Quantum properties of topological Yang-Mills theories: Symmetries, renormalizability, and Gribov copies

Octávio C. Junqueira

Instituto de Física

Universidade Federal Fluminense

Supervisor

Prof. Rodrigo F. Sobreiro

Foreign Supervisor: Prof. David Dudal

In partial fulfillment of the requirements for the degree of

PhD in Physics

January 22, 2019

J95q Junqueira, Octávio Costa
Quantum properties of topological Yang-Mills theories: Symmetries, renormalizability, and Gribov copies / Octávio Costa Junqueira ; Rodrigo Ferreira Sobreiro, orientador ; David Dudal, coorientador. Niterói, 2020.
252 f. : il.

Tese (doutorado)-Universidade Federal Fluminense, Niterói, 2020.

DOI: http://dx.doi.org/10.22409/PPGF.2020.d.12452757713

1. Teoria quântica de campos topológica. 2. Renormalização algébrica. 3. Cópias de Gribov. 4. Produção intelectual. I. Sobreiro, Rodrigo Ferreira, orientador. II. Dudal, David, coorientador. III. Universidade Federal Fluminense. Instituto de Física. IV. Título.

CDD -

Acknowledgements

First, I would like to thank my advisor Prof. Rodrigo F. Sobreiro. His guidance over these four years was essential, not only for the development of the thesis, which would not have been written without his constant intellectual support, but in a broader sense, to naturally inspire me to be a researcher, and even a person, much more collaborative. This is the great value that I take for my life, beyond the professional side, as the greatest learning during my PhD period at UFF under the supervision of Rodrigo. I still owe him a curtain from his living room. Sorry for that.

Undoubtedly, as a major result of my collaboration with the UFF Theoretical Physics Group, I made friends. I would like to thank Antônio D. Pereira, Guilherme Sadovski, and Anderson A. Tomaz, in alphabetical order for surnames, without a preferential order except this one, in the same way we have signed our published papers. Each piece of this thesis has precious collaborations from each of you. I have to say that I had the privilege of working in an environment without any petty competition. Given the nature of the human being, this is no small feat. Thank you, my friends, you are awesome. And I could not forget Tiago Ribeiro, who was very important for me at the beginning of my PhD, when he introduced me to the study of algebraic renormalization, which was of great value for my research. Thank you, Tiago, for your generosity. I also would like to thank

Suelen and Fernanda from the institute's secretariat for always solving my bureaucratic problems with infinite patience.

I have the opportunity of spending six months at KU Leuven, in Belgium, under the supervision of Prof. David Dudal, who was immensely important to the development of the thesis, specially concerning Gribov copies. Dudal is a great person, a kind of Brazilian-Belgian, with the best sense of the word *Brazilian*. He knows very well the UERJ and Mangueira, as he told me. Thank you, David, for the hospitality. There, I also met Caroline Felix, who was amazing to us, in all aspects. Thank you, Carol, for all.

From outside the scientific community, I am infinitely grateful to my mother, who has always been by my side, supporting me in every way, not only during these four years, but throughout my life. We went through great losses, but we remain firm. It is impossible not to think about my father at this moment, who is no longer with us. I miss you, father, thanks for everything. I would like to thank the second family that life gave me, my "second parents", Josenilda and Zé Paulo, and my brother João Paulo. We are in this together. You know how much I love you. Thank you for all. You are my real family. I also will never forget the time I lived with my brother-in-law and my mother-in-law, Daniel and Renata. I sincerely thank you for your support, the reserved corner of the room for the "Octavio's office", with a little table and everything, separated by a cupboard. It was everything I needed, besides your companionship. "Silence, Octávio is studying..." These small things are very big for me. Thank you. Suely Cordeiro, it is impossible to forget you, the "salvation army", who always appeared in the most difficult moments to save us! Thank you all for all the support!

Finally, I would like to thank my wife, Luana, who gave me the most beautiful thing of the world, my son, Guto. I have no words to express my gratitude and love to you, Lua, who has almost sacrificed great part of your time so that I could finish my PhD. I hope to make up for it soon. I love you, Lua. You are the woman of my life.

I am grateful to CNPq and CAPES for the financial support along these four years. We hope that the Brazilian government maintains its indispensable support. Without that, doing science in this country will become a "rich thing", which represents a very small part of Brazilian people. Even for me, who belong to the middle class, the conditions are no longer easy, especially with a son. By further decreasing the investment, the access to scientific research will be compromised, and the country has a lot to lose from it.

To my wife and son, Luana and Augusto

Abstract

We provide a comparative study between the Witten's topological quantum field theory (TQFT), which is based on the *twist* transformation of the $N = 2$ super Yang-Mills (SYM) action, with the Baulieu-Singer (BS) one, which, in turn, is based on the BRST gauge fixing of a non-Abelian action composed of topological invariants for four-manifolds. We analyze the *on-shell* character of Witten theory, and confront it to the *off-shell* Baulieu-Singer one in the self-dual Landau gauges. As it is well known in literature, both theories share the same observables given by the Donaldson polynomials.

Studying the Ward identities of the Baulieu-Singer theory in the self-dual Landau gauges, we first show that all two-point Green functions are tree-level exact in this model. In particular, the gauge field propagator vanishes to all orders as a consequence of the Ward identity associated to the vector supersymmetry. We then generalize this result by proving that not only the two-point functions but all n-point Green functions are tree-level exact, being this property protected by the topological BRST cohomology. In a few words, we prove the absence of radiative corrections in self-dual Landau gauges for the *off-shell* topological gauge theory of Baulieu-Singer type. Besides that, we demonstrate the existence of an extra non-linear bosonic symmetry that relates the Faddeev-Popov ghost with the topological one

derived from the shift symmetry. From the quantum stability condition, taking into account this new symmetry, we identify a kind of renormalization ambiguity concerning the system of Z-factors in the BS theory, and explain the origin of such an ambiguity by analyzing the discrete symmetries of the classical action. We relate this ambiguity to the non-physical character of the β-function in the *off-shell* model, as the coupling constant only appears in the trivial part of the BRST cohomology.

The quantum properties of the self-dual Landau gauges were used to prove that the BS β-function (β_g) vanishes to all orders, a different result from the twisted $N = 2$ SYM one, which is not zero (proportional to g^3) and receives contributions at one-loop. The Donaldson polynomials, however, are reproduced by the Witten's TQFT in the weak coupling limit ($g^2 \to 0$) of the twisted $N = 2$ SYM, *i.e.*, for $\beta_g \to 0$, which shows that the conformal property of the self-dual Landau gauges in the BS theory is in agreement with Witten's TQFT — an expected result as the BS and Witten theories possess the same observables in this energy regime.

Finally we study the Gribov problem in topological Yang-Mills theories of BS type in the self-dual Landau gauges. We show that the introduction of the usual Gribov horizon in ordinary Yang-Mills theory is sufficient to eliminate the infinitesimal gauge copies in the topological case, for the Fadeev-Popov and bosonic ghost sectors, preserving the global degrees of freedom that characterize the dimension of the instanton moduli space. After applying the no-pole condition, we could prove that the gap equation forbids the introduction of an infrared massive Gribov parameter in the gauge field propagator. In other words, the BRST symmetry structure and the conformal property of the self-dual Landau gauges hide a mechanism that protects the original topological properties of the BS model, in such a way that the elimination of the gauge copies in the Feynman path integral does not affect the infrared dynamics in the topological Yang-Mills theory.

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Chapter 1

Introduction

The whole extent of topological effects to the quantization of field theories is far from being completely understood. A general method for the computation of amplitudes involving topologically inequivalent configurations, taking into account nonperturbative aspects, and its quantum implications is, up to now, a great challenge in Physics and Mathematics. The most famous case in Yang-Mills theories must be undoubtedly the Pontryagin action in Euclidean four-dimensional spacetime which represents the tunnelling amplitude between topologically inequivalent configurations with different *winding numbers* known as instantons [1; 2]. These topological field configurations are present in the vacuum of Yang-Mills theories such as the Quantum Chromodynamics (QCD) — the theory that describes the strong interactions between quarks and gluons.

Another example, with a much more mathematical bias, is the computation of topology-changing amplitudes in (2+1)-dimensional gravity [3]. In 2+1 dimensions, gravity is a topological finite theory, and, in this paper, Witten showed that it is possible to compute amplitudes associated to the topology of spacetime itself if the cosmological constant is zero. In the 1980s, many concepts about quantum field theory and topology were developed, as the concept of wormholes (originally, a theory for nontrivial spacetime topology that could explain monopole-like *singularities* [4]), and its consequences to describe the behaviour of the cosmological constant. At the time, some physicists related wormholes to the vanishing of the cosmological constant [5; 6]. Hawking also speculated that quantum fluctuations in spacetime topology at small scales may shift the cosmological constant to zero [7; 8]. The presence of wormholes, however, was never detected. Nowadays we know that the Universe is accelerating with a non-vanishing cosmological constant.. In quantum field theories, the topology generally affects the theory observables at the quantum level, but not the classical equations of motion. It illustrates the difficulty in investigating topology in gravity as there is no consistent theory — unitary and renormalizable — of four-dimensional quantum gravity.

Despite the difficulty of studying topological effects in gravity, the connection between topology and Physics has become narrower. Today we are able to say that both theories walk together. Approximately during the same period, topological Abelian models were used to describe topological phases of electrons, and to explain the Physics of superconductors. Just to illustrate the success of topological models, J. M. Kosterlitz and D. J. Thouless, in 1972, identified a new type of phase transition in two-dimensional systems in the presence of topological defects [9; 10]. Their theory describes superconducting and superfluid films. In 1982, D. J. Thouless *et al* applied topology to explain the quantum Hall conductance of an electron gas in a two-dimensional periodic potential [11]. In 1983, D. Haldane proposed a model for spin chains taking into account topological effects based on a nonlinear field theory of large-spin antiferromagnets [12; 13]. All these models were later observed in experiments[1].

The success of topology in describing phases of matter should not seem sur-

[1] In 2016 D. J. Thouless was awarded with the Nobel prize due to his "theoretical discoveries of topological phase transitions and topological phases of matter".

prising. We can find physical evidence of topological properties in well-known experiments, such as the Aharonov-Bohm effect [14]. In this effect, it does not matter the shape of the electric circuit around the (infinite) solenoid. The circuit could be circular or square. The phase acquired by the electron that surrounds the solenoid depends on the number of loops, but not on the path shape. The magnetic field along the solenoid works as a *singularity* in the space, in such a way the paths that could be continuously deformed into the other represent a class of topologically equivalent configurations, *i.e.*, that describe the same Physics. The usual Feynman diagrams, for example, are composed of topologically inequivalent one-dimensional paths. In the same way, it is impossible to continuously deform one diagram into the other. The Feynman diagrams give a perturbative tool to compute the probabilistic amplitudes of particle scatterings for the four interactions in the Universe. It is not difficult to find topological properties in the mathematical structure of physical theories that describes Nature with high precision, and we must deal naturally with the occurrence of topological effects in many branches of Physics.

Our aim is to study the quantum properties of non-Abelian topological field theories. In this kind of theories the instantons play a crucial role. However, many problems involving instantons remain unsolved. Some topological field theories whose global observables are defined by instanton configurations are essentially based on supersymmetry. We would like to investigate four-dimensional topological gauge theories capable of producing the same global observables of supersymmetric models, in particular of the Donaldson-Witten topological quantum field theory, by employing the machinery of BRST (Becchi-Rouet-Stora-Tyutin) quantization.

1.1 Motivation

During the early eighties, Donaldson constructed a whole new class of topological invariants as integrals of differential forms over the moduli space of instantons [15; 16; 17]. The Donaldson polynomials are of utmost importance in the classification of four-manifolds as they keep track of the topologically inequivalent ways one may cover a topological space with local charts. This created a new toolbox to study the so-called "exotic" manifolds [18], a.k.a. manifolds with non-standard differential structures.

The classification of four-manifolds is not only an abstract topic reserved for mathematicians. The physics on exotic manifolds has also being investigated with results ranging from particle physics to cosmology, [19; 20; 21; 22; 23]. In theses works, topological structures showed to be capable of generating a cosmological constant from small exotic $\mathbb{R}^4$, and introducing fermions into general relativity by exotic smoothness structures. In the recent paper [21], the authors also applied a topological approach based on exotic smoothness to predict neutrinos masses, in very good agreement with experiments. They also used topology to speculate about the origin of an asymmetry between neutrinos and anti-neutrinos.

Moreover topology-changing processes might play a relevant role in quantum gravity and QCD, to name only these two examples. For instance, the knowledge of topologically inequivalent four-manifolds might be fundamental to define the physically inequivalent states in some quantum gravity models [24; 25], in which the classical theory of general relativity is recovered for large scales. On the other hand, the moduli space of instantons represents a huge degeneracy of the QCD vacuum. Topology-changing processes among these vacua, a famous non-perturbative effect, can explain the anomalous $U(1)$ axial symmetry [26] and it is related to the strong CP problem. Undoubtedly, the most famous solution to the strong CP problem was proposed by Peccei and Quinn (PQ) in 1977 [27; 28]. The

PQ model consists of an extended Standard Model with an extra $U_{PQ}(1)$ global symmetry, which is constructed through the introduction of two Higgs doublets — one that couples to up-type quarks, and the other to down-type ones. When the electroweak symmetry is broken, together with the Z boson, an *axion field* is produced [29], giving rise to a pseudoscalar field in the instanton sector of the action, that depends on the vacuum expectation values of the Higgs fields. The PQ mechanism solves the CP problem as parity is not violated anymore. Over the years, this model has aroused the interest of many researchers, as axions are appear to be effectively collisionless, *i.e.*, the only significant long-range interactions of axions are gravitational, providing a candidate for (cold) Dark Matter, the missing mass of the Universe [30; 31; 32; 33; 34; 35; 36; 37].

The challenge of constructing a topological phase in quantum field theory consists in how to built a mechanism to liberate the local degrees of freedom from the global ones, and provide a physical interpretation of it. In [38], the authors have demonstrated that inflation can arise from exotic smoothness. It is a model for a topological phase transition, in which the geometric observables are described in terms of topological invariants, calculated via path integral. The sum over all metrics in the Feynman path integral, together with the background dependence, represents an obstacle to finding a consistent quantum theory of gravity, and the topological models appear to be good candidates to solve this problem, since the observables in these models are constructed independently of the metric choice, and because the general covariance is built before integrating over the space of all metrics [39; 40].

All of these issues motivate our study of topological quantum field theories, where we could analyze, for instance, the topological Yang-Mills symmetries and their relation to the mass gap problem [41], following the Gribov procedure [42], in an attempt to shed fome light on the quantum properties of a possible topological phase in non-Abelian field theories.

1.2 Overview of the thesis

In the Chapters 2, 3 and 4, we study topological aspects of non-Abelian field theories based on the well-known literature results. In Chapter 5, we provide an overview of the Gribov quantization. Our results concerning the quantum analysis of topological Yang-Mills theories are in the Chapters 5, 6 and 8. The thesis was organized as follows.

In Chapter 2, we introduce the basic elements of non-Abelian topological field theory that will be widely used throughout the thesis, namely, the concept of topological invariants; the (anti-)self-dual field strength configurations — instantons and anti-instantons configurations — as the classical minima of the Yang-Mills action; the BPST instanton solution for SU(2) theories; the θ-vacuum term described by the Pontryagin action, as the result of tunneling between degenerate vacuum states with different winding numbers; etc. The study of the Pontryagin action is based on the semi-classical approach for transition amplitudes with imaginary time systems, that can be found in S. Coleman book, *The Uses of Instantons* [43], here presented in a direct way. Such an approach is inspired in the periodic structure of the instanton sector of QCD vacuum [44; 45]. In the last section of the chapter we qualitatively discuss the solution of the $U_A(1)$ *problem* in QCD theory, due to the presence of instantons in the vacuum, and justify the necessity of further investigation concerning non-Abelian topological configurations.

The Chapter 3 is dedicated to the study topological quantum field theories (TQFT), *i.e.*, quantum field theories that possesses a partition function[1] which

[1]By abuse of language, throughout the thesis we say partition *function* instead of partition *functional*.

is independent of the metric choice, therefore having only global observables, described by topological invariants that characterize the target manifold [40]. We present the definition of Schwarz and Witten type topological models, and how the observables are formally defined for both theories. Then we study the relativistic Witten's TQFT which is obtained through the *twist* transformation of the $N = 2$ super Yang-Mills theory in the Wess-Zumino gauge. We demonstrate how the observables given by the Donaldson polynomials are obtained in the weak coupling limit of Witten theory, following the original Witten paper [46]. We largely discuss the *on-shell* character of Witten's TQFT, and, qualitatively, its perturbative exact β-function, which only receives one-loop contributions, as can be demonstrated via algebraic analysis [47].

In Chapter 4, we start the study of the Baulieu-Singer approach [48], which consists of an anomaly-free Schwarz type TQFT, built form the BRST gauge-fixing of an action composed of topological invariants, in particular, the Pontryagin action. Summarizing, we discuss the *off-shell* character of Baulieu-Singer theory, described by topological BRST transformations that define a field space with trivial cohomology; we present a geometric interpretation of such a BRST quantization (which possesses a different nature of the BRST construction of Witten's TQFT, performed by Brooks *et al.* [49], as we discuss in details in this chapter) in an "extended" space; and relate its observables with the Witten ones (both possesses the same classical observables, given by the Donaldson polynomials [50; 51]), in terms of the equivariant cohomology, and the nth Chern class by which the observables are defined with respect to the universal curvature of this extended space [52]. Finally, by using arguments from the BRST cohomology, we compare the Baulieu-Singer and twisted $N = 2$ SYM theories, and show that, despite sharing the same observables (in the weak coupling limit of Witten theory), the quantum properties of each theory are not necessarily the same (for every energy regime).

We start the study of the quantum properties of the *off-shell* Baulieu-Singer theory in Chapter 5, where we describe the Ward identities of the model in self-dual Landau gauges, with the introduction of a new non-linear bosonic symmetry that relates the Faddeev-Popov ghost, $c^a(x)$, with the topological one, $\psi^a_\mu(x)$, derived from the topological shift symmetry. With this new Ward identity, we prove, by employing BRST algebraic techniques, that the theory is renormalizable to all orders in perturbation theory with only one independent renormalization parameter[1]. We then analyze the consequences of the Ward identities to the two-point functions, and conclude that the propagators of the theory are tree-level exact, as a consequence of the vector supersymmetry present in Landau gauges [53]. In fact, all two-point are tree-level exact, being this result associated to the fact that, in this gauge choice, the gauge field propagator vanishes to all orders in perturbation theory — all of theses results were published in [54].

In Chapter 6 we study the renormalizability of the model in generalized classes of gauges, where we verify the presence of a renormalization ambiguity concerning the system of Z-factors obtained from the quantum stability condition. We interpret this ambiguity as a consequence of the absence of certain discrete symmetries, and due to the non-physical character of the gauge field propagator in the Baulieu-Singer approach, see [55]. This ambiguity is transferred to the renormalization of the coupling constant. In self-dual Landau gauges, by analyzing the Feynman diagrams and the vertex structure, we prove that the BS theory does not receive radiative corrections, *i.e.*, that all n-point Green functions are tree-level exact, due to the BRST cohomology and the impossibility of closing loops with a vanishing gauge field propagator [56]. From this result, we analyze the

[1] The renormalizability of such theories is a well-known result in literature, see for instance [53]. With the new Ward identity, we were able to reduce the independent renormalization parameters from four to one.

non-physical character of the β-function in *off-shell* topological Yang-Mills theories, and conclude that the BS theory in the Landau gauges possesses a vanishing β-function.

