(\varphi, \phi) \ , \tag{5.89}$$

where $\phi \equiv \{J, \lambda_I, \xi_I, m_{IJ}, n_{IJ}\}$ and φ denotes all the other fields, sources and parameters. Then, it follows that $\bar{\Delta}^{-1}(\varphi)$ is identical to that of the vortex-free sector (Eq. (5.53)). Amazingly, after applying all remaining constraints in (5.85), one finds

$$\begin{aligned} \bar{\Delta}^{-1} &= \int d^4x \Big(a_1(\bar{c}^a_I \partial^2 \zeta^a_I + g f^{abc} \partial_\mu A^a_\mu \bar{c}^b_I \zeta^c_I + g^2 f^{acm} f^{dbm} A^a_\mu A^b_\mu \bar{c}^c_I \zeta^d_I) + a_2 f^{IJK} f_{abc} \kappa \bar{c}^a_I \zeta^b_J \zeta^c_K + \\ &\quad + a^{abcd}_{3,IJKL} \lambda \bar{c}^a_I \zeta^b_J \zeta^c_K \zeta^d_L + a_4 \mu^2 \bar{c}^a_I \zeta^a_I \Big) \qquad (5.90) \\ D^{-1} &= 0 \ , \qquad (5.91) \end{aligned}$$

with a_i being independent renormalization parameters. The tensor $a^{abcd}_{3,IJKL}$ has the same structure of γ^{abcd}_{IJKL}. Therefore,

$$\Sigma^{c.t.} = \Sigma^{c.t.}(\varphi) \ , \tag{5.92}$$

with $\Sigma^{c.t.}(\varphi)$ being the vortex-free counterterm, given by

$$\begin{aligned} \Sigma^{c.t.}(\varphi) &= \int d^4x \left(\frac{a_0}{2} (\partial_\mu A^a_\nu)^2 - \frac{a_0}{2} \partial_\nu A^a_\mu \partial_\mu A^a_\nu + \frac{a_0}{2} g f^{abc} A^a_\mu A^b_\nu \partial_\mu A^c_\nu + \right. \\ &+ \frac{a_0}{4} g^2 f^{abc} f^{cde} A^a_\mu A^b_\nu A^d_\mu A^e_\nu + a_1 (\partial_\mu b^a_I \partial_\mu \zeta^a_I + g f^{abc} b^a_I \partial_\mu \zeta^b_I A^c_\mu + g f^{abc} \partial_\mu b^a_I A^b_\mu \zeta^c_I \\ &+ g^2 f^{abe} f^{cde} A^a_\mu b^b_I A^c_\mu \zeta^d_I + \partial_\mu \bar{c}^a_I \partial_\mu c^a_I + g f^{abc} \bar{c}^a_I \partial_\mu c^b_I A^c_\mu + g f^{abc} \partial_\mu \bar{c}^a_I A^b_\mu c^c_I + \\ &+ g^2 f^{abe} f^{cde} A^a_\mu \bar{c}^b_I A^c_\mu c^d_I) + a_2 f^{IJK} f^{abc} (\mathcal{K} \bar{c}^a_I \zeta^b_J \zeta^c_K - 2\kappa \bar{c}^a_I c^b_J \zeta^c_K - \kappa b^a_I \zeta^b_J \zeta^c_K) + \\ &+ a^{abcd}_{3,IJKL} (\Lambda \bar{c}^a_I \zeta^b_J \zeta^c_K \zeta^d_L - 3\lambda \bar{c}^a_I c^b_J \zeta^c_K \zeta^d_L - \lambda b^a_I \zeta^b_J \zeta^c_K \zeta^d_L) + \\ &+ \left. a_4 (U^2 \bar{c}^a_I \zeta^a_I - \mu^2 \bar{c}^a_I c^a_I - \mu^2 b^a_I \zeta^a_I) \right) \ , \end{aligned} \tag{5.93}$$

5.8.3 Quantum Stability

To complete the renormalizability analysis, we have to check if the action $S_{\text{c.v.}}$ is stable under the quantum corrections given by $\Sigma^{c.t.}$. Specifically, we must show that all the divergences contained in $\Sigma^{c.t.}$ may be absorbed by a multiplicative redefinition of the fields Φ, parameters $\mathcal{P}$, and sources $\mathcal{S}$ contained in $S_{\text{c.v.}}$:

$$S_{\text{c.v.}}[\Phi,\mathcal{C},\mathcal{P}]+\hbar\Sigma^{c.t.}[\Phi,\mathcal{C},\mathcal{P}]=S_{\text{c.v.}}[\Phi_0,\mathcal{C}_0,\mathcal{P}_0]\ . \tag{5.94}$$

We follow the conventions of the vortex-free sector for the renormalization factors, i.e.

$$\begin{aligned}\Phi_0 &= \left(1+\frac{\epsilon}{2}z_\Phi\right)\Phi\ ,\\ \mathcal{S}_0 &= \left(1+\epsilon z_{\mathcal{S}}\right)\mathcal{S}\ ,\\ P_0 &= \left(1+\epsilon z_P\right)P\ .\end{aligned} \tag{5.95}$$

Then, as the vortex-free part Σ of the action was already shown to be stable, it follows that these terms renormalize in exactly the same way

$$\begin{aligned}
&z_A=a_0\ , && z_g=-\frac{a_0}{2}\ ,\\
&z_{\bar{c}_I}=2a_1\ , && z_{c_I}=0\ ,\\
&z_{\zeta_I}=0\ , && z_{b_I}=2a_1\ ,\\
&z_{\mu^2}=-a_1-a_3\ , && z_{U^2}=-a_1-a_3\ ,\\
&z_\kappa=-a_1-a_2\ , && z_{\mathcal{K}}=-a_1-a_2\ ,\\
&z_\lambda=-a_1-a_4\ , && z_\Lambda=-a_1-a_4\ ,\\
&z_{\bar{C}}=0\ , && z_b=0\ ,\\
&z_c=0\ , && z_{\bar{c}}=0\ ,\\
&z_L=-a_1\ , && z_{\bar{L}}=0\ ,\\
&z_K=-\frac{a_0}{2}\ , && z_B=-a_1\ ,\\
&z_M=0\ , && z_Q=0\ ,\\
&z_N=0\ .
\end{aligned} \tag{5.96}$$

Furthermore,

$$z_n=z_m=z_J=-\frac{z_{\lambda_I}}{2}=-\frac{z_{\xi_I}}{2}\ . \tag{5.97}$$

As there is no counterterm containing the source J, and the fields c_I,ζ_I do not renormalize, we may set $z_n=z_m=z_J=z_{\lambda_I}=z_{\xi_I}=0$. Therefore, we have established the all-orders perturbative renormalizability of sectors labeled by a general configuration of center vortices.

Chapter 6

A toy model to study quantum fluctuations around a center vortex

The results of the previous chapter establish the calculability of the Yang-Mills ensemble. This means that the partial contributions Z^{S_0} of Yang-Mills are in principle calculable. As already mentioned, in a sector labeled by a center vortex $S_0 = e^{i\chi\beta\cdot T}$, the off-diagonal components ψ_α satisfying $\alpha \cdot \beta$ will need to satisfy regularity conditions at the vortex worldsurfaces. More precisely, they should vanish at the vortex locations. The evaluation of the partition function in these nontrivial sectors is thus resemblant of Casimir energy problems, but the Dirichlet boundary conditions are imposed on surfaces instead of volumes, i.e. a codimension 2 problem. In this section we shall explore the influence of the codimension on the Casimir effect. We shall see that codimension 2 is indeed a special case [158].

6.1 A moving point with Dirichlet boundary conditions

Let us consider a scalar field φ in $d+1$ space-time dimensions, $d = 1, 2, 3$, with Dirichlet boundary conditions along a one-dimensional curve $\mathcal{C}$. The Euclidean effective action $\Gamma(\mathcal{C})$ is given by

$$e^{-\Gamma(\mathcal{C})} = \frac{\mathcal{Z}(\mathcal{C})}{\mathcal{Z}_0} \, , \tag{6.1}$$

with

$$\mathcal{Z}(\mathcal{C}) = \int D\varphi \; \delta_C(\varphi) \; e^{-\frac{1}{2}\int d^{d+1}x(\partial_\mu\varphi(x)\partial_\mu\varphi(x)+m^2\varphi^2(x))} \tag{6.2}$$

being the partition function with the Dirichlet boundary conditions, implemented by $\delta_C(\varphi)$, and $\mathcal{Z}_0$ the free partition function. It is convenient to consider a generalized problem, where the field is coupled to the curve $\mathcal{C}$ by means of a mass λ localized on $\mathcal{C}$. In this case,

$$e^{-\Gamma_\lambda(\mathcal{C})} = \frac{\mathcal{Z}_\lambda(\mathcal{C})}{\mathcal{Z}_0} \;, \tag{6.3}$$

with

$$\mathcal{Z}_\lambda(\mathcal{C}) = \int D\varphi \; e^{-\frac{1}{2}\int d^{d+1}x\,(\partial_\mu\varphi(x)\partial_\mu\varphi(x)+m^2\varphi^2(x))-\frac{\lambda}{2}\int d\tau\sqrt{g(\tau)}\varphi^2(y(\tau))} \;, \tag{6.4}$$

$y(\tau)$ being a parametrization of $\mathcal{C}$. Then, it is clear that the Dirichlet limit corresponds to the specific case $\lambda \to \infty$. By using an auxiliary field $\xi(\tau)$, $\mathcal{Z}_\lambda(\mathcal{C})$ may be cast in the alternative form [1]

$$\mathcal{Z}_\lambda(\mathcal{C}) = \det \hat{M}^{\frac{1}{2}} \int D\varphi D\xi \; e^{-\frac{1}{2}\int d^{d+1}x\,(\partial_\mu\varphi(x)\partial_\mu\varphi(x)+m^2\varphi^2(x))+i\int d^{d+1}x J_{\mathcal{C}}(x)\varphi(x)-\frac{1}{2\lambda}\int d\tau\sqrt{g(\tau)}\xi^2(\tau)} \;, \tag{6.5}$$

where

$$J_{\mathcal{C}} = \int d\tau\sqrt{g(\tau)}\xi(\tau)\delta(x-y(\tau)) \;, \tag{6.6}$$

$$\hat{M}(\tau,\tau') = \delta(\tau-\tau')\frac{\sqrt{g(\tau)}}{\lambda} \;. \tag{6.7}$$

Integrating φ, we arrive at

$$\mathcal{Z}_\lambda(\mathcal{C}) = \det \hat{M}^{\frac{1}{2}} \mathcal{Z}_0 \int D\xi \; e^{-\frac{1}{2}\int d\tau d\tau' \xi(\tau)\mathcal{K}(\tau,\tau')\xi(\tau')} \;, \tag{6.8}$$

with the kernel $\mathcal{K}(\tau,\tau')$ given by

$$\mathcal{K}(\tau,\tau') = \sqrt{g(\tau)}\left(\frac{\delta(s(\tau)-s(\tau'))}{\lambda} + \langle y(\tau)|(-\partial^2+m^2)^{-1}|y(\tau')\rangle\right)\sqrt{g(\tau')} \;, \tag{6.9}$$

where $s(\tau)$ denotes the arc-length of $\mathcal{C}$. Integrating over the auxiliary field, we get

$$\mathcal{Z}_\lambda(\mathcal{C}) = \det \hat{M}^{\frac{1}{2}} \mathcal{Z}_0 (\det \mathcal{K})^{-1/2} \;. \tag{6.10}$$

This implies that the effective action $\Gamma_\lambda(\mathcal{C})$ is given by

$$\Gamma_\lambda(\mathcal{C}) = \frac{1}{2}\mathrm{Tr}\log \mathcal{K} - \log \mathcal{Z}_0 + \frac{1}{2}\log\det \mathcal{K} \;. \tag{6.11}$$

[1] To arrive at (6.5), we discarded a contribution that gives rise to a renormalization of the tension of $\mathcal{C}$.