In Chapter 7, we provide an overview of the Gribov problem in Yang-Mills theories [42]. We discuss the Faddeev-Popov quantization [57]; the semi-classical method developed by Gribov to eliminate the infinitesimal gauge copies; how the physical content of the Feynman path integral is preserved inside the Gribov region; the non-perturbative character of the Gribov procedure, which only affects the infrared dynamics, by generating an infrared massive parameter in the gluon propagator; the Zwanziger generalization of Gribov horizon to all orders [58]; and the physical character of the massive Gribov parameter, that does not belong to trivial part of the BRST cohomology [59]. We finish the chapter with a discussion about the Fundamental Modular Region [60]. In this chapter we also argue that we do not have any physical motivation to introduce condensates, cf. [59], in the topological Yang-Mills case, as in the presence of such condensates the results of the next chapter would be the same.

Finally, Chapter 8 is dedicated to the study of Gribov copies in topological Yang-Mills theories of Baulieu-Singer type, worked out in [61]. We first prove the equivalence between the Fadeev-Popov gauge-fixing procedure and the topological BRST quantization in self-dual Landau gauges. Then we obtain the copy equations in this gauge choice, and we conclude that the infinitesimal Gribov copies can be eliminated trough the introduction of the usual Gribov horizon. We compute the no-pole condition at one-loop order for the Faddeev-Popov and bosonic sectors, and prove the triviality of the gap equation, in other words, that the symmetry structure of the topological Yang-Mills theory forbids the introduction of an infrared massive parameter of Gribov type in the gauge field propagator. After obtaining the one-loop result, we extended it to all orders, as a consequence of the absence of radiative corrections in the presence of the Gribov-Zwanziger horizon. We finalize the chapter with a discussion about the preservation of the original BRST-cohomological properties of the *off-shell* topological Yang-Mills theory. Chapter 9 contains our conclusions and perspectives.

Chapter 2

Non-Abelian field theory and topology

As mentioned in the overview of the thesis, this chapter will be used to introduce the basic elements and principles of non-Abelian topological theories that will appear throughout the thesis. In the the last section, we provide a qualitative analysis of the solution of the $U_A(1)$ *problem* in strong interactions, that indicates the necessity of further investigation concerning topological effects in non-Abelian theories.

2.1 The Yang-Mills vacuum: Instantons and the θ-vacuum in Quantum Chromodynamics

For a long time, the vacuum was treated in a secondary way as a state of little importance to the physical phenomenon. Almost unanimously, the physicists believed that only variations with respect to the vacuum energy (the lowest energy) could be experimentally observed. Only in 1998, Steve Lamoreaux, at the University of Washington, proved the unexpected [62]. He verified experimentally, using a system of two plates (a curved plate and a flat plate), the intriguing Casimir effect, proposed by Hendrik Casimir in 1948 [63]. Essentially he proved the Casimir force, which depends on the space between the plates in a closed box without air or source of heat. Following the Casimir explanation, the force is due to the residual energy of the empty space: the vacuum.

The Casimir force equals the electrical attraction holding an electron in a hydrogen atom. It's a tiny force, but it could directly affect the particle world. In Quantum Electrodynamics, the source of such a residual energy is interpreted as a soup of virtual photons[1]. In agreement with the Heisenberg's uncertainty principle, these *vacuum fluctuations* must prevent a particle from reaching the absolute rest. There is no perfect analogue to the Casimir experiment in QCD, but the Casimir effect has elucidated that the vacuum, which could have peculiar symmetry properties, represents a state of great physical significance in quantum field theory, that could explain the existence (or absence) of certain particles in Nature. In Yang-Mills theories, the vacuum, beyond other possible fluctuations, is filled by nontrivial topological field configurations called instantons, that can directly affect the quantum behaviour of the theory.

2.1.1 Classical minima of the Yang-Mills action

In this section, we would like to discuss the physical condition in which the Yang-Mills action must be finite over all space, and how this condition naturally leads to topological nontrivial vacuum solutions. Differently of Abelian theories like QED, which possesses $U(1)$ symmetry — being the gauge fields numbers —, in non-Abelian theories the $\mathcal{G}$-valued gauge fields are matrices given by

$$A_\mu = A_\mu^a T^a \,, \tag{2.1}$$

[1]Photons that are created and subsequently annihilated at the quantum level.

where A_μ^a are the components of the gauge field, and the matrices T^a are the generators of the Lie algebra of the group $\mathcal{G}$, which obey

$$\left[T^a, T^b\right] = f^{abc}T^c \, , \tag{2.2}$$

where f^{abc} are the structure constants characteristic of the group. (If all f^{abc} are zero, the group is Abelian. Otherwise, non-Abelian.) For a theory with $SU(N)$ symmetry, $a = \{1, \cdots, N^2 - 1\}$, and f^{abc} is completely antisymmetric, defined by

$$S^\dagger S = 1 \quad \text{and} \quad \det S = 1 \; , \tag{2.3}$$

where $S = e^{i\omega^a T^a}$, being ω^a the $\mathcal{G}$-valued parameters. The covariant derivative,

$$D_\mu = \partial_\mu - igA_\mu \; , \tag{2.4}$$

must obey

$$(D_\mu \Psi)' = SD_\mu \Psi \; , \tag{2.5}$$

where Ψ is a field in the fundamental representation of the group, which transforms as $\Psi' = S\Psi$, such that $S \in \mathcal{G}$. The equations (2.4) and (2.5) define the gauge transformation of the gauge field as[1]

$$A'_\mu = S^{-1}A_\mu S + S^{-1}\partial_\mu S \; , \tag{2.6}$$

(we are using the redefinition $A_\mu \to \frac{i}{g}A_\mu$, where g is the coupling constant). The curvature $F_{\mu\nu} = [D_\mu, D_\nu]$, also known as field strength, takes the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \, . \tag{2.7}$$

[1]In Abelian theories like QED, $\mathcal{G}$ is the $U(1)$ group, $S^\dagger S = 1$, where S are only phases (numbers) given by $e^{i\alpha}$, and we naturally get $A'_\mu = A_\mu - \partial_\mu \alpha$.

From (2.5), its is easy to see that $D'_\mu = SD_\mu S^{-1}$, consequently, the gauge transformation of the field strength is $F'_{\mu\nu} = [D'_\mu, D'_\nu] = SF_{\mu\nu}S^{-1}$. In four dimensions, the Lagrangian must have mass dimension equal to four. Hence the respective non-Abelian action, invariant under Lorentz and gauge transformations, takes the form

$$S_E(A) = \frac{1}{2g^2}\int d^4x \,\mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}) \ . \tag{2.8}$$

The trace appears to compensate the gauge transformation of $F_{\mu\nu}$, such that $\mathrm{tr}\left(F'_{\mu\nu}F'_{\mu\nu}\right) = \mathrm{tr}\,(SF_{\mu\nu}F_{\mu\nu}S^{-1}) = \mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu})$, using the cyclic property of the trace. The action S_E is the well-known Yang-Mills action in four-dimensional Euclidean spacetime, which could be thought as a theory in imaginary time, in other words, in Minkowski space after the Wick rotation $x_0 \to ix_0$. (Most calculations of scattering amplitudes in quantum field theory are calculated after a Wick rotation.)

The physical requirement is that the action must vanish at infinity, in such a way that S_E must be finite. This boundary condition reads

$$\lim_{|\mathbf{x}|\to\infty} F_{\mu\nu} = 0 \ , \tag{2.9}$$

in other words, that the field strength must vanishes at infinity. Normally we take this to mean $A_\mu(x) = 0$ at infinity, but this is too much restrictive. The condition (2.9) only requires that

$$\lim_{|\mathbf{x}|\to\infty} A_\mu(x) = S^{-1}\partial_\mu S \ , \tag{2.10}$$

which means that the gauge field is a *pure gauge* in the boundary (one can easily prove that $F_{\mu\nu}(S^{-1}\partial_\mu S) = 0$, using (2.3) and (2.7)). In the $SU(2)$ theory, eq. (2.10) represents a $S^3 \to S^3$ mapping: a mapping from the three-sphere of spacetime at infinity into the $SU(2)$ space which is also a three-sphere. To understand the latter statement, we must recall that the $SU(2)$ manifold is topologically equivalent to a three-sphere S^3. For $S \in SU(2)$, we have $S = e^{i\omega^a \sigma^a}$, being σ^a, for $a = \{1, 2, 3\}$, the three Pauli matrices. We can rewrite S, using the Pauli matrices identities, as

$$S = x_0 + x_i \sigma^i \ , \tag{2.11}$$

where x_0 and the vector components x_i are real. As S satisfies $S^\dagger S = 1$, we obtain $x_0^2 + x_i^2 = 1$, which is exactly the equation of a sphere with radius one in four-dimensional Euclidean space. The $S^3 \to S^3$ mapping consists of a mapping between the points of the S^3 in the boundary of spacetime into the elements of the $SU(2)$ group, since if $S \in SU(2)$, $S^{-1}\partial_\mu S$ also belongs to the algebra $\mathfrak{su}(2)$[1]. This kind of mappings characterizes the *winding number*. Before studying the four-dimensional $S^3 \to S^3$ mapping, let us analyze the one-dimensional case.

The $S^1 \to S^1$ mapping. We call an *homotopy* between two maps, $f_0(x)$ and $f_1(x)$, a continuous function $F(x,t)$, $t \in [0,1]$, which continuously deforms f_0 into f_1, *i.e.*, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. (In one dimension, the maps are paths.) If such $F(x,t)$ exists, we say that f_0 and f_1 are *homotopic*, in other words, f_0 and f_1 belong to the same *homotopic class*, which means that they are topologically equivalent. In the $S^1 \to S^1$ mapping, we start with a unit circle labelled by an angle θ, where the angles θ and $\theta + 2\pi$ are identified. This circle could be expressed by the complex number $z = e^{i\alpha}$. We can read this mapping as $\{\theta\} \to \{e^{i\alpha}\}$. The continuous functions

$$f_i^{(n)}(\theta) = \exp\left[i(n\theta + \omega_i)\right] \tag{2.12}$$

[1]These interpretation can be generalized to others non-Abelian groups. The $SU(2)$ group is only simpler and illustrative. Non-trivial configurations in $SU(N)$ are constructed through maps embedded into a suitable $SU(2)$ subgroup, that retains their winding numbers in higher rank gauge groups. For the $SU(N)$ generalization, see for instance [64; 65].

naturally form a homotopic class for different values of ω_i, being n integer numbers accordingly to the identification between θ and $\theta+2\pi$, *i.e.*, $f(\theta) = f(\theta+2\pi)$ which yields $e^{i2\pi n} = 1$. As we can see, there is a homotopy described by

$$F(\theta,t) = \exp\{i[n\theta + (1-t)\omega_i + t\omega_j]\}, \quad t \in [0,1] \ , \tag{2.13}$$

which continuously deforms $f_i^{(n)}$ into $f_j^{(n)}$. The integers n, also known as the *winding number* or *Pontryagin index*, denotes the number of times we walk around a unit circle, which maps $f_i^{(n)}$ into the same point of $f_j^{(n)}$. The *first group of homotopy*[1] (Π_1) of a S^1 sphere is then the integers: $\Pi_1(S^1) = \mathbb{Z}$, characterized by $n = \{0, \pm1, \pm2, \cdots\}$, where the "+" sign" means clockwise loops, and the "−" sign, counterclockwise loops. The winding number n can be written as

$$n = \int_0^{2\pi} \frac{d\theta}{2\pi}\left[\frac{-i}{f(\theta)}\frac{df(\theta)}{d\theta}\right] \ . \tag{2.14}$$

For the winding number $n = 1$, we have

$$f^{(1)}(\theta) = e^{i\theta} \ . \tag{2.15}$$

The mappings $[f^{(1)}(\theta)]^k$ will have winding number k. In Cartesian coordinates we can write an unit circle as

$$f(x,y) = x + iy \quad \text{with} \quad x^2 + y^2 = 1 \ . \tag{2.16}$$

Considering the identification between the end-points $-\infty$ and $+\infty$, *i.e*, $f(x = -\infty) = f(x = +\infty)$, we can generalize the domain from an unit circle to the

[1] In topology, a mapping (or map) is a continuous function. A *homotopy* is a continuous path between maps. In one dimension, the Π_1 maps are closed paths. The *first group of homotopy* counts the topologically inequivalent closed paths which can be mapped into a S^1 sphere (or a circle).

whole real line $-\infty < x < +\infty$. The both are topologically equivalent. Thus, in Cartesian coordinates the winding number is expressed by

$$n = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \left[\frac{-i}{f(x)} \frac{df(x)}{dx} \right] . \tag{2.17}$$

The corresponding winding number $n = 1$ in one dimension, for example, could be expressed by

$$f_1(x) = \exp\{\frac{i\pi x}{(x^2+\lambda^2)^2}\} , \tag{2.18}$$

where λ is an arbitrary parameter called *instanton size.*

In the four-dimensional case, the domain is the three-dimensional S^3 sphere with all points identified at infinity. The natural generalization of (2.12) and (2.16) in $S^1 \to S^1$ to $S^3 \to S^3$ mappings is

$$f(x_0, x_i) = x_0 + i x_i \cdot \sigma^i \quad \text{with} \quad x_0^2 + x_i^2 = 1 . \tag{2.19}$$

It can be shown that the generalization of the winding number as a volume integral takes the form

$$n = -\frac{1}{24\pi^2} \int d^3x \, \mathrm{tr}\{\varepsilon_{ijk}[f^{-1}(x)\partial_i f(x)][f^{-1}(x)\partial_j f(x)][f^{-1}(x)\partial_k f(x)]\} . \tag{2.20}$$

Looking at equation (2.17), the expression above reveals three *components* with a similar form, embed in a general topological structure. It counts the times the group wraps itself around the three-sphere S^3, such that the *third group of homotopy*[1] (Π_3) of S^3 is also the integers: $\Pi_3(S^3) = \mathbb{Z}$. The sign of n is determined by the sense we twisted it around S^3, like a plastic bag around a four-dimensional ball (that cannot be visualized). The winding number $n = 1$ for

[1]While the mapps of Π_1 are closed paths, for Π_3 the mapps are closed four-dimensional surfaces.

(2.20) is obtained with

$$f^{(1)}(x) = \exp\{\frac{i\pi x_i \sigma^i}{(x^2+\lambda^2)^{\frac{1}{2}}}\} \ . \tag{2.21}$$

Looking at (2.16), (2.18) and (2.18), this expression is the natural 4D generalization. (For a detailed study about the winding number generalization in 4D, see [43].)

Equations (2.19) and (2.20) contain exactly the same structure of the Yang-Mills vacuum, taking into account the physical condition of the gauge field as a pure gauge at infinity (2.10). To see that, we must first determine the classical minima of the Yang-Mills action. To this aim we must remember that the winding number can be expressed in terms of the gauge field. The volume integral

$$S_{\text{inst}} = \mathbf{Tr} \int d^4x F_{\mu\nu} \widetilde{F}_{\mu\nu} \ , \tag{2.22}$$

where $\widetilde{F} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual of the field strength, can be written as a total derivative of the Chern-Simons (unobservable) gauge dependent current,

$$K_\mu = 4\epsilon_{\mu\nu\alpha\beta}\text{Tr}\,[A_\nu\partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta] \ , \tag{2.23}$$

expressly,

$$\text{Tr} \int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{2}\int d^4x\, \partial_\mu K_\mu = \frac{1}{2}\int_{S^3_\infty} d^3S_\mu\, K_\mu \ , \tag{2.24}$$

where we applied the Stokes theorem, being the surface integral over S^3 at infinity. In this region the gauge field is given by the pure gauge (2.10), hence, using

$S^\dagger S = 1$, we obtain at S^3_∞

$$K_\mu = \frac{4}{3}\varepsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[(S^{-1}\partial_\nu S)(S^{-1}\partial_\alpha S)(S^{-1}\partial_\beta S)] \ . \tag{2.25}$$

The eq. (2.24) reveals that the Pontryagin action, $\mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$, is a Chern-Simons (CS) surface term in the 4D boundary. The Chern-Simons current K_μ establishes

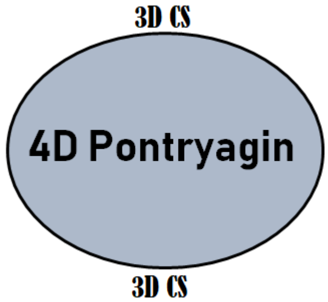

Figure 2.1: 4D Pontryagin is a 3D Chern-Simons in the boundary.

the conservation of a *topological charge*. This conservation law (different of the Noether's theorem concept) means that there is a conserved quantity whose characterization is the impossibility of classical transitions between field configuration with different winding numbers. These configurations are topologically inequivalent, and cannot be continuously deformed between each other. The proof is achieved by taking an infinitesimal transformation of the Lie group $S = e^{\omega^a T^a}$, *i.e.*,

$$\tilde{\delta} S = S\tilde{\delta}\omega^a T^a \equiv S\tilde{\delta}T \ . \tag{2.26}$$

Under this transformation, $\tilde{\delta}(S\partial_\mu S^{-1}) = -S\partial_\mu\tilde{\delta}TS^{-1}$, therefore, using $\partial_\mu S^{-1}S = -S^{-1}\partial_\mu S$, we find

$$\tilde{\delta}S_{\mathrm{inst}} = 0 \ , \tag{2.27}$$

due to the antisymmetric property of the Levi-Civita tensor, which shows that the Pontryagin action represents *topological invariants*, *i.e.*, invariant quantities under continuous deformations, thus defining a conserved *topological charge* accordingly to each winding number. For the winding number in $S^3 \to S^3$ mappings (2.20), identifying $f(x_0, x_i)$ with $S(x_0, x_i)$, and replacing (2.25) into (2.24), we get

$$\int d^4x \,\mathrm{Tr} F_{\mu\nu}\tilde{F}_{\mu\nu} = 16\pi^2 n \,. \tag{2.28}$$

This is the well-known relation which make it possible to write the topological winding number in terms of the non-Abelian gauge fields. The positivity condition in Euclidean space yields

$$\mathrm{Tr}\int d^4x \,(F_{\mu\nu} + \widetilde{F}_{\mu\nu})^2 \geq 0 \,. \tag{2.29}$$

By using $(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu})^2 = 2(F_{\mu\nu}F_{\mu\nu} \pm F_{mu\nu}\widetilde{F}_{\mu\nu})$, and eq. (2.28), we automatically obtain the inequality

$$\mathrm{Tr}\int d^4x F_{\mu\nu}F_{\mu\nu} \geq |\mathrm{Tr}\int d^4x F_{\mu\nu}\widetilde{F}_{\mu\nu}| = 16\pi^2|n| \,, \tag{2.30}$$

The equation above shows that the four-dimensional Yang-Mills action in a topological sector with winding number n is bounded by

$$S_E(A) \geq \frac{8\pi^2|n|}{g^2} \,. \tag{2.31}$$

The minimization of S_E occurs when this equation reaches the equality. From (2.29) we conclude that the instanton configurations

$$F_{\mu\nu} = \pm\widetilde{F}_{\mu\nu} \tag{2.32}$$

represent the classical minima of S_E. As we know the Feynman path integral is dominated by the classical configuration. The Yang-Mills path integral can be seen as quantum perturbations around instantons. For the "+" sign, the field which obeys (2.32) is called instanton, for the "−" sign, anti-instanton. The *Bogomoln'yi argument* states that the (anti-)self-dual configuration must solve the full equations of motion since it minimizes the action in some topological sector. In the case of instantons this is immediately satisfied, as $D_\mu F_{\mu\nu} = D_\mu(\pm\widetilde{F}_{\mu\nu}) = 0$ via Bianchi identity. In practice nontrivial topological instanton solutions define a *lowest level*, a kind of a energy source that cannot be "switched off" in the presence of instantons, like the vacuum fluctuation of virtual photons in QED[1]. The physical interpretation in Yang-Mills theory is that the vacuum is degenerate, filled by topological field configurations, in such a way that the tunneling between vacuum states with different winding numbers, intermediated by instantons, will affect the system at the quantum level.

The BPST instanton. In 1975, A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin (BPST) found a classical instanton solution with winding number 1, which obeys the equations of motion of $SU(2)$ Yang-Mills theory in Euclidean spacetime [1], namely,

$$A^a_\mu(x) = \frac{2}{g}\frac{\zeta^a_{\mu\nu}(x-X)_\nu}{(x-X)^2+\lambda^2}\ , \tag{2.33}$$

wherein X is the *instanton center* (an arbitrary parameter); λ (another parameter), the *instanton size*; and $\zeta^a_{\mu\nu}$, the anti-self-dual 't Hooft matrices,

$$\zeta^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \zeta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \zeta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.34}$$

[1]The comparison is not completely satisfactory, as the instantons are configurations of purely topological origin. However it is well-known that instantons are responsible for producing short-range attractive forces at the quantum level in strong interactions [66], basically between quarks. See for instance [67], about the attractive quantum force due to instantons acting on glueballs.

Due to the anti-self-duality of ζ^a, this field automatically satisfies the self-dual configuration in (2.32), in other words, it is a $n = 1$ solution that minimizes the Yang-Mills action. This solution is called *regular instanton*, and obeys the Landau gauge, $\partial_\mu A^a_\mu = 0$. The BPST instanton (2.33) will be of great importance for the thesis, in particular for the analysis of Gribov copies in topological quantum field theories.

Large ρ^2 solutions. The $SU(2)$ gauge transformation for winding number $n = 1$ has the form

$$S(x) = \frac{x_0 + ix_i\sigma^i}{\rho} \ , \quad \text{where} \quad \rho^2 = x_0^2 + x_i^2 \ . \tag{2.35}$$

For large ρ^2, it gives rise to the gauge field

$$A^{inst}_\mu(x) = \frac{\rho^2}{\rho^2 + \lambda^2} S^{-1}\partial_\mu S \ . \tag{2.36}$$

As we can see, for $\rho \gg \lambda$, $A_\mu \to S^{-1}\partial_\mu S$, showing that the instanton field (2.36) naturally reduces to the pure gauge configuration that satisfies the physical condition (2.10). Writing it in components,

$$A^{inst}_0(x) = \frac{-ix_i\sigma^i}{\rho^2 + \lambda^2} \ , \quad A^{inst}_i(x) = \frac{-i[x_0\sigma_i + (\overrightarrow{\sigma} \times \overrightarrow{x})_i]}{\rho^2 + \lambda^2} \ . \tag{2.37}$$

One can check that (2.36) yields the finite Euclidean action

$$S^{(1)}_E(A^{inst}) = \frac{8\pi^2}{g^2} \ , \tag{2.38}$$

i.e, it provides an explicit $n = 1$ solution for large ρ^2, that also minimizes S_E. This kind of instanton solution was the starting point for understanding the Yang-Mills quantum vacuum structure as a tunnelling event.

2.1.2 Tunneling between vacuum sates

When we include the condition (2.10) into the path integral, the non-Abelian gauge theory shows up a nontrivial vacuum structure, which corresponds to a superposition of vacuum states with different winding numbers. The instanton configurations can connect initial and final vacuum states, through vacuum-to-vacuum tunneling transitions. The first study in this direction was done by G. ‘t Hooft in 1976 [2], followed by the seminal studies on the periodic structure of Yang-Mills vacuum in the same year [44; 45].

2D system. In order to construct a two-dimensional analogy, let us analyze the system with only one spatial coordinate. In non-relativistic quantum mechanics, there is no classical transition between the two vacuum states, located in q_0 and $-q_0$ (see Fig. 1.2 (a) below), for a Higgs potential $V(q) = (q - q_0)^2$, being $q \equiv q(t)$ some generalized coordinate. The energy of the system is

$$E = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 + V(q) \, , \tag{2.39}$$

such that the quantum-tunnelling is the only possible transition between the states $|q_0\rangle$ and $|-q_0\rangle$.

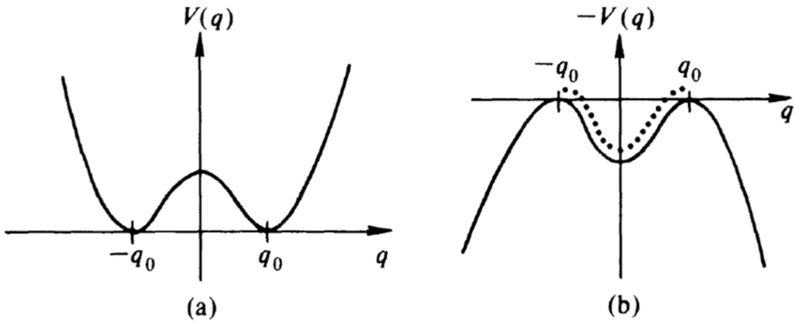

Figure 2.2: (a) Higgs potential; (b) the same potential in imaginary time.

In general, the ground (vacuum) state is given by the superposition

$$|0\rangle = \frac{1}{\sqrt{2}} \left(|q_0\rangle + |-q_0\rangle \right) \ . \tag{2.40}$$

The tunneling amplitude can be calculated in the corresponding time imaginary system, $t \to i\tau$, as it shows a classical particle trajectory for a particle moving in the potential $-V(q)$ (see Fig. 1.2 (b)), with the energy

$$-E = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q) \ . \tag{2.41}$$

For the vacuum state, $E = 0$, we immediately get the solution for the trajectory in the imaginary time as

$$q(\tau) = q_0 \tanh\left(2^{\frac{1}{2}} q_0 \tau\right) \ . \tag{2.42}$$

Consequently, the corresponding action can be calculated, and gives a finite value, namely,

$$S_\tau = \int_{-\infty}^{+\infty} d\tau \{ \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - [-V(q)]\} = \frac{4}{3}\sqrt{2} q_0^3 \ . \tag{2.43}$$

In the Feynman path-integral formalism, the transition amplitude in Euclidean space (imaginary time) is computed via

$$T = \langle q_f | e^{-\frac{H\tau}{\hbar}} | q_i \rangle = \int \mathcal{D}q \, e^{\frac{-S_E}{\hbar}} \ , \tag{2.44}$$

where S_E is the Euclidean action, and $\mathcal{D}q$ is the *integration measure* which denotes a sum over all paths between $|q_i\rangle$ and $|q_f\rangle$. The integral (2.44), for an expansion in powers of $\hbar$, is dominated by the path for which S_E is stationary. Naturally, in the semi-classical approximation $e^{\frac{-S_E}{\hbar}} \sim e^{\frac{-S_\tau}{\hbar}} \left[1 + \varphi(\hbar)\right]$, (an explicit calculation can be found in [43])

$$T \sim e^{-\frac{4}{3}\sqrt{2}q_0^3} \,, \tag{2.45}$$

in which the transition amplitude is clearly dominated by the vacuum-to-vacuum quantum tunneling.

4D Yang-Mills quantum vacuum. Going back to the non-Abelian four-dimensional case, we shall see that the amplitude transition between degenerate vacuum states is dominated by $|0, n\rangle \to |0, n+1\rangle$ transitions, being $|0, k\rangle$ the multiple vacuum states with different winding numbers k. (Hereafter, we will denote these vacuum states only by $|k\rangle$.)

Imposing the physical condition in a four-dimensional cylinder, we say that $F_{\mu\nu}(t, \overrightarrow{x})$ vanishes in the region

$$t < -\frac{T}{2} \,, \quad t > \frac{T}{2} \,, \quad \text{and} \quad |\overrightarrow{x}| > L \,, \tag{2.46}$$

for T and L very large. It means that outside the cylinder with length T and radius L, the gauge field behaves as a pure gauge. Henceforth, in the Feynman path integral, one sums over all field configurations including the vacuum states which are recognized as the ones outside the cylinder. In the gauge choice

$$A_0(x) = 0 \,, \tag{2.47}$$

only the space-like segments contribute. Moreover the gauge transformations $S(x)$ must be time independent, since, under a gauge transformation in the gauge (2.47), $A'(x) = S^{-1}\partial_0 S = 0$, *i.e.*, $\partial_0 S(x) = 0$. Thus the vacuum is described by the time-independent field

$$A_i(\overrightarrow{x}) = S(\overrightarrow{x})^{-1}\partial_i S(\overrightarrow{x}) \,. \tag{2.48}$$

By choosing $S(\overrightarrow{x}) = 1$ at initial time $t = \frac{T}{2}$, we have

$$A_i(\overrightarrow{x}) = 0 \quad \text{for} \quad t = \frac{T}{2} \ . \tag{2.49}$$

The vacuum condition $F_{\mu\nu} = 0$ for the equation above implies that $F_{0i} = \partial_0 A_i = 0$, where we use the gauge (2.47). From it, we conlcude that $A_i = 0$ throughout the vacuum (in the region outside the cylinder). Consequently, the vacuum at $\frac{T}{2}$ is identified. For large T, it corresponds to mappings from the $SU(2)$ gauge manifold into the three-dimensional space with infinities identified, in other words, the $S^3 \to S^3$ mapping[1]. As we saw in the previous section, the trivial solution with winding number $n = 0$ (2.49) is not the only one that satisfies the physical condition, in fact this kind of solution can be divided into inequivalent homotopic classes. For the same $A_i(\overrightarrow{x})$ we can construct the topological nontrivial solution (see eq. (2.21))

$$S(\overrightarrow{x}) = \exp\{\frac{i\pi x_i \sigma^i}{(x^2 + \lambda^2)^{\frac{1}{2}}}\} \tag{2.50}$$

with winding number $n = 1$. We conclude that the vacuum is degenerate formed by multiple topologically inequivalent states with different winding numbers. For completeness, we must verify if there is a field that connects these vacuum states making it possible a coherent Feynman path integral representation of the quantum tunneling between them.

The instanton is that field. In order to verify if the $n = 1$ instanton (2.36) could be gauge transformed into the gauge form (2.47), we must analyze the

[1] A good way to "visualize" this four-dimensional mapping is to consider the three-dimensional (2+1) case, with the time as the z-axis. In the top of the cylinder, we have a disc (a S^2 surface) with its edge identified by (2.49), which, at the large T limit, assuming the condition $F_{\mu\nu} = 0$ outside the cylinder, consists of a mapping from the $SU(2)$ group to infinities identified. In four dimensions, the top is a S^3 surface — then the $S^3 \to S^3$ mapping — although the 4D cylinder cannot be visualized.

solution for the gauge transformation

$$A_0^{inst} \to A^{inst\prime}{}_0 = 0 \; , \tag{2.51}$$

and interpret if such solution has a physical meaning. The condition above yields

$$U^{-1}(x)A_0^{inst}(x)U(x) + U^{-1}\partial_0 U(x) = 0 \; , \tag{2.52}$$

where $U(x) \in SU(2)$. Replacing (2.36) into (2.52), we get the equation

$$\partial_0 U(x) = \frac{ix_i\sigma^i}{x_0^2 + \overrightarrow{x}^2 + \lambda^2} U(x) \; ; \tag{2.53}$$

this equation can be easily integrated. The solution is

$$U(x) = \exp\{\frac{ix_i\sigma^i}{(\overrightarrow{x}^2 + \lambda^2)^{\frac{1}{2}}} \left(\tan^{-1}\left[\frac{x_0}{(x_0^2 + \lambda^2)^{\frac{1}{2}}}\right] + (n + \frac{1}{2})\pi \right)\} \; , \tag{2.54}$$

where $(n + \frac{1}{2})\pi$ is the integration constant. In order to obtain a solution also consistent with the space-like pure gauge (2.48) we require A_i^{inst} to be zero at $x_0 = \pm\infty$, so that

$$A^{inst\prime}{}_i = U^{-1}(x)\partial_i U(x) \; , \tag{2.55}$$

where

$$U(x_0 = -\infty) = \exp\{i\pi n \frac{x_i\sigma^i}{(\overrightarrow{x}^2 + \lambda^2)^{\frac{1}{2}}}\} \tag{2.56}$$

and

$$U(x_0 = +\infty) = \exp\{i\pi(n+1) \frac{x_i\sigma^i}{(\overrightarrow{x}^2 + \lambda^2)^{\frac{1}{2}}}\} \; , \tag{2.57}$$

from which we prove that the instaton connects two vacuum states that differ by one unit of winding number. (These solutions give, respectively, the Euclidean finite actions $\frac{8\pi^2 n}{g^2}$ and $\frac{8\pi^2(n+1)}{g^2}$.) It is possible to generalize this result for an instanton with winding number k that connects two states $|m\rangle$ and $|n\rangle$, being $k = n - m$, *i.e.*, the difference between the final and initial winding numbers of the vacuum states (see [43]).

The θ-vacuum. The transition amplitude between two neighboring states (2.45) in the 2D-system suggests that in the semi-classical approximation for the 4D non-Abelian system, the tunnelling amplitude is dominated by

$$T \sim e^{-\frac{8\pi^2}{g^2}} , \tag{2.58}$$

being the exponent the finite energy of the BPST instanton with winding number $n = 1$, which connects the states $|n\rangle$ and $|n+1\rangle$. We can think this problem as a vacuum constructed through a periodic potential [45], in which we can accommodate multiple vacuum states with different winding numbers, separated by finite-energy WKB (Wentzel-Kramers-Brillouin) barriers.

In general, for an arbitrary vacuum state, under a gauge transformation T_1 with winding number $n = 1$, we have

$$T_1|n\rangle = |n+1\rangle , \quad [T_1, H] = 0 , \tag{2.59}$$

where H is Hamiltonian of the system, being the second equation in (2.59) a consequence of the gauge invariance of the system. (This is the transition made by instantons; for (anti-)instantons, the transition is $|n\rangle \to |n-1\rangle$.) The eq. (2.59) reveals a Bloch structure for a periodic potential. In this case, the complete vacuum – the so-called θ-vacuum – is given by the superposition

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle , \tag{2.60}$$

which is an eigenstate of T_1,

$$T_1|\theta\rangle = e^{i\theta}|\theta\rangle \ . \tag{2.61}$$

The amplitude transition between classically distinct θ-vacuums takes the form

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \sum_{\text{all instatons}\,(m,n)} e^{im\theta'}e^{-in\theta}\langle m|e^{-iHt}|n\rangle_J \ , \tag{2.62}$$

where J is the external source. The term $\langle m|e^{-iHt}|n\rangle_J$ denotes a general transition amplitude between states with different winding numbers, $|n\rangle \to |m\rangle$, which are dominated by $|n\rangle \to |n\pm 1\rangle$ transitions. As we discussed before, theses states are connected by instantons with winding number $n-m$, therefore

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \sum_{\text{all instatons}\,(m,n)} e^{im(\theta'-\theta)}e^{-i(n-m)\theta}\int \mathcal{D}A_\mu^{(n-m)}e^{-i\int d^4x\,(\mathcal{L}_{YM}+J_\mu A_\mu)} \ , \tag{2.63}$$

where we just rearranged the exponential terms. Calling $n-m=k$, and summing over m, aftermath we get

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \delta(\theta-\theta')\sum_k\int \mathcal{D}A_\mu^{(k)}e^{-i\int d^4x\,(\mathcal{L}_{YM}+k\theta+J_\mu A_\mu)} \ , \tag{2.64}$$

in which we obtain an effective Lagrangian $\mathcal{L}_{eff} = \mathcal{L} + k\theta$, being k a general winding number. Using the expression of the winding number in the Pontryagin action form, we conclude that the sum over all instantons gives rise to the effective action

$$\mathcal{L}_{eff} = \mathcal{L}_{YM} + \frac{\theta}{16\pi^2}\mathrm{Tr}F_{\mu\nu}\widetilde{F}_{\mu\nu} \ , \tag{2.65}$$

where θ is, in principle, a free parameter associated to the inequivalent vacuum sates described by the superposition (2.60) in the non-Abelian Bloch-like model for a periodic potential[1]. We saw that the θ-vacuum term, despite being a total derivative of a CS current, admits nontrivial topological solutions through pure gauge fields for $S^3 \to S^3$ mappings at infinity. These instanton solutions were the first successfully vacuum theory to explain the $U_A(1)$ *problem* in QCD, related to a particular boson that does not appear in the QCD spectra after a chiral symmetry breaking.

2.2 Solution of the $U_A(1)$ ***problem*** and its relation with strong CP violation

. The description of strong interactions via a non-Abelian gauge theory with $SU(3)$ symmetry (QCD theory) was proposed by Murray Gell-Mann in 1961 [68]. The "Eightfold Way" refers to the eight gauge fields A^a_μ — the gluons — present in the theory, since $a = \{1, \cdots, 8\}$ in the adjoint representation of $SU(3)$ group. The QCD is a theory of interactions between quarks (spin-$\frac{1}{2}$ fermions) and gluons (spin-1 bosons), the elementary particles that make up composite hadrons (such as protons and neutrons). Gluons are the *force carrier* of the theory, like photons in QED that carries the electromagnetic force between electrons and positrons. The great difference is the self-interactions between gluons, which do not occur between photons. The QCD Lagrangian, invariant under local $SU(3)$ gauge transformations, is given by

$$\mathcal{L}_{QCD} = \frac{1}{g^2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu} + \frac{i}{2}\bar{q}^\alpha\gamma_\mu D^{\alpha\beta}_\mu q^A_\alpha - \frac{i}{2}\overline{D^{\alpha\beta}_\mu q^A_\alpha}\gamma_\mu q^A_\alpha - m_A q^A_\alpha q^A_\alpha \ , \quad (2.66)$$

[1]We emphasize that the construction of $\mathcal{L}_{eff}$ was done here in the semi-classical approximation. The non-perturbative analysis, that shows that this construction is consistent at the quantum level, was worked out by G. 't Hooft in [2].

where γ_μ are the Dirac matrices, m_A are the quark masses, and $D_\mu^{\alpha\beta} = \partial_\mu \delta^{\alpha\beta} + A_\mu^a \lambda^{\alpha\beta}$, the covariant derivative in color space, being $\lambda^{\alpha\beta}$ the generators of the Lie algebra of the $SU(3)$ color symmetry. The quantum number A, called *flavor*, refers to six types of quarks q_α^A of the Standard Model[1]. The QCD Lagrangian (2.66), however, does not have a $SU(6)$ flavor symmetry. This reflected in the fact that the quark masses for each flavor type are different[2].