The two last contributions may be absorbed by a renormalization of the tension of $\mathcal{C}$, and will thus be disregarded from now on.

6.1.1 Small-departure expansion for the massless field

Consider a curve $\mathcal{C}$ that is almost a line, so that it may be parametrized as $y_\mu = y_\mu(t) = (t, \eta_i(t))$, $i = 1, \ldots, d$, with small η_i. If we interpret the curve as the trajectory of a particle in space-time, this would correspond to nonrelativistic motion. Then, an expansion of the effective action in powers of $\eta_i(t)$ may be performed:

$$\Gamma_\lambda(\mathcal{C}) = \Gamma_\lambda^{(0)}(\mathcal{C}) + \Gamma_\lambda^{(1)}(\mathcal{C}) + \Gamma_\lambda^{(2)}(\mathcal{C}) + \ldots \tag{6.12}$$

where

$$\Gamma_\lambda^{(0)}(\mathcal{C}) = \frac{1}{2}\mathsf{Tr}\log \mathcal{K}^{(0)} \;, \tag{6.13}$$

$$\Gamma_\lambda^{(1)}(\mathcal{C}) = \frac{1}{2}\mathsf{Tr}\left[(\mathcal{K}^{(0)})^{-1}\mathcal{K}^{(1)}\right] \;, \tag{6.14}$$

$$\Gamma_\lambda^{(2)}(\mathcal{C}) = \frac{1}{2}\mathsf{Tr}\left[(\mathcal{K}^{(0)})^{-1}\mathcal{K}^{(2)}\right] - \frac{1}{4}\mathsf{Tr}\left[(\mathcal{K}^{(0)})^{-1}\mathcal{K}^{(1)}(\mathcal{K}^{(0)})^{-1}\mathcal{K}^{(1)}\right] \;. \tag{6.15}$$

Notice that the contribution $\Gamma_\lambda^{(0)}(\mathcal{C})$ is that due to a straight line (static particle), and will be disregarded. The kernels $\mathcal{K}^{(i)}$ are obtained by expanding $\mathcal{K}$ in powers of the fluctuations. The lowest order is given by

$$\mathcal{K}^{(0)}(t,t') = \frac{1}{\lambda}\delta(t-t') + \langle t, \mathbf{0}|(-\partial^2)^{-1}|t', \mathbf{0}\rangle = \int \frac{d\omega}{2\pi}\tilde{K}_\lambda^{(0)}(\omega) \;, \tag{6.16}$$

with

$$K_\lambda^{(0)}(\omega) = \frac{1}{\lambda} + I(\omega) \;, \tag{6.17}$$

$$I(\omega) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{\mathbf{k}^2+\omega^2} \;. \tag{6.18}$$

The first order contribution vanishes, as

$$\mathcal{K}_\lambda^{(1)}(t,t') = i\int \frac{d\omega}{2\pi}e^{i\omega(t-t')}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\mathbf{k}^2+\omega^2}k_j(\eta_j(t)-\eta_j(t')) = 0 \;. \tag{6.19}$$

The lowest nontrivial contribution is given by

$$\mathcal{K}_\lambda^{(2)}(t,t') = -\frac{1}{2d}\int \frac{d\omega}{2\pi}e^{i\omega(t-t')}\int \frac{d^dk}{(2\pi)^d}\frac{\mathbf{k}^2}{\mathbf{k}^2+\omega^2}(\eta_i(t)-\eta_i(t'))^2 \;, \tag{6.20}$$

which may be rewritten as

$$\mathcal{K}_\lambda^{(2)}(t,t') = \frac{1}{2d}\int \frac{d\omega}{2\pi} e^{i\omega(t-t')}\omega^2 I(\omega)(\eta_i(t)-\eta_i(t'))^2 = \tag{6.21}$$

$$\frac{1}{2d}((\eta_i(t))^2+(\eta_i(t'))^2)\int \frac{d\omega}{2\pi} e^{i\omega(t-t')}\omega^2 I(\omega) - \frac{1}{d}\eta_i(t)\eta_i(t')\int \frac{d\omega}{2\pi} e^{i\omega(t-t')}\omega^2 I(\omega)\ . \tag{6.22}$$

These results allow us to compute the terms in the expansion (6.12) up to second order. Since $\mathcal{K}_\lambda^{(1)} = 0$, the first order contribution $\Gamma_\lambda^{(1)}(\mathcal{C})$ vanishes, and the second order one is given by

$$\Gamma_\lambda^{(2)} = \int \frac{d\omega}{2\pi}(\tilde{K}_\lambda^{(0)}(\omega))^{-1} e^{i\omega(t-t')}\mathcal{K}_\lambda^{(2)}(t',t) = \tag{6.23}$$

$$\frac{1}{d}\int dt(\eta_i(t))^2 \int \frac{d\omega}{2\pi}(\tilde{K}_\lambda^{(0)}(\omega))^{-1}\omega^2 I(\omega) + \frac{1}{2}\int \frac{d\omega}{2\pi} f(\omega)|\tilde{\eta}_i(\omega)|^2\ , \tag{6.24}$$

where we defined

$$f(\omega) = -\frac{1}{d}\int \frac{d\nu}{2\pi}(\tilde{K}_\lambda^{(0)}(\nu+\omega))^{-1}\nu^2 I(\nu)\ . \tag{6.25}$$

The first term in Eq. (6.24) gives rise to a renormalization of the particle's mass. We now proceed to analyze the second one, for different codimensions.