The Lagrangian (2.66) is invariant under certain global transformations. One of them defines the baryonic charge, associated to the transformation $q_\alpha^A(x) \rightarrow {q'}_\alpha^A(x) = e^{-i\theta\mathbb{I}} q_\alpha^A(x)$, where θ is a real parameter and $\mathbb{I}$ is the unit matrix in color and flavor spaces. If we consider now global transformations acting only on flavor space, one can check that $\mathcal{L}_{QCD}$ is invariant under

$$q_\alpha(x) \quad \rightarrow \quad {q'}_\alpha(x) = e^{-i\theta^A T^A} q_\alpha(x)\,, \tag{2.67}$$

$$q_\alpha(x) \quad \rightarrow \quad {q'}_\alpha(x) = e^{-i\theta^A T^A \gamma_5} q_\alpha(x)\,, \tag{2.68}$$

if the quark masses are zero, $m_A = 0$. In eq.'s (2.67) and (2.68), θ^A are real parameters; T^A, with $A = \{1, \cdots, N_1^2 - 1\}$, are the generators of $SU(N_f)$ group in the fundamental representation, and $q_\alpha(x)$, vectors with N_f components. In the absence of quarks masses, it defines the left- and right-handed charges, Q_L^A and Q_R^A, which represents the global symmetry $SU_L(N_f) \times SU_R(N_f)$, where the

[1] The proton, for instance, is composed of two quarks up, and one down.

[2] The quarks which compose the hadrons always appear in Nature as *colorless* bound states. In order to separate two quarks, ever-increasing amounts of energy is required, capable of producing quark-antiquark pairs, but never an isolated color charge. This is the dogma of *color confinement* [69], one of the greatest unsolved problems in Physics since the last century. Another crucial feature of QCD theory is the *asymptotic freedom* of strong interactions, demonstrated by D. Gross and F. Wilczek [70], and independently by D. Politzer [71] both in 1973. (All three shared the 2004 Nobel Prize in Physics for their discovering.) As the energy scale increases, the strength of interactions between quarks and gluons decreases (and vice-versa). For very large energies (or very short distances), the interactions between quarks and gluons will be very weak, and they will behave almost like free particles (a state of matter at extremely high temperature and/or density, called quark-gluon plasma) — hence the name asymptotic freedom.

left and right quark fields are decoupled in the Lagrangian accordingly to their quiralities[1]. In Nature, the chiral symmetry $SU_L(N_f)\times SU_R(N_f)$ is spontaneously broken through

$$Q^A|0\rangle = 0 \;, \quad Q_5^A|0\rangle \neq 0 \;, \tag{2.69}$$

where the vacuum state is not invariant under the axial subgroup, being Q_5 the conserved charge for the transformation (2.68). The Goldstone's theorem states that to each generator of a continuous symmetry that does annihilate the vacuum there is an associated spin-zero massless particle [72; 73]. Eq. (2.69) implies the existence of a $(N_f^2 - 1)$-plet of massless pseudoscalars, and a set with massive multiplets with degenerate masses[2].

In addition to the chiral symmetry, the massless $\mathcal{L}_{QCD}$ possesses another global symmetry — the so-called $U_A(1)$ symmetry, given by the uniparametric transformations

$$q_\alpha(x) \;\rightarrow\; {q'}_\alpha(x) = e^{-i\theta\mathbb{I}\gamma_5} q_\alpha(x) \;, \tag{2.70}$$

where θ is a real constant, and $\mathbb{I}$ is the unit matrix in color and flavor indices. The $U_A(1)$ symmetry is broken by the same mass terms that break the approximate chiral symmetry $SU_L(2)\times SU_R(2)$. The Goldstone's theorem implies that we must observe in QCD spectra a pseudoscalar meson with a mass smaller than $\sqrt{2}m_{\text{pion}}$ [74]. In this case, the natural candidate is the η-particle, but $\frac{m_\eta}{m_{\text{pion}}} \simeq 4$. On the other hand, if we consider an exact $SU_L(3) \times SU_R(3)$ symmetry, η appears as a member of the 0^- octect that contains the pions, but due to the additional $U_A(1)$

[1]For a right-handed quark, $q_R = \frac{1}{2}(1+\gamma_5)q$, the spin points in the same momentum vector direction; for a left-handed one, $q_L = \frac{1}{2}(1-\gamma_5)q$, it points against the momentum.

[2]In the $N_f = 3$ case the O^- octet is the one of Goldtone bosons, and the massive multiplets are $\frac{1}{2}^+$ octet, $\frac{3}{2}^+$ decuplet, etc. The pion mass, however, is small if compared to other hadrons, and this observation is attributed to the existence of an approximate chiral $SU(2)_L \times SU(2)_R$ symmetry. The diagonal part of $SU(2)_L \times SU(2)_R$ group is the isospin group, and its invariance is well realized in Nature, as the masses of the up and down quarks is very small if compared to pertubative QCD mass scale Λ.

symmetry, a singlet meson with mass smaller than $\sqrt{3}m_{\rm pion}$ is missing. Moreover, a Wigner-Weyl realization[1] would imply parity doublets of all massive hadrons, but this was also never observed. This inconsistency is known in literature as the $U_A(1)$ *problem* [2; 26; 74; 75; 76; 77; 78; 79].

The current associated to the $U_A(1)$ symmetry is not conserved at the quantum level, due to an Adler-Bell-Jackiw anomaly [80; 81] present in the Ward identity related to the axial current, which yields

$$\partial_\mu j^5_\mu = \frac{g^2 N_f}{16\pi^2} F_{\mu\nu}\widetilde{F}_{\mu\nu}\ . \tag{2.71}$$

At this point we may think that there is no $U_A(1)$ *problem* to be concerned with, since there is no $U_A(1)$ symmetry to be broken at the quantum level. Nevertheless the right term in (2.71) can be written as a total divergence of the CS current (2.24), then we can define a gauge-variant current $\bar{j}^5_\mu = j_\mu - K_\mu$ such that

$$\partial_\mu \bar{j}^5_\mu = 0\ . \tag{2.72}$$

When we integrate (2.72) over whole space, the charge corresponding to $\bar{j}^5_\mu$, $\bar{Q}_5$, will be conserved in the absence of instantons [82], and the unwanted Goldstone bosons are not eliminated. In the presence of instantons, however, G. 't Hooft has proved in his seminal paper [2] that the Goldstone boson associated to the gauge-variant current $\bar{j}^5_\mu$ does not appear as a Kogut-Susskind zero mass pole [83] of physical gauge-invariant Green functions. In principle, the instanton solved the $U_A(1)$ *problem*[2] [26].

[1] In the Wigner-Weyl realization both charges annihilate the vacuum, $Q^A|0\rangle = 0$, $Q^A_5|0\rangle = 0$.

[2] In one sense, the instanton does not fully explain the QCD spectra, as the mass of the η meson — much heavier than expected — remains a mystery. Some physicist tried to explain the $U_A(1)$ *problem* making use of alternative approaches: Witten explained the problem from the large N_c point of view [78], while Veneziano introduced an additional ghost state, and showed the possibility of computing the mass of the η particle without introducing instantons [79]. This approach, however, has problems concerning BRST invariance (as it breaks the BRST

The strong CP problem. We must point out that the introduction of the θ-vacuum term brought to light another problem: the violation of CP symmetry [86], as the Pontryagin action is odd under parity transformation, and such a violation was never observed in strong interactions. The original QCD Lagrangian (2.66) is CP invariant in accordance with experimental data. A realistic model capable of explaining all the aspects related to the $U_A(1)$ problem and its relation with CP violation is considered an open problem in Physics up to now [34; 87; 88; 89; 90; 91; 92]. An electric dipole moment (d_n) for the neutron is one of the consequence of strong CP violation that could be observed, as $|d_n| \sim 10^{-16}\theta\, e\,\mathrm{cm}$, where e is the electric charge. The experimental upper limit is

$$|d_n| \leq .3 \times 10^{-26}\, e\,\mathrm{cm} \;, \tag{2.73}$$

so that $|\theta| \leq 10^{-9}$, approximately. The small value of θ lies at the heart of the matter — how to give a rationale for such a small value, if θ is a strong interaction parameter? The natural value of θ is expected to be of order one (see for instance [34]). In the early 80's, some physicists believed that θ would be effectively zero as a symmetry requirement, however this argument is not enough since higher-order CP-violating weak interactions generate γ_5-dependent quark mass terms, and to eliminate them one has to apply a chiral symmetry rotation which induces a θ-vacuum term. If at least one of the quarks of the Standard Model were massless, then θ will become unobservable, and to set $\theta = 0$ would be consistent, but this solution has proved to be fragile, since empirical evidence strongly suggests that none of the quarks are massless — see [93].

Recently, at HERA (Hadron-Electron Ring Accelerator at DESY-Deutsches Elektronen-Synchrotron in Hamburg), with the effort of a great collaboration, physicists have been trying to detect a quantum state induced by instantons

symmetry) and, consequently, problems concerning renormalizability [84; 85].

known as *fireball*. An instanton is likely to create a miniature fireball giving rise to quarks and gluons, in addition to a quark jet, in a electron-proton deep inelastic scattering [94] — see Fig. 2.4 below[1].

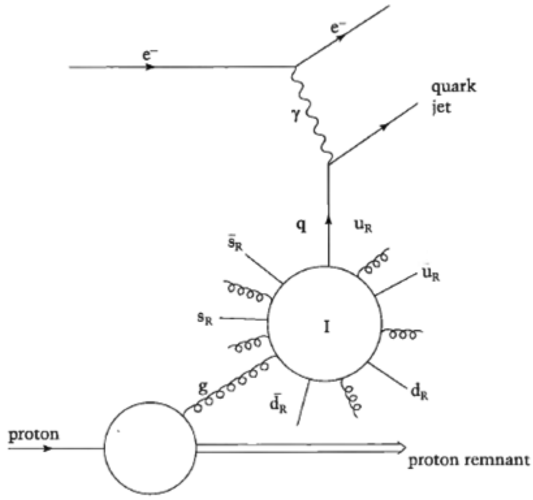

Figure 2.3: Schematic diagram of an instanton contribution to the electron-proton deep inelastic scattering.

No evidence for the production of QCD instanton-induced events was observed up to now. Anyway, the instantons play a crucial role in order to comprehend the QCD spectra, once they explain the absence of pseudoscalars that was never observed in any experiment — pseudoscalars that would have a mass bigger then the pion via chiral symmetry breaking, and we must remember that the chiral symmetry breaking is responsible for producing 95% of the mass in the Universe [95]. The (partial) solution of the $U_A(1)$ *problem* through the introduction of instanton represents an indirect evidence of the presence of non-Abelian topological configurations in Nature. A further investigation on the topological structure of such a configuration, and its quantum properties showed to be necessary in order to shed some light on unsolved problems concerning topological effects in the quantization of non-Abelian field theories.

[1]Extracted from "The Quantum Quark" by A. Watson.

Chapter 3

Topological quantum field theories

Essentially, a topological quantum field theory (TQFT) on a smooth manifold is a quantum field theory which is independent of the metric on the basis manifold. Such a theory has no dynamics, no local degrees of freedom, and is only sensitive to topological/differential invariants that describes the manifold in which the theory is defined. The observables of a TQFT are naturally metric independent. The latter statement leads to the main property of topological field theories, namely, the metric independence of the vacuum expectation values of the observables,

$$\langle \mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\rangle = \int [D\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)e^{-S[\phi]} \ , \quad (3.1)$$

which reads

$$\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\rangle = 0 \ , \quad (3.2)$$

where $g_{\mu\nu}$ is the metric tensor, $\phi_i(x)$ are quantum fields, and $\mathcal{O}_\alpha$, functional operators of the fields that compose the global observables. A typical operator $\mathcal{O}_\alpha$ is integrated over the whole space in order to capture the global structures of the manifold. There are no local particles, no energy available for particle scatterings. The only nontrivial observables are of global nature defined by the cohomology of the target manifold, as all BRST-invariant local operator belong to the trivial part of cohomology, which means that the theory has no local observables [96; 97].

As a particular result of (3.2), the partition function of a topological theory is itself a topological invariant,

$$\frac{\delta}{\delta g_{\mu\nu}} Z[J] = 0 \;, \tag{3.3}$$

insofar as $Z[J]$ represents the expectation value of the vacuum in the presence of a external source, $Z[J] = \langle 0|0\rangle_J$. In literature, if the action is explicitly independent of the metric, the topological theory is said to be of *Schwarz type*[1]; otherwise, if the variation of the action with respect to the metric gives a BRST-exact term, one says the theory is of *Witten type*. More precisely, being δ an infinitesimal transformation that denotes a symmetry of the action S, $\delta S = 0$, if the following properties are satisfied,

$$\delta \mathcal{O}_\alpha(\phi_i) = 0 \,, \quad T_{\mu\nu}(\phi_i) = \delta G_{\mu\nu}(\phi_i) \;, \tag{3.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the model,

$$\frac{\delta}{\delta g_{\mu\nu}} S = T_{\mu\nu} \;, \tag{3.5}$$

and $G_{\mu\nu}$ some tensor, then the quantum field theory can be regarded as topological. Obviously, in this case eq. (3.3) is also satisfied, since the expectation value

[1]The Pontryagin action, which represents the tunnelling between vacuum sates with different winding numbers, is typically a Schwarz type action.

of a "BRST-exact term" vanishes[1] [46; 48]. As we can see, by using (3.5) and (3.4), and assuming that the measure $[D\phi_i]$ is invariant under δ,

$$\begin{aligned}
\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\rangle &= -\int [D\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)T_{\mu\nu}e^{-S} \\
&= \langle \delta[\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_2}(\phi_i)G_{\mu\nu}]\rangle \\
&= 0\ . \qquad (3.7)
\end{aligned}$$

In the equation above we assumed that all $\mathcal{O}_\alpha$ are metric independent. Nevertheless this is not a requirement of the theory, as we can have

$$\delta_{g_{\mu\nu}}\mathcal{O}_\alpha = \delta\mathcal{Q}_{\mu\nu} \neq 0\ , \qquad (3.8)$$

that preserves the topological structure of $\delta_{g_{\mu\nu}}\langle O_{\alpha_1}\cdots\mathcal{O}_{\alpha_p}\rangle = \langle\delta(\cdots)\rangle = 0$ [101]. Analogously to the BRST operator, eq. (3.7) only makes sense if the δ operator is nilpotent[2].

From the physical point of view, topological quantum field theories provide mathematical tools capable of revealing the topological structure of field theories that are independent of the metric, and of the background choice, together with the set of symmetries behind these properties. One of the major obstacles to construct a quantum theory of gravity is the integration over all metrics. An introduction of a topological phase in gravity would have the power to make a theory of gravity arises, after a spontaneous breaking of general covariance, without having to integrate over the space of all metrics [40; 46]. We must say that

[1]Broadly speaking, the vacuum expectation values are invariant under a BRST transformation, so that

$$s\langle(\cdots)\rangle = \langle s(\cdots)\rangle - \langle(\cdots)sS\rangle = 0\ , \qquad (3.6)$$

being $(\cdots)$ an arbitrary operator, s the BRST operator, and S the action. As $sS = 0$, the equation above yields $\langle s(\cdots)\rangle = 0$. For a further analysis of its renormalization properties, and definition of physical observables, see for instance [98; 99; 100].

[2]In the Witten theory, for instance, such an operator is on-shell nilpotent, *i.e.*, $\delta^2 = 0$ by using the equations of motion.

the introduction of such a topological phase is one of the intricate problems in topological quantum theories, since one must develop a mechanism for spontaneously breaking the topological symmetry, but, by construction, these theories have no dynamics. A realistic mechanism for a symmetry breaking in topological field theory is still a challenge. On the other hand, we can also investigate conformal properties of field theories via topological models, based on the connection between three-dimensional Chern-Simons theory and two-dimensional conformal theories [102]. In the Mathematics/Physics frontier, TQFT's are intimately connected with the AdS/CFT correspondence [103; 104].

In practice, TQFT's have the power to reproduce topological invariants of the basis manifold. The first one to obtain topological invariants from a quantum field theory was A. S. Schwarz in 1978 [105]. He showed that the Ray-Singer analytic torsion [106] can be represented as a partition function of the Abelian Chern-Simons action, which is invariant by diffeomorphisms. The Schwarz topological theory was the prototype of Witten theories in the 1980's. Indeed the well-known Witten paper in which he reproduces the Jones polynomial of knot theory [102] is the non-Abelian generalization of [105]. In his work Witten was actually able to represent invariants of three-manifolds as the partition function of the non-Abelian CS theory. Besides the knot invariants, the computation of the expectation value of the Wilson loop in $U(1)$ CS gauge theory, in a loop C,

$$W_C = e^{i \oint_C A_\mu(x) dx^\mu}, \tag{3.9}$$

which is a gauge invariant observable in this case, gives the Gauss's linking integral representation of the *linking number*. In Mathematics, the linking number is a topological invariant in three-dimensional space, which represents the number of times that each curve winds around the other — see Figure 3.1 below. As we can easily see, the linking number is invariant under continuous deformations in the curves.

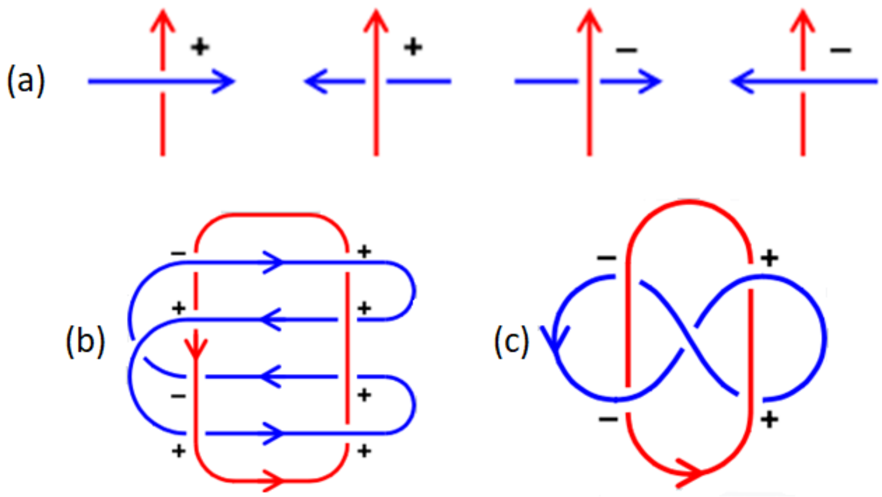

Figure 3.1: The algorithm to compute the linking number consists of labeling each crossing as positive or negative, according to the rule in figure (a); the total number of positive crossings minus the total number of negative crossings is equal to twice the linking number. Examples: (b) two curves that have linking number two; (c) the *Whitehead link* with linking number zero.

Explicitly, the partition function of Abelian CS theory in $\mathbb{R}^3$ is

$$Z[J] = \int A_\mu \exp\left(\frac{i}{4\pi}\int d^3x\,\varepsilon^{\lambda\nu\mu}A_\lambda\partial_\nu A_\mu + i\int d^3x\,J_\mu A_\mu\right)\,, \tag{3.10}$$

where $\varepsilon^{\lambda\nu\mu}$ is the three-dimensional Levi-Civita antisymmetric tensor, and A_μ, the gauge field. The theory is pure Gaussian, as the CS action is quadratic in the fields, and so exact soluble at one-loop order. For a source J_μ describing a point particle moving in a loop C_1,

$$J_\mu(x) = \oint_{C_1} dx_i^\mu\,\delta^3(x - x_i(t))\,, \quad \text{and} \quad \int d^3x\,J_\mu A_\mu = \oint_{C_1} A_\mu dx^\mu\,. \tag{3.11}$$

Therefore, computing the expectation value of the Wilson loop (3.9) in a loop C_2,

$$Z(C_1, C_2) = \langle W_{C_2}\rangle_{C_1}\,, \tag{3.12}$$

we get exactly

$$Z(C_1, C_2) = \exp\left[2\pi i \zeta(C_1, C_2)\right] \ , \tag{3.13}$$

where

$$\zeta(C_1, C_2) = \oint_{C_1} dx^\lambda \oint_{C_2} dy^\mu \frac{(x-y)^\nu}{|x-y|^3} \varepsilon_{\lambda\mu\nu} \tag{3.14}$$

is the expression of Gauss's linking integral, that counts the linking number between two non-intersecting differentiable curves, C_1 and C_2, in $\mathbb{R}^3$. This is the simplest case in which we can represent a topological invariant by using the Feynman path integral of a quantum field theory. In $\mathbb{R}^3$ we can visualize its topological invariants. In four dimensions the topological/differential invariants cannot be visualized, and are defined with the effort of differential geometry. Topological quantum field theories that obey eq. (3.2) have the power of reproducing these differential invariants in higher dimensions, as the Witten's TQFT that describes the Donaldson invariants in $\mathbb{R}^4$ through a metric independent partition function of a relativistic non-Abelian action, namely, the *twisted* version of the $N = 2$ super Yang-Mills (SYM) action.

3.1 Witten's topological quantum field theory

Throughout the 1980s, based on the self-dual Yang-Mills equations introduced by A. Belavin, A. Polyakov, A. Schartz, and Y. Tyupkin in their study of instantons [1], S. K. Donaldson discovered and described topological structures of polynomial invariants for smooth four-manifolds [15; 16; 17]. The connection between the Floer theory for three-manifolds [107; 108] and Donaldson invariants for four-manifolds with a non-empty boundary, *i.e.*, that assumes values in Floer groups, has led to the Atiyah's conjecture, in which he proposed that the Floer homology must lead to a relativistic quantum field theory. This conjecture was the motivation for the Witten theory in four dimensions, as Witten himself admits [46]. Answering Atiyah's conjecture, Witten found a relativistic formulation of [109], capable of reproducing the Donaldson polynomials in the the weak coupling limit.

3.1.1 The *twist* transformation: A mapping between $N = 2$ super and topological Yang-Mills theories

The eight supersymmetric charges $(Q^i_\alpha,\, \bar{Q}_{j\dot{\alpha}})$ of $N = 2$ SYM theories obey the susy algebra

$$\begin{aligned} \{Q^i_\alpha,\, \bar{Q}_{j\dot{\alpha}}\} &= \delta^i_j (\sigma_\mu)_{\alpha\dot{\alpha}} \partial_\mu \;, \\ \{Q^i_\alpha,\, Q_{j\alpha}\} &= \{\bar{Q}^i_{\dot{\alpha}},\, \bar{Q}_{j\dot{\alpha}}\} = 0 \;, \end{aligned} \tag{3.15}$$

where the indices $(i, \alpha, \dot{\alpha})$ both run from one to two. The index $i = \{1, 2\}$ denotes the internal $SU(2)$ symmetry accordingly to the susy algebra above, and $(\alpha, \dot{\alpha}) = \{1, 2\}$ are Weyl spinor indices: α denotes right-handed spinors, and $\dot{\alpha}$, left-handed ones. The fact that both indices equally run form one to two suggest the identification between spinor and supersymmetry indices,

$$i \equiv \alpha \;. \tag{3.16}$$

The $N = 2$ SYM action theory possesses a gauge group symmetry given by

$$SU_L(2) \times SU_R(2) \times SU_I(2) \times U_R(1) \;, \tag{3.17}$$

where $SU_L(2) \times SU_R(2)$ is the rotation group, $SU_I(2)$ is the internal supersymmetry group labeled by i, and $U_R(1)$, the so-called $\mathcal{R}$-symmetry defined by the supercharges $(Q^i_\alpha,\, \bar{Q}_{j\dot{\alpha}})$ which are assigned eigenvalues $(+1,\, -1)$, respectively. The identification performed in eq. (3.16) amounts to a modification of the rotation group,

$$SU_L(2) \times SU_R(2) \to SU_L(2) \times SU_R(2)' \ , \tag{3.18}$$

where $SU_R(2)'$ is the diagonal sum of $SU_R(2)$ and $SU_I(2)$. The *twisted* global symmetry of $N = 2$ SYM takes the form $SU_L(2) \times SU_R(2)' \times U_R(1)$, with the corresponding *twisted* supercharges

$$Q^i_\alpha \to Q^\beta_\alpha \, , \quad \bar{Q}_{i\bar{\alpha}} \to \bar{Q}_{\alpha\dot{\alpha}} \ , \tag{3.19}$$

which can be rearranged as

$$\frac{1}{\sqrt{2}}\epsilon^{\alpha\beta}Q_{\alpha\beta} \ \equiv \ \delta \ , \tag{3.20}$$

$$\frac{1}{\sqrt{2}}\bar{Q}^{\alpha\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}\alpha} \ \equiv \ \delta_\mu \ , \tag{3.21}$$

$$\frac{1}{\sqrt{2}}(\sigma_{\mu\nu})^{\dot{\alpha}\alpha}Q_{\dot{\alpha}\alpha} \ \equiv \ \delta_{\mu\nu} \ , \tag{3.22}$$

where we adopt the conventions for $\epsilon^{\alpha\beta}$, $(\sigma^\mu)^{\alpha\dot{\alpha}}$ and $(\sigma_{\mu\nu})^{\dot{\alpha}\alpha}$ as the same of [110]. The operator $\delta_{\mu\nu}$ is manifestly self-dual due to the structure of $\sigma_{\mu\nu}$,

$$\delta_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\delta^{\lambda\rho} \ , \tag{3.23}$$

reducing to three the number of its independent components. The operators δ, δ_μ and $\delta_{\mu\nu}$ possess eight independent components in which the eight original supercharges $(Q_{\beta\alpha}, \bar{Q}_{\alpha\dot{\alpha}})$ are mapped into. These operators obey the following *twisted* supersymmetry algebra

$$\delta^2 = 0 \,, \tag{3.24}$$

$$\{\delta, \delta_\mu\} = \partial_\mu \,, \tag{3.25}$$

$$\{\delta_\mu, \delta_\nu\} = \{\delta_{\mu\nu}, \delta\} = \{\delta_{\mu\nu}, \delta_{\lambda\rho}\} = 0 \tag{3.26}$$

$$\{\delta_\mu, \delta_{\lambda\rho}\} = -(\varepsilon_{\mu\lambda\rho\sigma}\partial^\sigma + g_{\mu\lambda}\partial_\rho - g_{\mu\rho}\partial_\lambda) \,. \tag{3.27}$$

The nilpotent scalar supersymmetry charge δ defines the cohomology of Witten's TQFT, as its observables appear as cohomology classes of δ, which is invariant under a generic differential manifold. It is implicit in the anti-commutation relation (3.25) the topological nature of the model, as it allows to write the common derivative as a δ-exact term.

The gauge multiplet of the $N = 2$ SYM in Wess-Zumino gauge is given by the fields

$$(A_\mu, \psi^i_\alpha, \bar{\psi}^i_{\dot{\alpha}}, \phi, \bar{\phi}) \,, \tag{3.28}$$

where ψ^i_α is a Majorana spinor (the supersymmetric partner of the gauge connection A_μ), and ϕ, a scalar field, all of them belonging to the adjoint representation of the gauge group. The *twist transformation* is defined by the identification eq. (3.16), and thus only acts on the fields $(\psi^i_\mu, \bar{\psi}^i_\mu)$, leaving the bosonic fields $(A_\mu, \phi, \bar{\phi})$ unaltered. Expressly, the *twist transformation* is given by the linear transformations

$$\psi^i_\beta \;\rightarrow\; \psi_{\alpha\beta} = \frac{1}{2}\left(\psi_{(\alpha\beta)} + \psi_{[\alpha\beta]}\right) \,, \tag{3.29}$$

$$\bar{\psi}^i_{\dot{\alpha}} \;\rightarrow\; \bar{\psi}_{\alpha\dot{\alpha}} \;\rightarrow\; \psi_\mu = (\sigma_\mu)^{\alpha\dot{\alpha}}\bar{\psi}_{\alpha\dot{\alpha}} \,, \tag{3.30}$$

together with

$$\psi_{(\alpha\beta)} \quad \to \quad \chi_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)} \ , \tag{3.31}$$

$$\psi_{[\alpha\beta]} \quad \to \quad \eta = \varepsilon^{\alpha\beta}\psi_{[\alpha\beta]} \ . \tag{3.32}$$

The *twist* consists of a mapping of degrees of freedom. The field $\bar{\psi}_{\alpha\dot{\alpha}}$ has four independent components as $(\alpha, \dot{\alpha}) = \{1, 2\}$, and is mapped into the field ψ_μ that also has four independent components, as the Lorentz index $\mu = \{1, 2, 3, 4\}$ in four dimensions. In the other mappings occurs the same, as the symmetric part of $\psi_{\alpha\beta}$, *i.e.*, $\psi_{(\alpha\beta)}$ has three independent components mapped into the self-dual field $\chi_{\mu\nu}$, and the antisymmetric part, $\psi_{[\alpha\beta]}$, with only one independent component, into η, a scalar field. We must note that $(\psi_\mu, \chi_{\mu\nu}, \eta)$ anticommute due to their spinor origin.

Because it is a linear transformation, the *twist* simply corresponds to a change of variables with trivial Jacobian that could be absorbed in the normalization factor, in other words, both theories (before and after the twist) are perturbatively indistinguishable. Finally, twisting the $N = 2$ SYM action ($S^{N=2}_{SYM}$), in Euclidean space, we obtain the Witten four-dimensional topological Yang-Mills action (S_W),

$$S^{N=2}_{SYM}[A_\mu, \psi^i_\alpha, \bar{\psi}^i_{\dot{\alpha}}, \phi, \bar{\phi}] \ \to \ S_W[A_\mu, \psi_\mu, \chi_{\mu\nu}, \bar{\phi}, \phi] \ , \tag{3.33}$$

where

$$\begin{aligned} S_W \ = \ & \frac{1}{g^2}\mathrm{Tr}\int d^4x \left(\frac{1}{2}F^+_{\mu\nu}F^{+\mu\nu} - \chi_{\mu\nu}\left(D_\mu\psi_\nu - D_\nu\psi_\mu\right)^+ + \eta D_\mu\psi^\mu \right. \\ - \ & \frac{1}{2}\bar{\phi}D_\mu D^\mu\phi + \frac{1}{2}\bar{\phi}\{\psi_\mu, \psi_\mu\} - \frac{1}{2}\phi\{\chi_{\mu\nu}, \chi_{\mu\nu}\} - \frac{1}{8}\left[\phi, \eta\right]\eta \\ - \ & \left. \frac{1}{32}\left[\phi, \bar{\phi}\right]\left[\phi, \bar{\phi}\right]\right) \ , \end{aligned} \tag{3.34}$$

wherein $F^+_{\mu\nu}$ is the self-dual field

$$F^+_{\mu\nu} = F_{\mu\nu} + \widetilde{F}_{\mu\nu}\,, \quad (\widetilde{F}^+_{\mu\nu} = F^+_{\mu\nu})\,, \tag{3.35}$$

and, analogously,

$$(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ = D_\mu\psi_\nu - D_\nu\psi_\mu + \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\left(D_\alpha\psi_\beta - D_\beta\psi_\alpha\right)\,. \tag{3.36}$$

The Witten action[1] (3.34) possesses an usual Yang-Mills gauge invariance, see eq. (2.6), denoted by

$$\delta^{\rm YM}_{\rm gauge} S_W = 0\,. \tag{3.37}$$

The theory, however, does not possess gauge anomalies [112]. The symmetry that defines the cohomology of the theory, also known as *equivariant cohomology*, is the fermionic scalar supersymmetry which, acting on the fields, has the form:

$$\begin{aligned} \delta A_\mu &= -\varepsilon\psi_\mu\,, \quad \delta\phi = 0\,, \quad \delta\lambda = 2i\varepsilon\eta\,, \quad \delta\eta = \frac{1}{2}\varepsilon[\phi,\bar{\phi}]\,, \\ \delta\psi_\mu &= -\varepsilon D_\mu\phi\,, \quad \delta\chi_{\mu\nu} = \varepsilon F^+\,, \end{aligned} \tag{3.38}$$

where ε is the supersymmetry fermionic parameter that carries no spin, ensuring that the propagating modes of commuting and anticommuting fields have the

[1]Technically, the Witten action (3.34) is the four-dimensional generalization of the non-relativistic topological quantum field theory [109], whose construction is based on the Floer theory for three-manifolds $\mathcal{M}_{3D}$, in which the Chern-Simons action is taken as a Morse function on $\mathcal{M}_{3D}$, see Floer's original paper [108]. In few words, the critical points of CS action (W_{CS}) yield the curvature free configurations, as $\frac{\delta W_{CS}}{\delta A^a_i} = -\frac{1}{2}\varepsilon^{ijk}F^{jk}$, where F^{jk} is the 2-form curvature in three dimensions, which defines the gradient flows of a Morse function, see [40]. In the supersymmetric formulation of [109], the Hamiltonian (H) is obtained via the "supersymmetric charges" d_t and d^*_t, from the well-known relation $d_t d^*_t + d^*_t d_t = 2H$, see [111], whereby $d_t = e^{-tW_{CS}} d e^{tW_{CS}}$ and $d^*_t = e^{tW_{CS}} d^* e^{-tW_{CS}}$, for a real number t, being d the exterior derivative on the space of all connections $\mathcal{A}$, according to the transformation $\delta A^a_i = \psi^a_i$, and d^* its dual. Before identifying the *twist* transformation, this formulation (in four-dimensions) was employed by Witten in his original paper [102] to obtain the relativistic topological action (3.34).

same helicities[1]. This symmetry relates bosonic and fermionic degrees of freedom, which are identical — an inheritance of the supersymmetric original action[2]. The price of working in Wess-Zumino gauge is the fact that, disregarding gauge transformations, one needs to use the equations of motion to recover the nilpotency of δ [97]. One can easily verify that (see [46])

$$\delta^2\Phi = 0 \ , \quad \text{for} \quad \Phi = \{A, \psi, \phi, \bar{\phi}, \eta\} \ , \tag{3.39}$$

but

$$\delta^2\chi = \text{equations of motion} \ . \tag{3.40}$$

Considering the result of eq. (3.40), hereafter we will say that the Witten fermionic symmetry is *on-shell* nilpotent. This symmetry is associated to an on-shell nilpotent "BRST charge", $\mathcal{Q}$, according to the definition of the variation $\delta\mathcal{O}$ of any functional $\mathcal{O}$ under the fermionic symmetry eq. (3.38) as a linear transformation on the space of all functionals of field variables, namely,

$$\delta\mathcal{O} = -i\varepsilon \cdot \{\mathcal{Q}, \mathcal{O}\} \ , \quad \text{such that} \quad \mathcal{Q}^2|_{\textit{on-shell}} = 0 \ . \tag{3.41}$$

In order to verify that Witten theory is valid in curved spacetimes, it is worth noting that the commutators of covariant derivatives always appears acting in the scalar field ϕ, like in $\delta Tr\{D_\mu\psi_\nu \cdot \bar{\chi}_{\mu\nu}\} = \frac{1}{2}Tr([D_\mu, D_\nu]\phi \cdot \bar{\chi}^{\mu\nu})$, so the Riemann tensor does not appear, and the theory could be extended to any Riemannian

[1]Precisely, the propagating modes of A_μ have helicities $(1,-1)$, and of $(\phi, \bar{\phi})$, $(0,0)$; while of the fermionic fields (η, ψ, χ), helicities $(1,-1,0,0)$.

[2]The action S_W is also invariant under global scaling with dimensions $(1,0,2,2,1,2)$ for $(A, \phi, \bar{\phi}, \eta, \psi, \chi)$, respectively; and preserves an additive U symmetry for the assignments $(0,2,-2,-1,1,-1)$. In the BRST formalism, the latter is naturally recognized as ghost numbers, as we will see in Section 4.

manifold. In practice one can take

$$\int d^4x \to \int d^4x\sqrt{g} \ , \tag{3.42}$$

if one wants to work in a curved spacetime. Such a theory has the property of being invariant under infinitesimal changes in the metric. This property characterizes the Witten model as a topological quantum field theory. Such a property is verified by the fact that the energy-momentum tensor can be written as the anti-commutator

$$T_{\mu\nu} = \{\mathcal{Q}, V_{\mu\nu}\} \ , \tag{3.43}$$

which means that $T_{\mu\nu}$ is an *on-shell* BRST-exact term,

$$T_{\mu\nu} = \delta V_{\mu\nu} \ , \quad \delta^2|_{on\text{-}shell} = 0 \ , \tag{3.44}$$

with

$$\begin{aligned} V_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\{F_{\mu\sigma}\chi_\nu{}^\sigma + F_{\nu\sigma}\chi_\mu{}^\sigma - \frac{1}{2}g_{\mu\nu}F_{\sigma\rho}\chi^{\sigma\rho}\} + \frac{1}{4}g_{\mu\nu}\mathrm{Tr}\eta[\phi,\bar{\phi}] \\ &+ \frac{1}{2}\mathrm{Tr}\{\psi_\mu D^\nu\bar{\phi} + \psi_\nu D^\mu\bar{\phi} - g_{\mu\nu}\psi_\sigma D^\sigma\bar{\phi}\} \ . \end{aligned} \tag{3.45}$$

Equation (3.44) together with $\delta S_W = 0$ means that Witten theory satisfies (on-shell) the second condition displayed in eq. (3.4), that allows to say that S_W automatically leads to a four-dimensional topological field model, in other words,

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}}Z_W &= \int \mathcal{D}\Phi(-\frac{\delta}{\delta g_{\mu\nu}}\mathcal{S}_W)\exp(-\mathcal{S}_W) \\ &= -\frac{1}{g^2}\langle\{\mathcal{Q}, \int_M d^4x\sqrt{g}V_{\mu\nu}\}\rangle = 0 \ , \end{aligned} \tag{3.46}$$

as all expected value of a BRST-exact term vanish. It remains to know which kind of topological/differential invariants can be represented by the Feynman path integral of Witten's TQFT. As it is well-known, it will naturally reproduce the Donaldson invariants for four-manifolds.

3.1.2 Donaldson polynomials

An important feature of Witten's TQFT is the fact that the theory can be interpreted as quantum fluctuations around classical instanton configurations. To find the nontrivial classical minima one must note that the gauge field terms in S_W are

$$S_W^{gauge}[A] = \frac{1}{2}\mathrm{Tr}\int d^4x(F_{\mu\nu} + \widetilde{F}_{\mu\nu})(F^{\mu\nu} + \widetilde{F}^{\mu\nu}) \ , \tag{3.47}$$

which is positive semidefinite, and only vanishes if the field strenght $F_{\mu\nu}$ is anti-self-dual,

$$F_{\mu\nu} = -\widetilde{F}_{\mu\nu} \ , \tag{3.48}$$

the same nontrivial vacuum configuration that minimizes the Yang-Mills action in the case of anti-instantons fields, see (2.32). We conclude that Witten action has a nontrivial classical minima for $F = -\widetilde{F}$ and $\Phi_{\text{other fields}} = 0$. Being precise, the evaluation of the Witten's TQFT path integral computes quantum corrections to classical anti-instantons solutions.