Codimension one

This case corresponds to the most widely studied situation in the context of the Casimir effect, which is when boundary conditions are imposed on a hypersurface of codimension one with respect to the full space-time [159]. The main difference is that this effect is usually studied with the presence of more than one conductor, in order to calculate the forces between them. This specific problem with just one conductor was studied in Ref. [160]. In that reference, $I(\omega)$ was shown to be convergent, and given by

$$I(\omega)|_{d=1} = \frac{1}{2|\omega|}\ , \tag{6.26}$$

which implies

$$f(\omega) = -\frac{\lambda^2}{8\pi}\left(2|\omega| - \lambda\left(1+\frac{2}{\lambda}|\omega|\right)\ln\left(1+\frac{2}{\lambda}|\omega|\right)\right)\ . \tag{6.27}$$

The first term amounts to a renormalization of the kinetic energy of the particle, and the second, which survives in the $\lambda\to\infty$ limit, is the well-known result in the case of Dirichlet boundary conditions.

Codimension three

In this case, $I(\omega)$ is divergent, and so is $f(\omega)$. Regularizing with a frequency cutoff Λ, the integral evaluates to

$$I(\omega)|_{d=3} = \frac{\Lambda}{2\pi^2} - \frac{|\omega|}{4\pi} \; . \tag{6.28}$$

This implies that

$$\tilde{K}^{(0)}_{\lambda}(\omega) = \frac{1}{\lambda_r} - \frac{|\omega|}{4\pi} \; , \tag{6.29}$$

with

$$\frac{1}{\lambda_r} = \frac{1}{\lambda} + \frac{\Lambda}{2\pi^2} \; . \tag{6.30}$$

The kernel $f(\omega)$ may be evaluated as well. It turns out that all terms of even power in ω are divergent, and the odd ones are finite, given by:

$$f(\omega) = -\frac{1}{256}|\omega|^3 + \frac{\pi^2}{\lambda_r^2}|\omega| - \frac{64\pi^4}{3\lambda_r^4}|\omega|^{-1} + O\left(\frac{1}{\lambda_r^6}\right) \; . \tag{6.31}$$

Performing the rotation back to real time, we have

$$Im(\Gamma^{(2)}_{\lambda}) = \frac{1}{2}\int \frac{d\omega}{2\pi}|\tilde{\eta}_i(\omega)|^2 \times \tag{6.32}$$

$$\left(-\frac{1}{256}|\omega|^3 + \frac{\pi^2}{\lambda_r^2}|\omega| - \frac{64\pi^4}{3\lambda_r^4}|\omega|^{-1} + O\left(\frac{1}{\lambda_r^6}\right)\right) \; . \tag{6.33}$$

The imaginary part of the effective action in this case of higher codimension may thus be rendered finite by a renormalization of λ. The presence of this imaginary part is expected, as it is associated to the creation of particles out of the vacuum, which is induced by the motion of the particle (dynamical Casimir Effect [161]).

Codimension two

This is a special case, as the coupling λ is dimensionless. Indeed, the integral $I(\omega)$ is logarithmically divergent:

$$I(\omega)|_{d=2} = \frac{1}{2\pi}\log\left|\frac{\Lambda}{\omega}\right| \; . \tag{6.34}$$

To be able to deal with this divergence, we introduce a mass scale μ, and write

$$\tilde{K}_\lambda^{(0)}(\omega) = \frac{1}{\lambda_r} + I(\omega,\mu) \; , \tag{6.35}$$

with the renormalized coupling

$$\frac{1}{\lambda_r} = \frac{1}{\lambda} + \frac{1}{2\pi} \log \left| \frac{\Lambda}{\omega} \right| \; , \tag{6.36}$$

and the $\mu-$ dependent integral

$$I(\omega,\mu) = -\frac{1}{2\pi} \log \left| \frac{\omega}{\mu} \right| \; . \tag{6.37}$$

This implies the following expression for the kernel $f(\omega,\mu)$

$$f(\omega,\mu) = -\frac{1}{4\pi} \int_0^\infty d\nu \left(\frac{\log\left|\frac{\nu+\omega}{\mu}\right|}{\log\left|\frac{\nu}{\mu}e^{-\frac{2\pi}{\lambda_r}}\right|}(\nu+\omega)^2 + \frac{\log\left|\frac{\nu-\omega}{\mu}\right|}{\log\left|\frac{\nu}{\mu}e^{-\frac{2\pi}{\lambda_r}}\right|}(\nu-\omega)^2 \right) \; , \tag{6.38}$$

which is still divergent. To deal with this, we subtract from the integrand its Taylor expansion around $\omega = 0$, up to order 2. The subtracted terms give rise to renormalizations of the mass and kinetic energy of the particle. Then, the subtracted integral $f_s(\omega)$ may be cast in the form

$$f_s(\omega) = |\omega|^3 \psi\left(\left|\frac{\omega}{\mu}\right| e^{-\frac{2\pi}{\lambda_r}} \right) \; , \tag{6.39}$$

with

$$\psi(y) = -\frac{1}{4\pi} \int_0^\infty \frac{dx}{\log|xy|} \left((x^2+1)\log\left|1-\frac{1}{x^2}\right| + 2x\log\left|\frac{x+1}{x-1}\right| - 3 \right) \; . \tag{6.40}$$

The function $\psi(y)$ may be evaluated numerically, and turns out to be finite and smooth for all $y > 0$ (see Ref. [158]). We have thus succeeded in renormalizing the codimension 2 problem.

6.1.2 The massive case in 2 dimensions

To analyze the massive case for the special situation of $d = 2$, it is convenient to express the effective action in the form

$$\Gamma_\lambda(\mathcal{C}) = \frac{1}{2} \mathrm{Tr} \log(\hat{1} + \lambda \mathcal{H}_{\mathcal{C}}) \; . \tag{6.41}$$

We have discarded constants and contributions which renormalize the tension of the string to arrive at this expression. It will be most convenient to use a representation in the arc length s space, where the operator $\hat{H}_{\mathcal{C}}$ has the components

$$\mathcal{H}_{\mathcal{C}}(s,s') = \langle y(s)|(-\partial^2+m^2)^{-1}|y(s')\rangle = \frac{1}{4\pi}\frac{e^{-m|y(s)-y(s')|}}{|y(s)-y(s')|} \ . \tag{6.42}$$

As this object is singular when $s \to s'$, a regularization is necessary. In this regard, it is convenient to define the object

$$\mathcal{H}^{\epsilon}_{\mathcal{C}}(s,s') = e^{-m|y(s)-y(s')|}\mathcal{I}^{\epsilon}_{\mathcal{C}}(s,s') \quad , \quad \mathcal{I}^{\epsilon}_{\mathcal{C}}(s,s') = \frac{1}{4\pi}\frac{\mu^{\epsilon}}{|y(s)-y(s')|^{1-\epsilon}} \ , \tag{6.43}$$

which may be rewritten as

$$\mathcal{H}^{\epsilon}_{\mathcal{C}} = D^{\epsilon} + \mathcal{H}^{\epsilon}_{l} \quad , \quad D^{\epsilon} = \mathcal{H}^{\epsilon}_{\mathcal{C}} - \mathcal{H}^{\epsilon}_{l} \ . \tag{6.44}$$

The kernel $D^{\epsilon}(s,s)$ may be shown to be finite when $\epsilon \to 0$, so that the regulator may be removed from this term. However, $\mathcal{H}^{\epsilon}_{l}$, which is the kernel associated to a straight line, must be analyzed more carefully. It is given by

$$\mathcal{H}^{\epsilon}_{l}(s,s') = e^{-m|s-s'|}\mathcal{I}^{\epsilon}_{l}(s,s') \quad , \quad \mathcal{I}^{\epsilon}_{l}(s,s') = \frac{1}{4\pi}\frac{\mu^{\epsilon}}{|s-s'|^{1-\epsilon}} \ . \tag{6.45}$$

The distribution $|x|^{\alpha}$ is well-known to have a pole at $\alpha = -1$, with residue $2\delta(x)$ [162]. This results allows us to add and subtract the pole of $\mathcal{H}^{\epsilon}_{l}(s,s') =$:

$$\mathcal{H}^{\epsilon}_{l}(s,s') = R^{\epsilon}(s,s') + \frac{1}{2\pi\epsilon}\delta(s-s') \ , \tag{6.46}$$

$$R^{\epsilon}(s,s') = \mathcal{H}^{\epsilon}_{l}(s,s') - \frac{1}{2\pi\epsilon}\delta(s-s') \ . \tag{6.47}$$

With these definitions, the distribution $R^{\epsilon}(s,s')$ is regular. This can be checked by applying it to a test function $f(s)$

$$\int_{-\infty}^{+\infty} ds' R^{\epsilon}(s,s')f(s') = \int_{|s-s'|\leq\frac{1}{\mu}} ds'\,\frac{1}{4\pi}\frac{\mu^{\epsilon}}{|s-s'|^{1-\epsilon}}(e^{-m|s-s'|}f(s')-f(s))$$
$$+\int_{|s-s'|>\frac{1}{\mu}} ds'\frac{1}{4\pi}\frac{\mu^{\epsilon}}{|s-s'|^{1-\epsilon}}f(s') \ . \tag{6.48}$$

The singularity when $s \to s'$ of the first contribution is eliminated because $e^{-m|s-s'|}f(s') - f(s) = O(s-s')$, and the second term is always regular. Then, the pole in Eq. (6.46) may be absorbed in a renormalization of λ:

$$\frac{1}{\lambda_r} = \frac{1}{\lambda} + \frac{1}{2\pi\epsilon} \ , \tag{6.49}$$

and the effective action is obtained from

$$\Gamma_\lambda(\mathcal{C}) = \frac{1}{2}\mathsf{Tr}\log\left(\frac{1}{\lambda_r} + D + R^\epsilon\right) \tag{6.50}$$

in the limit $\epsilon \to 0$.

The limit of small curvature

If the curve $\mathcal{C}$ were associated to a vortex world-line, the presence of a curvature term in the effective action would indicate a nonzero stiffness. This is an important property in the context of ensembles of center vortices. To be able to study this type of contribution, it is useful to expand $\Gamma_\lambda(\mathcal{C})$ in powers of D, which goes to zero as $\mathcal{C} \to l$. In this regard, it is convenient to rewrite Eq. (6.50) as

$$\Gamma_\lambda(\mathcal{C}) = \frac{1}{2}\mathsf{Tr}\log(\lambda_r^{-1} + R^\epsilon) + \frac{1}{2}\mathsf{Tr}\log(1 + (\lambda_r^{-1} + R^\epsilon)^{-1}D) \ . \tag{6.51}$$