Another important property of Witten theory is the invariance under infinitesimal changes in the coupling constant. The variation of Z_W with respect to g^2 yields, for similar reasons,

$$\delta_{g^2} Z_W = \delta_{g^2}(-\frac{1}{g^2})\langle\{Q, X\}\rangle = 0 \ , \tag{3.49}$$

where

$$X = \frac{1}{4}\mathrm{Tr}F_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\mathrm{Tr}\psi_\mu D^\mu\bar{\phi} - \frac{1}{4}\mathrm{Tr}\eta[\phi, \bar{\phi}] \ . \tag{3.50}$$

The Witten partition function, Z_W, is independent of the gauge coupling g^2, therefore we can evaluate Z_W in the weak coupling limit, *i.e.*, in the regime of very small g^2, in which Z_W is dominated by the classical minima.

Instanton moduli space. The *instanton moduli space*, $\mathcal{M}_{k,N}$, is defined to be the space of all solutions to $F = \widetilde{F}$ for a giving winding number k and gauge group $SU(N)$. By perturbing $F = \widetilde{F}$ nearby the solution A_μ via a gauge transformation $A_\mu \to A_\mu + \delta A_\mu$, we obtain the self-duality equation

$$D_\mu \delta A_\nu + D_\mu \delta A_\nu + \varepsilon_{\mu\nu\alpha\beta} D^\alpha \delta A^\beta = 0 \ . \tag{3.51}$$

Solutions to equation above are called zero modes. Requiring the orthogonal gauge fixing condition, $D_\mu A^\mu = 0$, one gets

$$D_\mu (\delta A_\mu) = 0 \ . \tag{3.52}$$

The Atiyah-Singer index theorem [113; 114] counts the number of solutions to eq. (3.51) and eq. (3.52). In Euclidean spacetimes, for instance, the index theorem gives

$$\dim(\mathcal{M}) \equiv \mathcal{M}_{k,N} = 4kN \ , \tag{3.53}$$

where the modes due to global gauge transformations of the group were included. Looking at fermion zero modes, the χ equation for S_W gives

$$D_\mu \psi_\nu + D_\nu \psi_\mu + \varepsilon_{\mu\nu\alpha\beta} D^\alpha \psi^\beta = 0 \ , \tag{3.54}$$

and from the η equation,

$$D_\mu \psi^\mu = 0 \ . \tag{3.55}$$

These are the same equations obtained for the gauge perturbation around an instanton in the orthogonal gauge fixing, so the number of ψ zero modes is also given by $\mathcal{M}_{k,N}$[1]. In order to get a non-vanishing partition function, Witten assumed that the moduli space consists of discrete, isolated instantons, in other words, that the dimension of the moduli space vanishes[2].

In expanding around an isolated instanton, in the weak coupling limit $g^2 \to 0$, the action is reduced to quadratic terms,

$$S_W^{(2)} = \int_M d^4x\sqrt{g}\left(\Phi^{(b)} D_B \Phi^{(b)} + i\Psi^{(f)} D_F \Psi^{(f)}\right) \ , \tag{3.57}$$

where $\Phi^{(b)} \equiv \{A, \phi, \bar{\phi}\}$ are the bosonic fields, and $\Psi^{(f)} \equiv \{\eta, \psi, \chi\}$, the fermionic ones. The Gaussian integral over D_B and D_F gives

$$Z_W|_{g^2\to 0} = \frac{\text{Pfaff}(D_F)}{\sqrt{\det(D_B)}} \ , \tag{3.58}$$

where Pfaff(D_F) is the Pfaffian of D_F, *i.e.*, the square root of the determinant of D_F up to a sign. The supersymmetry relates the eigenvalues of the operators D_B and D_F. The relation is a standard result in instanton calculus [115], which yields

$$Z_W|_{g^2\to 0} = \pm\prod_i \frac{\lambda_i}{\sqrt{|\lambda_i|}^2} \ , \tag{3.59}$$

with i running over all non-zero eigenvalues of D_B (D_F). Therefore, for the k^{th}

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[1] As Witten himself admits in his paper [46], "this relation between the fermion equations and the instanton moduli problem was the motivation for introducing precisely this collection of fermions".

[2] Otherwise, it occurs a net violation of the $U(1)$ global symmetry of S_W, and Z_W vanishes due to the fermion zero modes, see [2; 46]. The dimension of the intanton moduli spaces depends on the bundle, E, and the manifold, M. In the $SU(2)$ gauge theory, it can be written as

$$\dim(\mathcal{M}) = 8k(E) - \frac{3}{2}(\chi(M) + \sigma(M)) \ , \tag{3.56}$$

where $k(E)$ is the first Pontryagin (or winding) number of the bundle E, and $\chi(M)$ and $\sigma(M)$ are the Euler characteristic and signature of M [114]. (For $M = R^4$, $\chi(R^4) = \sigma(R^4) = 0$.) Thus one can choose a suitable E and M in order to get a vanishing dimension, $\dim(\mathcal{M}) = 0$.

isolated instanton, $Z_W^{(k)} = (-1)^{n_k}$, where $n_k = 0$ or 1 according to the orientation convention of the moduli space (Donaldson proved the orientability of the moduli space, *i.e.*, that the definition of the sign of Pfaff(D_F) is consistent, without global anomalies [16; 46]). In the end, summing over all isolated instantons,

$$Z_W|_{g^2 \to 0} = \sum_k (-1)^{n_k} \;, \tag{3.60}$$

which is precisely one of topological invariant for four-manifolds described by Donaldson.

The other metric independent observables are constructed in the context of eq. (3.8), in which they should appear as BRST-exac terms. These observables can be generated by exploring the descent equations defined by the equivariant cohomology, *i.e.*, the supersymmetry δ-cohomology. For that, being U_i the global charge of the operator $\mathcal{O}_i$ (see footnote on page 44), it must be understood that, for the observable $\prod_i O_i$, $\dim(\mathcal{M}) = \sum_i U_i$[1]. The simplest BRST invariant operator, that does not depend explicitly on the metric, and cannot be written as $\delta(X) = \{\mathcal{Q}, X\}$ (due to the scaling dimensions) is

$$W_0(x) = \frac{1}{2}\mathrm{Tr}\phi^2(x) \,, \quad U(W_0) = 4 \;. \tag{3.61}$$

Although W_0 is not a BRST-exact operator, taking the derivative of W_0 with respect of the coordinates, we find

$$\frac{\partial}{\partial x_\mu} W_0 = i\{\mathcal{Q}, \mathrm{Tr}\phi\psi_\mu\} \;, \tag{3.62}$$

[1] In order to construct topological invariants, the net U charge must equal the dimension of the moduli space, see [40; 46].

which is BRST exact. Using the exterior derivative[1], d, we can rewrite (3.62) as

$$dW_0 = i\{\mathcal{Q}, W_1\} \ , \tag{3.63}$$

where $W_!$ is the 1-form $\mathrm{Tr}(\phi\psi_\mu)dx^\mu$. A straightforward calculation gives

$$\begin{aligned} dW_1 &= i\{\mathcal{Q}, W_2\} \ , \quad dW_2 = i\{\mathcal{Q}, W_3\} \ , & (3.64)\\ dW_3 &= i\{\mathcal{Q}, W_4\} \ , \quad dW_4 = 0 \ , & (3.65)\end{aligned}$$

with

$$\begin{aligned} W_2 &= \mathrm{Tr}(\frac{1}{2}\psi \wedge \psi + i\phi \wedge F) \ , & (3.66)\\ W_3 &= i\mathrm{Tr}\psi \wedge F \ , & (3.67)\\ W_4 &= -\frac{1}{2}\mathrm{Tr}F \wedge F \ , & (3.68)\end{aligned}$$

where "$\wedge$" is the wedge product, the total charge is $U = 4 - k$ for each W_k, and ϕ, ψ, and F are zero, one, and two forms on M, respectively. F is the field strenth in the -p-form formalism, $F = dA + AA$, where A is the 1-form $A_\mu dx^\mu$. Considering now the integral

$$I(\gamma) = \int_\gamma W_k \ , \tag{3.69}$$

being γ a k-dimensional homology cycle on M, we have

$$\{\mathcal{Q}, I\} = \int_\gamma \{\mathcal{Q}, W_k\} = i\int_\gamma dW_{k+1} = 0 \ . \tag{3.70}$$

It proves that $I(\gamma)$ is BRST invariant and, then, a possible observable. To be

[1]See Section 4.1.1 in Chapter 4 for the definitions of the geometric elements concerning the p-form formalism.

a global observable of the topological theory, we just have to prove that $I(\gamma)$ is BRST exact, which can be immediately verified taking γ as the boundary $\partial\beta$, and applying the Stokes theorem,

$$I(\gamma) = \int_{\partial\beta} W_k = \int_\beta dW_k = i\{\mathcal{Q}, \int_\beta W_{k+1}\} \ . \tag{3.71}$$

We conclude, from equations (3.70) and (3.71), that $I(\gamma)$ are the global observables of the model as their expectation values produce metric independent quantities, *i.e.*, topological invariants for four-manifolds. Finally, the general path integral representation of Donaldson invariants in Witten's TQFT takes the form

$$Z(\gamma_1, \cdots, \gamma_r) = \int \mathcal{D}\Phi \left(\prod_i \int_{\gamma_i} W_{k_i} \right) e^{-S_W} = \langle \prod_i \int_{\gamma_i} W_{k_i} \rangle \ , \tag{3.72}$$

with moduli space dimension

$$\dim(\mathcal{M}) = \sum_i^r (4 - k_r) \ . \tag{3.73}$$

One of the beautiful results is the appearing of W_4 in the descent equations. Up to a sign, the observable

$$\int_\gamma W_4 = -\frac{1}{2} \int_\gamma F \wedge F \tag{3.74}$$

is the Pontryagin action written in the formalism of p-forms. The Pontryagin action, a well-known topological invariant of four-manifolds, naturally appear as one of the Donaldson polynomials — with a trivial winding number in this case, since $U(W_4) = 0$, and consequently the dimension of the moduli space vanishes.

3.2 Perturbative β-function of $N = 2$ super Yang-Mills via *twist*

We would like to present some quantum properties of Witten's TQFT that are well known in literature. This will serve as a basis for comparison between Witten *on-shell* model and Baulieu-Singer *off-shell* approach[1] [48], which may provide a broader understanding of the quantum behavior of topological Yang-Mills theories, according to the particularities of each theory.

The authors in [47] employed the algebraic renormalization techniques, which give results valid to all orders in perturbation theory, to study the twisted $N = 2$ SYM, and to prove that the β-function of Witten's TQFT (β_g) is one-loop exact, as a consequence of the non-renormalization of the composite operator $\mathrm{Tr}\phi^2(x)$ [116]. To this aim they considered the fact that the operator $\delta_{\mu\nu}$ (3.22) is redundant to define the theory [117], and provide the quantum extension through the definition of an extended BRST operator, namely,

$$\mathcal{S} = s_{YM} + \omega\delta + \varepsilon_\mu\delta_\mu \ , \tag{3.75}$$

where s_{YM} is the usual Yang-Mills BRST operator, ω and ε_μ are global ghosts, and δ and δ_μ are defined in equations (3.20) and (3.21). The relevant property of the operator $\mathcal{S}$ is that it is on-shell nilpotent in the space of integrated local functionals, since

$$\mathcal{S}^2 = \omega\varepsilon_\mu\partial_\mu + \text{eqs of motion} \ . \tag{3.76}$$

Such a property allows for a standard application of algebraic BRST techniques.

[1] Throughout the thesis we will denote the theories as *on-shell* or *off-shell* according to their BRST charges: *on-shell* for theories in which the BRST charge is only nilpotent through the use of equations motion, and *off-shell*, for the ones in which the equations of motion are not needed to prove its nilpotency.

(We would like to point out here that such a BRST construction requires the equations of motion to obtain a nilpotent BRST operator — a standard behavior of the BRST quantization of Witten theory.) Considering the non-renormalization of $\mathrm{Tr}\phi^2$ and eq. (3.76), the result is that the β-function only receives contributions to one-loop order, and is given by

$$\beta_g = -Kg^3 \ , \quad (K \equiv \text{constant}) \ , \tag{3.77}$$

differently of the $N = 4$ SYM, which possesses a vanishing β-function. The $N = 2$ β_g is one-loop exact, as all higher order loop corrections vanish. The computation of β_g via Feynman diagrams was performed in [49] by evaluating the one-loop contributions to the gauge field propagator (where the Landau gauge was used to fix the Yang-Mills symmetry of Witten action (3.37)). The behavior of one-loop exactness of the β-function had been usually understood in terms of the analogous Adler-Bardeen theorem for the $U(1)$ axial current in the $N = 2$ SYM [118]. In [47], we may say, the authors developed a formal proof to all orders based on the Ward identities of the model.

Despite the independence of the Witten partition function under infinitesimal changes in the coupling constant, such a result should not be surprising. In its twisted version, we can see that the trace of the energy-momentum is not zero, but given by

$$g_{\mu\nu}T^{\mu\nu} = \mathrm{Tr}\{D_\mu\phi D^\mu\bar{\phi} - 2iD_\mu\eta\psi^\mu + 2i\bar{\phi}[\psi_\mu, \psi^\mu] + 2i\phi[\eta, \eta] + \frac{1}{2}[\phi, \bar{\phi}]^2]\} \ , \tag{3.78}$$

meaning that S_W is not conformally invariant under the transformation

$$\delta g_{\mu\nu} = h(x)g_{\mu\nu} \ , \tag{3.79}$$

for an arbitrary real function $h(x)$ on M. Nonetheless, the trace of the energy-momentum tensor can be written as a total divergence,

$$g_{\mu\nu}T^{\mu\nu} = D_\mu R^\mu \ , \tag{3.80}$$

where $R^\mu = \mathrm{Tr}(\bar{\phi}D^\mu\phi - 2i\eta\psi^\mu)$, which means, in turn, that S_W is invariant under a global rescaling of the metric: $\delta g_{\mu\nu} = w g_{\mu\nu}$, with w constant – see [46]. The liberty of choosing $g^2 \to 0$ in the partition function, treating the problem in the weak coupling limit, does not eliminate the possibility of loop corrections to the effective action (Γ), since there is no Ward identity, or a particular property of the vertices and propagators of S_W capable of eliminating these quantum corrections. In the *off-shell* Baulieu-Singer approach, the situation is considerably distinct, as we shall see in the following sections.

Chapter 4

Baulieu-Singer approach

In 1988, L. Baulieu and I. M. Singer (BS) proposed a topological *off-shell* theory based on the BRST symmetry of non-Abelian topological gauge models [48]. The BS approach is not built through a linear transformation of a supersymmetric gauge theory, like Witten's TQFT. It is built through a gauge-fixing procedure of a topological invariant action, in such a way that the BRST operator naturally appears as nilpotent without requiring the use of equations of motion. The geometric interpretation of such an approach is that the non-Abelian topological theory lie in an universal space graded as a sum of the ghost number and the form degree, where the vertical direction of this double complex is determined by the ghost number, and the horizontal one, by the form degree. In this space the topological BRST transformations is written in terms of an universal connection, and its curvature naturally explains the BS approach as a topological Yang-Mills theory with the same global observables of Witten's TQFT.

4.1 BRST symmetry in topological gauge theories

The four-dimensional spacetime is assumed to be Euclidean and flat[1]. The non-Abelian topological action $S_0[A]$ in four-dimensional spacetime that represents topological invariants is the Pontryagin action[2],

$$S_0[A] = \frac{1}{2}\int d^4x\, F^a_{\mu\nu}\widetilde{F}^a_{\mu\nu} \;, \tag{4.1}$$

that labels topologically inequivalent field configurations, as $S_0[A] = 32\pi^2 n$, in which n is the topological charge known as winding number — see Chapter 2. We must note that the Pontryagin action has three different gauge symmetries to be fixed, these are:

(i) the gauge field symmetry,

$$\delta A^a_\mu = D^{ab}_\mu \omega^b + \alpha^a_\mu \;; \tag{4.2}$$

(ii) the topological parameter symmetry,

$$\delta \alpha^a_\mu = D^{ab}_\mu \lambda^b \;; \tag{4.3}$$

[1]Throughout the thesis we consider flat Euclidean spacetime. Although the topological action is background independent, the gauge-fixing term entails the introduction of a background. Ultimately, background independence is recovered at the level of correlation function due to BRST symmetry [48; 119; 120].

[2]It is worth mentioning that the action $S_0[A]$ encompasses a wide range of topological gauge theories. The Pontryagin action is the most common case because it can be defined for all semi-simple Lie groups. Nevertheless, other cases can also be considered. For instance, Gauss-Bonnet and Nieh-Yang topological gravities can be formulated for orthogonal groups [121].

(iii) the field strength symmetry[1],

$$\delta F^a_{\mu\nu} = -gf^{abc}\omega^b F^c_{\mu\nu} + D^{ab}_{[\mu}\alpha^b_{\nu]} \ ; \tag{4.4}$$

where $D^{ab}_\mu \equiv \delta^{ab}\partial_\mu - gf^{abc}A^c_\mu$ is the covariant derivative in the adjoint representation of the Lie group G, g is the coupling constant, f^{abc} are the structure constants of the gauge group and ω^a, α^a_μ and λ^a are the infinitesimal G-valued gauge parameters. The first parameter (ω^a) reflects the usual Yang-Mills symmetry of $S[A]$, whereas the second one (α^a_μ) is the topological shift associated to the fact that $S[A]$ is a topological invariant, *i.e.*, invariant under continuous deformations, see (2.27). The third gauge parameter (λ^a) is due to an internal ambiguity present in the gauge transformation of the gauge field (4.2). The transformation of the gauge field is composed by two independent symmetries. If the space has a boundary, the parameter $\alpha^a_\mu(x)$ must vanish at this boundary but not $\omega^a(x)$, what explains the internal ambiguity described by (4.3) in which $\alpha^a_\mu(x)$ is absorbed into $\omega^a(x)$, and not the other way around.

Following the BRST quantization procedure, the gauge parameters present in the gauge transformations (4.2)-(4.4) are promoted to ghost fields: $\omega^a \to c^a$, $\alpha^a_\mu \to \psi^a_\mu$, and $\lambda^a \to \phi^a$; c^a is the well-known Faddeev-Popov (FP) ghost; ψ^a_μ is a topological fermionic ghost; and ϕ^a is a bosonic ghost. The corresponding BRST transformations are

$$\begin{aligned} sA^a_\mu &= -D^{ab}_\mu c^b + \psi^a_\mu, \\ sc^a &= \frac{g}{2}f^{abc}c^b c^c + \phi^a, \\ s\psi^a_\mu &= gf^{abc}c^b\psi^c_\mu + D^{ab}_\mu\phi^b, \\ s\phi^a &= gf^{abc}c^b\phi^c, \end{aligned} \tag{4.5}$$

[1] The antisymmetrization index notation here employed means that, for a generic tensor, $S_{[\mu\nu]} = S_{\mu\nu} - S_{\nu\mu}$.

from which one can easily check the nilpotency of the BRST operator,

$$s^2 = 0 \ , \tag{4.6}$$

by applying two times the BRST operator s on the fields. Naturally $S_0[A]$ is invariant under the BRST transformations (4.5). The nilpotency property of s defines the cohomology of the theory, which allows for the gauge fixing of the Pontryagin action. Furthermore such a property reveals the geometric structure of the BRST transformations in non-Abelian topological gauge theories, which elucidates the nature of the global observables through a generalization of the gauge connection.