Then, an expansion may be performed:

$$\Gamma_\lambda(\mathcal{C}) = \Gamma_\lambda(l) + \frac{1}{2}\mathsf{Tr}((\lambda_r^{-1} + R^\epsilon)^{-1}D) - \frac{1}{4}\mathsf{Tr}((\lambda_r^{-1} + R^\epsilon)^{-1}D(\lambda_r^{-1} + R^\epsilon)^{-1}D) + \dots \tag{6.52}$$

Defining $Q(s-s')$ as the kernel of the operator $(\lambda_r^{-1} + R^\epsilon)^{-1}$, the lowest nontrivial contribution is given by

$$\Gamma_\lambda(\mathcal{C}) - \Gamma_\lambda(l) = \frac{1}{2}\int ds \int ds'\, Q(s-s')D(s,s') \ , \tag{6.53}$$

$$D(s,s') = \frac{1}{4\pi}\left(\frac{e^{-m|y(s)-y(s')|}}{|y(s)-y(s')|} - \frac{e^{-m|s-s'|}}{|s-s'|}\right) \ . \tag{6.54}$$

In the arc-length parametrization, the curvature is proportional to $\dot{e}^2(s)$, $e(s)$ being the normalized tangent vector. This type of contribution may be obtained by using the expansion

$$\frac{e^{-m|y(s)-y(s')|}}{|y(s)-y(s')|} = \frac{e^{-m|s-s'|}}{|s-s'|}\frac{e^{\frac{m}{24}\dot{e}^2|s-s'|^3}}{(1-\frac{1}{24}\dot{e}^2|s-s'|^2)} + \cdots = \tag{6.55}$$

$$\frac{e^{-m|s-s'|}}{|s-s'|} + \dot{e}^2(s)P(s-s') \ , \tag{6.56}$$

$$P(s-s') = \frac{1}{24}(|s-s'| + m|s-s'|^2)e^{-m|s-s'|} \ . \tag{6.57}$$

These results allow us to write $D(s,s')$ as

$$D(s,s') = \frac{\dot{e}^2(s)}{4\pi}P(s-s') \ . \tag{6.58}$$

Substituting these in the expression (6.53) for the effective action, we obtain

$$\Gamma_\lambda(\mathcal{C}) - \Gamma_\lambda(l) = \int ds\, \frac{\dot{e}^2(s)}{4\pi}\int ds'\, Q(s-s')P(s-s') = \int ds \frac{\dot{e}^2(s)}{4\pi}\int du\, Q(u)P(u) \ . \tag{6.59}$$

We have thus succeeded in obtaining a contribution proportional to the curvature of the curve $\mathcal{C}$. It remains to study the dependence of the numerical coefficient

$$\chi(m,\mu) = \frac{1}{4\pi}\int du\, Q(u)P(u) \tag{6.60}$$

on m and μ. By introducing the Fourier transforms

$$P(u) = \int \frac{d\zeta}{2\pi}\tilde{P}(\zeta)e^{i\zeta u} \quad , \quad R^\epsilon(u) = \int \frac{d\zeta}{2\pi}\tilde{R}^\epsilon(\zeta)e^{i\zeta u} \ , \tag{6.61}$$

we may write

$$\int du Q(u)P(u) = \int \frac{d\zeta}{2\pi}\frac{\tilde{P}(\zeta)}{\lambda_r^{-1} + \tilde{R}^\epsilon(\zeta)} \ . \tag{6.62}$$

The function $\tilde{P}(\zeta)$ may be easily shown to be

$$\tilde{P}(\zeta) = \frac{3m^4 - 6m^2\zeta^2 - \zeta^4}{12(m^2+\zeta^2)^3} \ . \tag{6.63}$$

Then, we need to obtain the Fourier transform of

$$R^\epsilon(s-s') = \frac{\mu^\epsilon}{4\pi}\frac{e^{-m|s-s'|}}{|s-s'|^{1-\epsilon}} - \frac{1}{2\pi\epsilon}\delta(s-s') \ . \tag{6.64}$$

For $\epsilon > 0$, we have that [162]

$$\mathcal{F}(e^{-m|x|}|x|^{\epsilon-1}) = ie^{i(\epsilon-1)\pi/2}\Gamma(\epsilon)(-\zeta+im)^{-\epsilon} + c.c. \tag{6.65}$$

which implies

$$\tilde{R}^{\epsilon}(\zeta) = -\frac{\gamma}{2\pi} - \frac{1}{4\pi}\log\frac{\zeta^2+m^2}{\mu} + O(\epsilon) \tag{6.66}$$

which is well-defined in the limit $\epsilon \to 0^+$. We may now substitute Eqs. (6.66), (6.63) in Eqs. (6.62), (6.59) to obtain

$$\Gamma_\lambda(\mathcal{C}) - \Gamma_\lambda(l) = \chi(m,\mu)\int ds\,\dot{e}^2(s)\;, \tag{6.67}$$

$$\chi(m,\mu) = \frac{1}{96\pi}\int_{-\infty}^{+\infty} d\zeta \frac{6m^2\zeta^2+\zeta^4-3m^4}{(m^2+\zeta^2)^3\log\left(\frac{\sqrt{\zeta^2+m^2}}{\mu}e^{-\frac{2\pi}{\lambda_r}}\right)}\;, \tag{6.68}$$

where we have redefined $\mu \to \mu e^{-\gamma}$. For $m >> \mu e^{\frac{2\pi}{\lambda_r}}$, the coefficient χ is finite and negative, implying in a negative stiffness contribution for the curve $\mathcal{C}$.

It would be important to generalize the calculations presented in this chapter for gauge fields. In this case, we expect that the stiffness contribution will be positive, in accordance with Eq. (3.17).