4.1.1 Geometric interpretation

In the p-form formalism, the fields c and ϕ are 0-forms, ψ is the 1-form $\psi_\mu dx_\mu$, and F, the following 2-form

$$F = dA + AA = \frac{1}{2} F_{\mu\nu} dx_\mu \wedge dx_\nu \ , \tag{4.7}$$

where "$\wedge$" is the *wedge product* which indicates that the tensor product is completely antisymmetric, and $d = dx_\mu \frac{\partial}{\partial x_\mu}$ is the exterior derivative whose operation in the space of smooth p-forms, Λ_p, $d : \Lambda_p \to \Lambda_{p+1}$, on a generic p-form ω_p,

$$\omega_p = \omega_{i_1, i_2, \ldots, i_p} dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p} \ , \tag{4.8}$$

is locally defined by

$$d\omega_p = \frac{\partial \omega_{i_1, i_2, \ldots, i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p} \ . \tag{4.9}$$

Being ω_p a p-form, $d\omega_p$ is a $(p+1)$-form. It follows that the exterior derivative is nilpotent, $d^2 = 0$, due to the antisymmetric property of the indices. One assumes that s anticommutes with d, $\{s, d\} = 0$. We can then write the BRST transformations in the form

$$\begin{aligned} sA &= -Dc + \psi \, , \\ sc &= -\frac{1}{2}[c,c] + \phi \, , \\ s\psi &= -D\phi - [c,\psi] \, , \\ s\phi &= -[c,\phi] \, . \end{aligned} \tag{4.10}$$

The geometric meaning of the topological BRST transformations showed up through the definition of the extended exterior derivative as the sum of the ordinary exterior derivative with the BRST operator,

$$\widetilde{d} = d + s \, , \tag{4.11}$$

and the generalized connection

$$\widetilde{A} = A + c \, . \tag{4.12}$$

The space is graded as a sum of form degree and ghost number, in which the BRST operator is the exterior differential operator in the moduli space direction $\mathcal{A}/\mathcal{G}$, where the gauge fields that differ by a gauge transformation are identified. The whole space is then $M \times \mathcal{A}/\mathcal{G}$, being M a compact oriented Riemannian four-dimensional manifold. By direct inspection one sees that the BRST transformations can be written in terms of the generalized curvature[1]

$$\mathcal{F} = F + \psi + \phi \ , \tag{4.13}$$

such that

$$\mathcal{F} = \widetilde{d}\widetilde{A} + \frac{1}{2}[\widetilde{A}, \widetilde{A}] \ , \tag{4.14}$$

with the Bianchi identity

$$\widetilde{D}\mathcal{F} = \widetilde{d}\mathcal{F} + [\widetilde{A}, \mathcal{F}] = 0 \ . \tag{4.15}$$

In the definition (4.12) and following equations we are adding quantities with different form degrees and ghost numbers as though they were of the same nature. Obviously this is not being done directly. We must see equations (4.14) and (4.15) as an expansion in form degrees and ghost numbers in which the elements with the same nature on both sides have to be compared.

The topological Yang-Mills theory appear as an extension of the ordinary Yang-Mills theory in an appropriate extended space, where the group of gauge transformations $\mathcal{G}$ acts on $P \times \mathcal{A}$ where $\mathcal{A}$ is the set of all vector potentials on the principle bundle P over M. In this sense $\mathcal{G}$ has a connection in the M direction, and an orthogonal complement in the direction $\mathcal{A}/\mathcal{G}$. (For a detailed study on the geometric interpretation of the universal fibre bundle and its curvature, we suggest [52].) The relevant cohomology is defined by the cohomology of $M \times \mathcal{A}/\mathcal{G}$, $\widetilde{d}^2 = 0$, as the nilpotency property of s follows from the Bianchi identity (4.15), being valid without requiring equations of motion. Such a geometric structure reveals the BRST *off-shell* character of the BS approach. We will discuss in the

[1]The nature of ϕ as the “curvature” in the in instanton moduli space direction is implicit in the BRST transformation of the FP ghost, that can be rewritten in the geometric mnemonic form $sc + \frac{1}{2}[c, c] = \phi$.

last section how the universal curvature $\mathcal{F}$ generates the same global observables of Witten theory, *i.e.*, the Donaldson polynomials.

4.1.2 Doublet theorem and gauge fixing: BS gauges

Let us recall the *doublet theorem* which will be indispensable later on, in order to fix the gauge ambiguities without changing the physical content of the theory. Suppose a theory that contains a pair of fields or sources that form a doublet, *i.e.*,

$$\begin{aligned} \hat{\delta}\mathcal{X}_i &= \mathcal{Y}_i \ , \\ \hat{\delta}\mathcal{Y}_i &= 0 \ , \end{aligned} \tag{4.16}$$

where i is a certain index, and $\hat{\delta}$ is a fermionic operator. The field (source) $\mathcal{X}_i$ is assumed to be fermionic. As the operator $\hat{\delta}$ increases the ghost number in one unity by definition, if $\mathcal{X}_i$ is an anti-commuting quantity, $\mathcal{Y}_i$ is a commuting one. The doublet structure of $(\mathcal{X}_i, \mathcal{Y}_i)$ in eq. (4.16) implies that such fields (or sources) belong to the trivial part of the cohomology of $\hat{\delta}$. The proof is as follows. Firstly we define the operators

$$\hat{N} = \int dx \left(\mathcal{X}_i \frac{\partial}{\partial \mathcal{X}_i} + \mathcal{Y}_i \frac{\partial}{\partial \mathcal{Y}_i} \right) \ , \tag{4.17}$$

$$\hat{A} = \int dx \, \mathcal{X}_i \frac{\partial}{\partial \mathcal{Y}_i} \tag{4.18}$$

$$\hat{\delta} = \mathcal{Y}_i \frac{\partial}{\partial \mathcal{X}_i} \ , \tag{4.19}$$

which obey the algebra

$$\{\hat{\delta}, \hat{A}\} = \hat{N} \ , \tag{4.20}$$

$$\left[\hat{\delta}, \hat{N}\right] = 0 \ , \tag{4.21}$$

where $\hat{\delta}$ is a nilpotent operator as it is fermionic, $\hat{\delta}^2 = 0$. The operator $\hat{N}$ counts the number of $\mathcal{X}_i$ and $\mathcal{Y}_i$. Being $\triangle$ a polynomial in the fields, sources and parameters, the cohomology of the nilpotent operator $\hat{\delta}$, as we know, is given by the the solutions of

$$\hat{\delta}\triangle = 0 \ , \tag{4.22}$$

that is not exact, *i.e.*, that cannot be written in the form

$$\triangle = \hat{\delta}\Sigma \ . \tag{4.23}$$

The general expression of $\triangle$ is then

$$\triangle = \widetilde{\triangle} + \hat{\delta}\Sigma \ , \tag{4.24}$$

where $\widetilde{\triangle}$ belongs to the non-trivial part of the cohomology, in other words, it is closed, $\hat{\delta}\widetilde{\triangle} = 0$, but not exact, $\widetilde{\triangle} \neq \hat{\delta}\widetilde{\Sigma}$. One can expand $\triangle$ in eigenvectors of $\hat{N}$,

$$\triangle = \sum_{n\geq 0} \triangle_n \ , \tag{4.25}$$

such that $\hat{N}\triangle_n = n\triangle_n$, where n is the total number of $\mathcal{X}_i$ and $\mathcal{Y}_i$ in $\triangle_n$. Such a expansion is consistent as each $\triangle_n$ is a polynomial in $\mathcal{X}_i$ and Y_i, and $\delta\triangle_n = 0$ inclusive for $\forall n \geq 1$, according to (4.16) and the commuting properties of $\mathcal{X}_i$ and $\mathcal{Y}_i$. Finally, rewriting (4.25) as

$$\triangle = \triangle_0 + \sum_{n\geq 1} \frac{1}{n}\hat{N}\triangle_n \ , \tag{4.26}$$

then, using the commuting relation (4.20), we get

$$\triangle = \triangle_0 + \hat{\delta}\left(\sum_{n\geq 1}\frac{1}{n}\hat{A}\triangle_n\right) , \tag{4.27}$$

which shows that all terms which contain at least one field (source) of the doublet never enter the non-trivial part of the cohomology of $\hat{\delta}$, being thus non-physical.

In order to fix the three gauge symmetries of the non-Abelian topological theory we introduce the following three BRST doublets:

$$\begin{aligned} s\bar{c}^a &= b^a \,, & sb^a &= 0 \,, \\ s\bar{\chi}^a_{\mu\nu} &= B^a_{\mu\nu} \,, & sB^a_{\mu\nu} &= 0 \,, \\ s\bar{\phi}^a &= \bar{\eta}^a \,, & s\bar{\eta}^a &= 0 \,, \end{aligned} \tag{4.28}$$

where $\bar{\chi}^a_{\mu\nu}$ and $B^a_{\mu\nu}$ are (anti-)self-dual fields according to the (negative) positive sign in (4.31), see below. The $\mathcal{G}$-valued Lagrange multiplier fields b^a, $B^a_{\mu\nu}$ and $\bar{\eta}$ have respectively ghost numbers 0, 0, and -1; while the antighost fields $\bar{c}^a$, $\bar{\chi}^a_{\mu\nu}$ and $\bar{\phi}^a$, ghost numbers -1, -1 and -2. (For completeness and further use, the quantum numbers of all fields are displayed in Table 4.1.)

Field	A	ψ	c	ϕ	$\bar{c}$	b	$\bar{\phi}$	$\bar{\eta}$	$\bar{\chi}$	B
Dim	1	1	0	0	2	2	2	2	2	2
Ghost n^o	0	1	1	2	-1	0	-2	-1	-1	0

Table 4.1: Quantum numbers of the fields.

Working in Baulieu-Singer gauges amounts to considering the constraints [48]

$$\partial_\mu A^a_\mu = -\frac{1}{2}b^a \,, \tag{4.29}$$

$$D^{ab}_\mu \psi^a_\mu = 0 \,, \tag{4.30}$$

$$F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu} = -\frac{1}{2}\rho B^a_{\mu\nu} \,, \tag{4.31}$$

where ρ is a real parameter. Beyond the gauge fixing of the topological ghost (4.30), we must interpret the requirement of two extra gauge fixings due to the fact that the gauge field possesses two independent gauge symmetries. In this sense the condition (4.29) fixes the usual Yang-Mills symmetry $\delta A^a_\mu = D^{ab}_\mu \omega^b$, and the second one, (4.31), the topological shift $\delta A^a_\mu = \alpha^a_\mu$. The (anti-)self-dual condition for the field strength (in the limit $\rho \to 0$) is convenient to identify the well-known observables of topological theories in four dimensions (see [40]) known as Donaldson polynomials [15; 17], see Chapter 3, that are described in terms of the instantons — in which we are interested in here. This condition on $F_{\mu\nu}$ (4.31), which is indirectly a condition on the gauge field as $F_{\mu\nu}$ only depends on A^a_μ, corresponds to the gauge fixing of the field strength itself, because $F^a_{\mu\nu}$ also transforms as a gauge field, cf. (4.4). The first gauge condition on A^a_μ fixes the information about its divergence while the second one restricts its curl freedom, in such a way that, from the point of view of the four-dimensional Helmholtz theorem [122], the gauge field is well-defined — disregarding the Gribov copies for a moment.

The partition functional of the topological action in BS gauges (4.29) takes the form

$$Z_{BS} = \int [dc][d\bar{c}][d\psi_\mu][d\bar{\chi}_{\mu\nu}][dB_{\mu\nu}][d\phi][d\bar{\phi}][d\eta] e^{-S_{BS}} \ , \tag{4.32}$$

whereby

$$S_{BS} = S_0[A] + S^{BS}_{gf} \ , \tag{4.33}$$

being S^{BS}_{gf} the gauge-fixing action which belongs to trivial part of the cohomology, given by

$$\begin{aligned}
S^{BS}_{gf} &= s\,\mathrm{Tr}\int d^4x \left[\bar{\chi}_{\mu\nu}\left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu} + \frac{1}{2}\rho B_{\mu\nu}\right) + \bar{\phi} D_\mu \psi_\mu + \bar{c}\left(\partial_\mu A_\mu - \frac{1}{2}b\right)\right] \\
&= \mathrm{Tr}\int d^4x \left[B_{\mu\nu}\left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu} + \frac{1}{2}\rho B_{\mu\nu}\right) + \bar{\chi}_{\mu\nu}\left(D_{[\mu}\psi_{\nu]} \pm \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} D_{[\alpha}\psi_{\beta]}\right)\right. \\
&- \bar{\chi}_{\mu\nu}\left[c, F_{\mu\nu} \pm \widetilde{F}_{\mu\nu}\right] + \eta D_\mu \psi_\mu + \bar{\phi}\left[\psi_\mu, \psi_\mu\right] + \bar{\phi} D_\mu D_\mu \phi - b\left(\partial_\mu A_\mu - \frac{1}{2}b\right) \\
&- \left.\bar{c}\partial_\mu D_\mu c - \bar{c}\partial_\mu \psi_\mu\right] \; . \qquad (4.34)
\end{aligned}$$

A key observation is that, for $\rho = 1$, one can eliminate the the topological term $S_0[A]$, *i.e.*, the Pontryagin action, by integrating out the field $B_{\mu\nu}$, such that

$$\mathrm{Tr}\{B_{\mu\nu}\left(F_{\mu\nu} + \widetilde{F}_{\mu\nu}\right) + \frac{1}{2}B_{\mu\nu}B_{\mu\nu}\} \sim \mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu} + F_{\mu\nu}\widetilde{F}_{\mu\nu}\} \; , \qquad (4.35)$$

and

$$\int [dc][d\bar{c}][d\psi_\mu][d\bar{\chi}_{\mu\nu}][dB_{\mu\nu}][d\phi][d\bar{\phi}][d\eta] \to \int [dc][d\bar{c}][d\psi_\mu][d\bar{\chi}_{\mu\nu}][d\phi][d\bar{\phi}][d\eta] \; . \qquad (4.36)$$

In this case we obtain a classical topological action which is equivalent to a Yang-Mills action plus ghost interactions. Such an action, however, does not produce local observables as the cohomology of the theory remain the same, as we will discuss in more detail later. The Green functions of local operators in (4.32) does not depend on the choice of the background metric. Let S^g_{BS} be an action with metric choice $g_{\mu\nu}$, and $S^{g+\delta g}_{BS}$, the same action up to the change of $g_{\mu\nu}$ into $g_{\mu\nu} + \delta g_{\mu\nu}$. As the only terms that depends on the metric belong to the trivial part of cohomology we conclude immediately that S^g_{BS} and $S^{g+\delta g}_{BS}$ only differ by a BRST-exact term,

$$S^g_{BS} - S^{g+\delta g}_{BS} = s\int d^4x \triangle^{(-1)} \; , \qquad (4.37)$$

where $\triangle^{(-1)}$ is a polynomial of the fields, with ghost number -1. It means that the expectation values of local operators are the same if computed with a background metric $g_{\mu\nu}$ or $g_{\mu\nu} + \delta g_{\mu\nu}$,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \prod_p \mathcal{O}_{\alpha_p}(\phi_i) \rangle = 0 \ , \tag{4.38}$$

where $\mathcal{O}_{\alpha_p}(\phi_i)$ are functional operators of the quantum fields $\phi_i(x)$ – see Chapter 3, eq. (3.7). An anomaly in the topological BRST symmetry would break the equation above. However there is no 4-form with ghost number 1, $\triangle^{(1)}_{4-\text{form}}$, defined modulo s- and d- exact terms which obeys (cf. [48])

$$s \, \triangle^{(1)}_{4-\text{form}} + d \triangle^{(2)}_{3-\text{form}} = 0 \ , \tag{4.39}$$

therefore radiative corrections that could break the topological property (4.38) at the quantum level are not expected. The formal proof of the absence of gauge anomalies to all orders in the topological BS theory is achieved by employing the isomorphism described in [50; 123].

4.1.3 Absence of gauge anomalies

The proof of the absence of gauge anomalies for the Slavnov-Taylor identity,

$$\mathcal{S}(S) = 0 \ , \tag{4.40}$$

consists in proving that the cohomology of $\mathcal{S}$ is empty. In equation above, S is the classical action for a given gauge choice, and

$$\mathcal{S} = \int d^4x \, (s\Phi^{\sigma}) \frac{\delta}{\delta \Phi^{\sigma}} \ , \tag{4.41}$$

where Φ^σ represents all fields. As $\mathcal{S}$ is a Ward identity, in the absence of anomalies the symmetry (4.40) is also valid at the quantum level, *i.e.*, $\mathcal{S}(\Gamma) = 0$, being Γ the quantum action with loop corrections — see Appendix A.

In eq. (4.41), $s\Phi^\sigma$ represents the BRST transformation of each field Φ^σ. The fields $\bar{c}$, b, $\bar{\chi}_{\mu\nu}$, $B_{\mu\nu}$, $\bar{\phi}$ and $\bar{\eta}$ transform as doublets, cf. eq. (4.28). Changing the variables according to the redefinitions

$$\begin{aligned} \psi &\to \psi' = \psi - Dc \,, \\ \phi &\to \phi' = \phi - \frac{1}{2}[c,c] \,, \end{aligned} \tag{4.42}$$

the BRST transformations (4.10) are reduced to the doublet transformations

$$\begin{aligned} sA &= \psi' \,, \\ s\psi' &= 0 \,, \\ sc &= \phi' \,, \\ s\phi' &= 0 \,. \end{aligned} \tag{4.43}$$

It configures a reduced transformation in which the non-linear part of the BRST transformations in the Slavnov-Taylor identity were eliminated. The complete transformation in this space is given by the reduced operator

$$\mathcal{S}_{doublet} = \int d^4x \, (s\Phi'^\sigma) \frac{\delta}{\delta \Phi'^\sigma} \,, \tag{4.44}$$

where $\Phi' = \{A, \psi', c, \phi', \bar{c}, b, \bar{\chi}_{\mu\nu}, B_{\mu\nu}, \bar{\phi}, \eta\}$, which is composed of five doublets. It means that $\mathcal{S}_{doublet}$ has vanishing cohomology (H),

$$H(\mathcal{S}_{doublet}) = \emptyset \,, \tag{4.45}$$

in other words, that any polynomial of the fields Φ', $\triangle(\Phi')$, that satisfies

$$\mathcal{S}_{doublet}(\triangle(\Phi')) = 0 \ , \tag{4.46}$$

belongs to the trivial part of the cohomology of $\mathcal{S}_{doublet}$ — see the doublet theorem in previous section. The crucial point here is the fact that the cohomology of $\mathcal{S}$ in the space of local integrated functionals in the fields and sources is isomorphic to a subspace of $H(\mathcal{S}_{doublet})$. Consequently $\mathcal{S}$ has also vanishing cohomology [123],

$$H(\mathcal{S}) = \emptyset \ . \tag{4.47}$$

(For an algebraic demonstration of the isomorphism between $H(\mathcal{S}_{doublet})$ and $H(S)$, see [50]. An alternative algebraic proof of the $H(\mathcal{S})$ triviality can be found in [96].) The result (4.47) shows that there is no room for an anomaly in the Salvnov-Taylor identity (4.40). All counterterms at the quantum level will belong to the trivial part of cohomology of the linearized Slavnov-Taylor operator, see Appendix A, and the condition (4.39) for the existence of an anomaly capable of breaking the topological property (4.38) never occurs, being the background metric independence valid to all orders in perturbation theory.

The second point, and not least, is the conclusion that the BS theory has no local observables. Due to its vanishing cohomology (4.47), all BRST-invariant quantities must belong to the non-physical (or trivial) part of the cohomology of s, and the only possible observables are the global ones, *i.e.*, topological invariants for four-manifolds. Such observables are characterized by the so-called *basic cohomology* of s [96; 124], in which the observables are globally defined in agreement with the supersymmetric formulation of J. H. Horne [125]. A simple way to identify theses observables is accomplished by studying the cohomology of the extended space $M \times \mathcal{A}/\mathcal{G}$, where the metric independent observables, known as

Chern classes, are constructed in terms of the universal curvature $\mathcal{F}$ (4.13). The Donaldson polynomials are naturally recovered, characterized by the so-called *equivariant cohomology*, that relates the BS approach to Witten theory.