Chapter 7

Conclusions

In this thesis we have presented recent advances on the understanding of the emergence of a confining flux tube in $SU(N)$ Yang-Mills theory, both in 3 and 4 spacetime dimensions. Initially, based on lattice results, we adopted the point of view that averages of Wilson Loops in continuum YM theory may be described by ensembles of center vortices. In 3 dimensions these objects are localized in closed worldlines, so that the resulting effective description is a field theory. In 4 dimensions the vortices are localized in closed worldsurfaces, and hence an effective description for the general case would be in terms of a string field theory, or a matrix model. We reviewed how the percolating phase of this theory in $3+1$d is related to the existence of an effective gauge field, corresponding to the Goldstone modes of the string field, and the possibility of an $SU(N) \to Z(N)$ Spontaneous Symmetry Breaking phase. By considering non-Abelian information and appropriate correlations between vortices and chains, the resulting effective description in 3 dimensions accommodates domain walls which interpolate the different discrete vacua. In 4 dimensions the relevant classical solutions are given by topologically stable strings, which exist due to the nontrivial topological structure of the vacuum. Then, both in $2+1$d and in $3+1$d, the Wilson Loop may be approximated by a saddle-point expansion, where the leading order is compatible with the observed asymptotic Casimir Law, and the lowest order fluctuations give rise to the Lüscher term. Moreover, the predicted chromoeletric field profiles are compatible with lattice simulations for asymptotic distances in $3+1$d. In $2+1$d, the results presented indicate profiles of the Sine-Gordon type. Additionally, both for $2+1$d and $3+1$d, we predict that the transverse profiles for the confining flux should depend on the representation of the quarks only via the asymptotic scaling law. It would be interesting to test these predictions with lattice simulations.

It is thus clear that center-vortex ensembles are successful for the description of flux tubes at asymptotic distances. Regarding the regime of validity of these ensembles, it is illuminating to consider the analysis of Refs. [56, 57, 58] of the energy-momentum tensor $T_{\mu\nu}$ of the confining flux tube. This was done for 4d YM theory at intermediate and nearly asymptotic distances. It was shown that the $T_{\mu\nu}$ of the Abelian Nielsen-Olesen vortex is not compatible with that of the flux tube for $SU(3)$, for $L = 0.46$ fm. This can be interpreted as a lower bound to the regime of validity of the ensembles of thin center vortices. A possible modification that may extend the regime of validity of the ensembles is the introduction of vortex thickness. Indeed, as reviewed in Chapter 3, the results are compatible with the observed intermediate Casimir Scaling when thickness is considered, even for a simple model. It would be interesting to investigate ensembles of thick non-oriented configurations. In the best case scenario, this description would also lead to an effective theory accommodating solitons compatible with the intermediate Casimir scaling and observed $T_{\mu\nu}$.

Next, we discussed a recent proposal to overcome the Gribov problem in $3+1$ spacetime dimensions, where the configuration space of YM theory is partitioned into sectors labeled by center vortices, and then the gauge is fixed by a condition that is local in configuration space. We showed that this approach not only has the potential to overcome the Gribov problem, but also could provide a path from pure Yang-Mills theory to ensembles of center vortices. We then showed the renormalizability of a sector containing an arbitrary number of center vortices relying on the algebraic method, which establishes the validity of the YM ensemble as a calculational tool. Then, we presented the computation of the effective action of a scalar field in the presence of boundary conditions of different codimensions, giving special emphasis to the $d = 2$ case, which would arise in the calculation of a sector labeled by a center vortex. Surprisingly, even in this simple case, a term proportional to the stiffness of the "vortex" worldline $\mathcal{C}$ was obtained.

These ideas offer a glimpse of how the flux tube could emerge in first principles YM theory. Starting from the YM ensemble, we would evaluate the contribution of each individual sector to the Wilson Loop, thus obtaining a weight factor for each configuration of center vortices. This weight would contain stiffness and tension terms, and perhaps other more complicated contributions. Then, the full average of the observable would be obtained by summing over all possible labels with the appropriate weight, thus making contact with ensembles of percolating center vortices described by an effective field theory in $2+1$ and in $3+1$ dimensions. Finally, the field content and SSB properties of these models are expected to support the formation of a soliton-like confining string. Indeed, domain walls in $2+1$d and flux tubes in $3+1$d are able to accommodate the asymptotic properties of confinement observed in Monte Carlo simulations.

Appendix A

Lie Algebra conventions

It is important to settle some conventions and notations for the Lie Algebra of this group. This is a real vector space, denoted as $su(N)$, of dimension $N^2 - 1$, spanned by the generators T^A, which are hermitian in our convention. These generators are closed under the operation of commutation, i.e.

$$[T^A, T^B] = if^{ABC} T_C \ , \tag{A.1}$$

where the real numbers f^{ABC} are known as the structure constants of the Lie Algebra. It is possible to define an inner product in this vector space, the Killing product, defined as

$$\langle X, Y \rangle = Tr(Ad(X)Ad(Y)) \ , X, Y \in su(N) \ , \tag{A.2}$$

where $Ad()$ stands for the adjoint representation of the Lie Algebra. This representation is induced by the adjoint representation of the Lie group ad_g

$$ad_g(u) = gug^{-1} \ , g, u, \in G \ . \tag{A.3}$$

This induces the definition of the Adjoint action of X on Y as $Ad_X(Y) = [X, Y]$. From this, it is possible to show that $Ad(T^A)|^{BC} = -if^{ABC}$. The normalization of the generators is chosen so as to assure that

$$\langle T^A, T^B \rangle = \delta^{AB} \ . \tag{A.4}$$

We also define the expansion in components of the elements X and Y of the Lie Algebra as $X = X^A T^A$, $Y = Y^A T^A$. Then, their Killing product is simply

$$\langle X, Y \rangle = X^A Y^A \ . \tag{A.5}$$

It is also possible to show the following useful properties, for $X, Y, Z \in su(N)$,

$$\langle X, Y \rangle = \langle Y, X \rangle \ , \tag{A.6}$$

$$\langle X, [Y, Z] \rangle = \langle Z, [X, Y] \rangle = \langle Y, [Z, X] \rangle \ . \tag{A.7}$$

The first one is obvious, as it is an inner product. The second follows from the cyclic property and from the definition of the adjoint representation of the algebra.

A.0.1 Cartan decomposition of $SU(N)$

A convenient choice of basis for the generators of the Lie algebra of $SU(N)$ is the Cartan basis. The first step is to find a set of $N-1$ elements satisfying

$$[T_q, T_p] = 0 \ . \tag{A.8}$$

The elements of this maximally commuting subspace are said to belong to the Cartan subalgebra. The remaining $N^2 - N$ elements are known as the root vectors E_α, which are eigenvectors of the adjoint action of the elements of the Cartan subalgebra

$$[T_q, E_\alpha] = \alpha|_q E_\alpha \ . \tag{A.9}$$

The $N-1$ dimensional vectors α are known as the roots of $su(N)$. The following commutation relations between root vectors hold

$$[E_\alpha, E_{-\alpha}] = \alpha_q T_q = \alpha \cdot T \ . \tag{A.10}$$

Also, for $\alpha \neq -\beta$,

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \ , \tag{A.11}$$

where N_α, β is zero if $\alpha + \beta$ is not a root, and is equal to

$$N_{\alpha,\beta} = \frac{1}{2N} \tag{A.12}$$

otherwise. These numbers also satisfy

$$N_{\beta,\alpha} = N_{-\alpha,-\beta} = -N\alpha, \beta \ , \tag{A.13}$$

and

$$N_{\alpha,\beta} = N_{\gamma,\alpha} = N_{\beta,\gamma} \ , \tag{A.14}$$

if $\alpha + \beta + \gamma = 0$. In this basis, the root generators are not hermitian, as $E_\alpha^\dagger = E_{-\alpha}$. It is possible to define an hermitian basis, where $E_\alpha, E_{-\alpha}$ are replaced by $T_\alpha, T_{\bar{\alpha}}$. Explicitly,

$$T_\alpha = \frac{E_\alpha + E_{-\alpha}}{\sqrt{2}} \ , \tag{A.15}$$

$$T_{\bar{\alpha}} = \frac{E_\alpha - E_{-\alpha}}{i\sqrt{2}} \ . \tag{A.16}$$

As is evident from their definitions, the hermitian generators satisfy

$$T_{-\alpha} = T_\alpha \ , \tag{A.17}$$

$$T_{-\bar{\alpha}} = -T_{\bar{\alpha}} \ . \tag{A.18}$$

Their commutation relations read

$$\begin{aligned} &[T_q, T_\alpha] = i\alpha_q T_{\bar{\alpha}} \ , [T_q, T_{\bar{\alpha}}] = -i\alpha_q T_\alpha \ , [T_\alpha, T_{\bar{\alpha}}] = i\alpha_q T_q \ , \\ &[T_\alpha, T_\beta] = \tfrac{i}{\sqrt{2}}(N_{\alpha,\beta} T_{\bar{\alpha+\beta}} + N_{\alpha,-\beta} T_{\bar{\alpha-\beta}}) \ , \\ &[T_{\bar{\alpha}}, T_{\bar{\beta}}] = -\tfrac{i}{\sqrt{2}}(N_{\alpha,\beta} T_{\bar{\alpha+\beta}} - N_{\alpha,-\beta} T_{\bar{\alpha-\beta}}) \ . \end{aligned} \tag{A.19}$$

From these algebraic properties, we see that the set $\alpha \cdot T, T_\alpha, T_{\bar{\alpha}}$, for each α, has the same algebraic properties of the usual $su(2)$ angular momentum algebra, and is thus identified as an $su(2)$ subalgebra of $su(N)$.

A.0.2 Representations of SU(N)

The group $G = SU(N)$ is defined as the set of NxN unitary matrices with determinant equal to 1. A representation of this group is obtained through a map $R : G \to O(V)$, $O(V)$ being the set of linear operators that act on a d dimensional vector space, that preserves the structure of the group. As the only structure of a group is its product, this condition means that, for two elements $g_1, g_2 \in G$,

$$R(g_1 g_2) = R(g_1) R(g_2) \ . \tag{A.20}$$

The representation is said to be irreducible if the only subspace of V that is left invariant by the group action is its trivial element. Similarly, a representation of the Lie algebra of G is a map $\rho : su(N) \to O(V)$ which preserves the commutator:

$$\rho([X, Y]) = [\rho(X), \rho(Y)] \ , \tag{A.21}$$

for $X, Y \in su(N)$.

The number of representations of $SU(N)$, for any $N > 1$, is infinite. An interesting fact is that the building blocks for any representation are the irreducible ones. Despite there being an infinite number of them as well, they may be classified in a relatively straightforward way through their weights. These are defined through the eigenvalues of the simultaneous eigenvectors of $R(T_q)$, i.e.

$$R(T_q)v^A = w^A|_q v^A \ , \tag{A.22}$$

where there is no sum over A. The most important examples are the fundamental representation, which has N weights $w_1, \ldots, w_N$. From those, it is possible to define the fundamental weights λ^k, $k = 1, \ldots, N-1$ of $SU(N)$ through the formula

$$\lambda^k = \sum_{j=1}^{k} w_k \ . \tag{A.23}$$

Then, it is possible to show that the highest weight w^h of any irreducible representation may be written as

$$w^h = \sum_{j=1}^{N-1} d_j \lambda^j \ , \tag{A.24}$$

where $d_j \geq 0 \in N$ are known as the Dynkin numbers, which characterize an irrep uniquely.

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