4.2 Baulieu-Singer approach versus Donaldson-Witten theory

We would like to emphasize that the Baulieu-Singer gauge-fixing procedure does not exactly recover the Witten action. The BRST charge in the Baulieu-Singer approach is off-shell nilpotent, as a consequence of the Bianchi identity for the universal curvature $\mathcal{F}$ of the space $M \times \mathcal{A}/\mathcal{G}$ (4.15). Such a cohomological property does not depend on the gauge choice, while the BRST charge in Witten theory is only on-shell nilpotent. It is possible to obtain exactly the Witten action following the BRST gauge-fixing construction of Brooks *et al.* [49]. In this construction, one first gauge fixes a Lagrangian, $\mathcal{L}_0$, which is assumed to be Yang-Mills invariant, and also invariant under a topological shift

$$\delta_1 A^a_\mu = \alpha^a_\mu \; . \tag{4.48}$$

The Lagrangian that satisfies both gauge conditions is "zero" or, in our case, a topological invariant for four-manifolds, namely, the Pontryagin action (4.1), considering that the parameter α^a_μ asymptotically drops off as one power faster than the gauge field in order to not change the winding number. By choosing the anti-self-dual gauge constraint (4.31) with $\rho = 0$ to fix $\mathcal{L}_0 \sim \mathrm{Tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$, one gets

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{L}_0 + \mathcal{L}^{(1)}_{gf+FP} \\ &= \mathcal{L}_0 + \mathrm{Tr}\{\frac{1}{4} i B_{\mu\nu}(F_{\mu\nu} + \widetilde{F}_{\mu\nu}) - i\chi_{\mu\nu} D_\mu \psi_\nu\} \; , \end{aligned} \tag{4.49}$$

where the antisymmetric field $\bar{\chi}_{\mu\nu}$ in (4.28) was redefined through $\bar{\chi}_{\mu\nu} \to i\chi_{\mu\nu}$. The resulting Lagrangian, $\mathcal{L}_1$, contains a BRST symmetry given by the set of doublet transformations

$$\begin{aligned} s_1 A^a_\mu &= \psi^a_\mu \,, \quad s_1 \psi^a_\mu = 0 \,, \\ s_1 \chi^a_{\mu\nu} &= B^a_{\mu\nu} \,, \quad s_1 B^a_{\mu\nu} = 0 \,, \end{aligned} \tag{4.50}$$

being s_1 off-shell nilpotent. Without a new restriction, ψ possesses four degrees of freedom (in Lorentz index), while the anti-self-dual antisymmetric $\chi_{\mu\nu}$ field, only three. In order to equal their degrees of freedom (a particular feature of Witten theory), another restriction on ψ is required. Fortunately, $\mathcal{L}_1$ has an extra symmetry given by

$$\delta_2 \psi^a_\mu = i(D\phi)^a \,, \quad \delta_2 B^a_{\mu\nu} = ig[\phi, \chi_{\mu\nu}]^a \,, \tag{4.51}$$

where the scalar field ϕ^a is a bosonic ghost (the same as before, present in the s operator). To gauge fix this extra symmetry, the authors of [49] started with an ansatz given by the gauge-fixing Lagrangian

$$\mathcal{L}^{(2)}_{gf+FP} = (\delta_1 + \delta_2) \mathrm{Tr}\left\{ic_0 \bar{\phi}(D_\mu \psi_\mu + c_1 \zeta) + c_2 \chi_{\mu\nu} B_{\mu\nu}\right\} \,, \tag{4.52}$$

where c_i are arbitrary real constants, $\bar{\phi}^a$ is the bosonic anti-ghost field, and ζ^a, a fermionic auxiliary field. In order to obtain a final action with a global scaling and U symmetries with the same weights of $(A, \phi, \bar{\phi}, \psi, \chi)$ in Witten theory, which are $(1, 0, 2, 1, 2)$ and $(0, 2, -2, 1, -1)$, respectively, we identify $\zeta = g[\phi, \eta]$, being η the anti-commuting field with weights 2 and -1, which is the transform of $\bar{\phi}$, *i.e.*, $\delta_2 \bar{\phi} = 2i\eta$, cf. eq. (3.38). One observes that $(\bar{\phi}, \eta)$ is not a doublet, as $\delta_2 \eta = -\frac{i}{2} g[\phi, \bar{\phi}] \neq 0$, an structure considerably different to what one would expect from traditional BRST gauge fixing, but in complete agreement with Witten's TQFT [103], as the algebra of (4.51) only closes up to an ordinary Yang-Mills gauge transformation [49]. In order to maintain the time-reversal symmetry of the final Lagrangian, $\mathcal{L}$, the natural and unique choice of c_i are $c_0 = c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{8}$. After using the auxiliary field equations of motion, one finally obtains

$$\begin{aligned}\mathcal{L} - \mathcal{L}_0 &= \mathcal{L}^{(1)}_{gf+FP} + \mathcal{L}^{(2)}_{gf+FP} \\ &= \mathcal{L}_W \ ,\end{aligned} \tag{4.53}$$

where $\mathcal{L}_W$ is the full Witten Lagrangian in eq. (3.34), which shows that Witten theory can be obtained through a BRST gauge-fixing construction. This procedure, however, is not equivalent to Baulieu-Singer approach. The final Witten Lagrangian possesses a remaining Yang-Mills ambiguity. The gauge-fixing construction of Brooks *et al.* is based on a class of gauges in which the independence of the Faddeev-Popov ghosts is imposed. The gauge fixing is performed in two "steps", and the final action cannot be written in the form $s\mathcal{W}$, being $\mathcal{W}$ a polynomial in the fields, and s the on-shell BRST operator. The closure of Brooks *et al.* algebra requires the equations of motion, as it just reproduces the Witten action. In the BS approach, all symmetries are fixed at once through s, characterized by an on-shell BRST charge, and the final BS Lagrangian does not possess a remaining Yang-Mills ambiguity.

It is possible to choose gauges in the BS approach in order to obtain the Witten action plus ghost interactions, but never the Witten action alone. These new ghost interactions are needed as the topological Yang-Mills symmetry are fully gauge-fixed, inclusive the ordinary Yang-Mills one, according to the cohomology of the complete s. In particular, taking into account ghost numbers and mass dimensions, we can add to BS action (4.33) the BRST-exact term

$$
\begin{aligned}
s\mathrm{Tr}\left\{c[\bar\chi_{\mu\nu},\bar\chi_{\mu\nu}]-\frac{1}{2}\bar c[\phi,\bar\phi]\right\} &= \mathrm{Tr}\left\{\phi[\bar\chi_{\mu\nu},\bar\chi_{\mu\nu}]-\frac{1}{2}[c,c][\bar\chi_{\mu\nu},\bar\chi_{\mu\nu}]\right.\\
&+ 2B_{\mu\nu}[c,\bar\chi_{\mu\nu}]+b[\phi,\bar\phi]\\
&- \left.\bar c[[\phi,c],\bar\phi]+\bar c[\phi,\bar\eta]\right\} . \qquad (4.54)
\end{aligned}
$$

After integrating out the bosonic fields $B_{\mu\nu}$ and b, cubic and quartic interactions involving $\chi_{\mu\nu}$, ϕ, $\bar\phi$ and η are produced[1]. These interactions are present in Witten action. In short, together with the BRST-exact term above, the Baulieu-Singer approach recovers the Witten action accompanied by quadratic ghost terms and ghost interactions,

$$
\begin{aligned}
S^{(W)}_{BS} &= S_{BS}+s\mathrm{Tr}\left\{c[\bar\chi_{\mu\nu},\bar\chi_{\mu\nu}]-\frac{1}{2}\bar c[\phi,\bar\phi]\right\}\\
&= S_W+\Sigma_{\mathcal{G}} , \qquad (4.56)
\end{aligned}
$$

where $\Sigma_{\mathcal{G}}$ represents the ghost quadratic terms and interactions mentioned above. The inclusion of the BRST-term (4.54) only amounts to a change of the equations of motion of the Lagrange multipliers, *i.e.*, $B_{\mu\nu}=F_{\mu\nu}\pm\widetilde F_{\mu\nu}$ into $B_{\mu\nu}=F_{\mu\nu}\pm$

[1]The elimination of $B_{\mu\nu}$ and b partially breaks the nilpotency of s, giving rise to a new BRST operator, s_0,

$$
\begin{aligned}
s_0A_\mu &= -D_\mu c+\psi_\mu ,\\
s_0c &= \phi-\frac{1}{2}[c,c] ,\\
s_0\psi_\mu &= -D_\mu\phi-[c,\psi_\mu] ,\\
s_0\phi &= -[c,\phi] ,\\
s_0\bar c &= -\partial_\mu A_\mu ,\\
s_0\bar\chi &= F_{\mu\nu}+\widetilde F_{\mu\nu} ,\\
s_0\bar\phi &= \bar\eta ,\\
s_0\bar\eta &= 0 , \qquad (4.55)
\end{aligned}
$$

where s_0^2 does not annihilate the antighosts $\bar\chi$ and $\bar c$, being proportional, in contrast, to antighost equations of motion, a standard property in BRST quantization.

$\widetilde{F}_{\mu\nu} + [\bar{c}, \bar{\chi}_{\mu\nu}]$, and $b = \partial_\mu A_\mu$ into $b = \partial_\mu A_\mu + [\phi, \bar{\phi}]$, that is obtained through a simple modification of the gauge constraints in (4.29). Eq. (4.56) shows that the supersymmetric Witten action appears as a sector of the topological Yang-Mills gauge theory characterized by a larger BRST symmetry. The extra ghost action, $\Sigma_{\mathcal{G}}$, does not belong to the trivial cohomology of s, as part of the BRST-exact term included in (4.56) — cubic and quartic interactions — was incorporated in S_W to obtain the full Witten action. Consequently,

$$S_W - S_{BS}^{(W)} \neq s\mathcal{W} \;, \tag{4.57}$$

with $\mathcal{W}$ some polynomial in the fields, which shows that Witten and Baulieu-Singer actions do not differ by a BRST-exact term. This relation does not depend on the gauge choice. In principle, it is not clear that Witten and Baulieu-Singer theories share the same observables. In spite of relation (4.57), the fact that BS theory also has the Donaldson polynomials as observables is a well-known result in topological gauge theories [50; 51; 52]. Such a behavior can be explained by the equivariant cohomology, defined as a cohomology for invariant quantities under ordinary Yang-Mills gauge transformations, which are independent of Faddeev-Popov ghost fields. This cohomology also showed up in the BS topological case, specifically in the Chern classes defined on the extended space $M \times \mathcal{A}/\mathcal{G}$.

4.2.1 Equivariant cohomology and global observables

Witten's topological theory is constrcuted without fixing its remaining ordinary Yang-Mills symmetry. Witten works all the time in the instanton moduli space $\mathcal{A}/\mathcal{G}$. A generic observable of his theory, $\mathcal{O}_{\alpha_i}^{(W)}$, is naturally gauge invariant under

Yang-Mills gauge transformations,

$$s_{YM}\mathcal{O}^{(W)}_{\alpha_i} = 0 \ , \tag{4.58}$$

where s_{YM} is the nilpotent BRST operator in ordinary Yang-Mills symmetry, *i.e.*, without including the topological shift:

$$\begin{aligned} s_{YM}A_\mu &= -D_\mu c \ , \\ s_{YM}\Phi_{adj} &= -[c, \Phi_{adj}] \ , \end{aligned} \tag{4.59}$$

where Φ_{adj} is a generic field in adjoint representation that suffers a group rotation. We conclude that we can add an ordinary Yang-Mills gauge transformation (in the $\mathcal{A}/\mathcal{G}$ direction) to Witten fermionic symmetry based on the "topological shift" $\delta A_\mu \sim \psi_\mu$,

$$\delta \rightarrow \delta_{eq} = \delta + s_{YM} \ , \tag{4.60}$$

that the descent equations for $\delta \sim \{\mathcal{Q}, \cdot\}$ will remain the same, see (3.41) and (3.64)-(3.68). The operator δ_{eq} is nilpotent when acting on gauge-invariant quantities under YM transformations, thus defining a cohomology in a space where the fields that differ by a Yang-Mills gauge transformations are identified, known as *equivariant cohomology*. Such a property indicates that there is a link between Witten theory and BS approach in which the BRST operator, s, is naturally defined taking into account the topological shift and the ordinary Yang-Mills transformation in a single formalism.

To prove the link between both, we must remember that the universal curvature in the space $M \times \mathcal{A}/\mathcal{G}$, $\mathcal{F}$, is given by the sum $F + \psi + \phi$. The difference between the on-shell BRST operator, s, and the Witten fermionic symmetry, δ, for $\mathbb{X} = (F, \psi, \phi)$ is of the form

$$s\mathbb{X} = \delta\mathbb{X} + [\mathbb{X}, c] \; , \tag{4.61}$$

in other words, in the space of the fields (F, ψ, ϕ), s and δ differ by an ordinary Yang-Mills transformation, as (F, ψ, ϕ) transform in the adjoint representation of the gauge group. These fields are the only ones we need to obtain the Donaldson polynomials as the observables of the BS theory, since in the space $M \times \mathcal{A}/\mathcal{G}$ they are constructed in terms of $\mathcal{F}$, which are composed of a sum of these three fields. This allows for identifying the equivariant operator with the BRST one,

$$\delta_{eq} \equiv s \; , \tag{4.62}$$

according to the construction of the observables in Witten and BS theory, respectively.

To understand the above statement, we must invoke the n'th Chern class, $\widetilde{\mathcal{W}}_n$, defined in terms of the universal curvature by

$$\widetilde{\mathcal{W}}_n = \mathrm{Tr}\,(\underbrace{\mathcal{F} \wedge \mathcal{F} \wedge \cdots \wedge \mathcal{F}}_{\text{n times}}) \tag{4.63}$$

where $n = \{1, 2, 3, \cdots\}$ is the number of wedge products[1]. (The Polyakov loop,

$$W_P^{(C)} = \mathrm{Tr}\{\mathcal{P} e^{i \oint_C A_\mu dx_\mu}\} \; , \tag{4.64}$$

unlike the Wilson loop which is a gauge-invariant observable obtained from the holonomy of the Abelian gauge connection (3.9), is not an observable in the non-

[1] It is not possible to construct topological observables using the Hodge product, as it is metric dependent. For this reason we never obtain Yang-Mills terms of the type $\{\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \mathrm{Tr}(F_{\mu\nu}F^{\nu\sigma}F^{\mu}{}_{\sigma}), \cdots\}$, without Levi-Civita tensors in the internal product, in the place of metric tensors.

Abelian topological BS case, as it is not gauge-invariant due to the topological shift symmetry. In any case, it does not make sense to discuss confinement in the BS theory, as it is not confining to any energy scale. So that the only possibilities for topological invariants are the wedge products in $\widetilde{\mathcal{W}}_n$.) The Weyl theorem ensures that $\widetilde{\mathcal{W}}_n$ is closed with respect to the extended differential operator $\widetilde{d} = d + s$ [48; 126], *i.e.*,

$$\widetilde{d}\,\widetilde{\mathcal{W}}_n = 0 \,. \tag{4.65}$$

If we choose the first Chern class

$$\widetilde{\mathcal{W}}_1 = \mathrm{Tr}\,(\mathcal{F} \wedge \mathcal{F}) \,, \tag{4.66}$$

the expansion in ghost numbers of equation (4.65) yields

$$s\mathrm{Tr}\,(F \wedge F) \;=\; d\mathrm{Tr}\,(-2\psi \wedge F) \,, \tag{4.67}$$

$$s\mathrm{Tr}\,(\psi \wedge F) \;=\; d\mathrm{Tr}\,(-\frac{1}{2}\psi \wedge \psi - \phi F) \,, \tag{4.68}$$

$$s\mathrm{Tr}\,(\psi \wedge \psi + 2\phi F) \;=\; d\mathrm{Tr}\,(2\psi\phi) \,, \tag{4.69}$$

$$s\mathrm{Tr}\,(\psi\phi) \;=\; d\mathrm{Tr}\,(-\frac{1}{2}\phi\phi) \,, \tag{4.70}$$

$$s\mathrm{Tr}\,(\phi\phi) \;=\; 0 \,, \tag{4.71}$$

which are the same descent equations obtained in (3.64)-(3.68) following Witten analysis, only replacing δ (or δ_{eq}) by s, proving that the Baulieu-Singer and Witten (in the weak coupling limit) theories possess the same observables given by the Donaldson invariants (3.72).

It should not seem surprising the fact that the observables in the BS approach are naturally invariant under ordinary Yang-Mills symmetry, as the n'th Chern class is Yang-Mills invariant itself (4.63) since $\mathcal{F}$ transforms in the adjoint representation of the gauge group. Equation (4.65) provides a powerful tool to obtain

Donaldson polynomials in any ghost number. One must note that we do not have to worry about with the independence of Fadeve-Popov ghosts to construct the observables in the BS approach. Although the gauge-fixed BS action has FP ghosts due to the gauge fixing of the Yang-Mills ambiguity, the $(c, \bar{c})$ independence of $\widetilde{\mathcal{W}}_n$ is a direct consequence of the fact that the universal curvature of the sapce $M \times \mathcal{A}/\mathcal{G}$ does not depend on FP ghosts, but only on F, and the ghosts ψ and ϕ.

in the weak coupling limit of Witten's TQFT, the observables of both theories are undoubtedly the same: the topological Donaldson invariants. We might ask if the quantum behavior are also compatible, once BS and Witten actions does not differ by a BRST-exact term, cf. (4.57), in other words, we cannot say, in principle, that BS and Witten partition functions are quantically correspondent, as

$$Z_{BS} = \int \mathcal{D}\Phi e^{-S_{BS}} = \int \mathcal{D}\Phi e^{-S_W - \Sigma_{\mathcal{G}}} \;, \tag{4.72}$$

wherein $\Sigma_{\mathcal{G}}$ does not belong to the trivial part of the s cohomology. At a first view, $Z_{BS} \neq Z_W = \int \mathcal{D}\Phi e^{-S_W}$. If we could write $\Sigma_{\mathcal{G}}$ as a BRST-exact term, *i.e.*, $\Sigma_{\mathcal{G}} = s\mathcal{W}$ for $\mathcal{W}$ a generic polynomial in the fields, then we would get

$$e^{-\Sigma_{\mathcal{G}}} = 1 + s\left(\sum_{n=0}^{\infty} W(s\mathcal{W})^n\right) \;, \tag{4.73}$$

due to the nilpotency of s; and therefore including $e^{-\Sigma_{\mathcal{G}}}$ would be equivalent to introduce an unit in the path integral, since the expectation values of BRST-exact terms vanish, — but this is not the case. In fact $\Sigma_{\mathcal{G}} \neq s\mathcal{W}$, which opens the possibility for both theories to have different quantum properties. The one-loop exactness of twisted $N = 2$ SYM is a well-known result in literature [47]. We will carefully analyse the Ward identities of the BS theory in self-dual Landau gauges, in order to compare the quantum behavior between *on-shell* and *off-shell* approaches in topological Yang-Mills theories. We conclude that the BS and twisted $N = 2$ SYM theories are not quantically equivalent — the β-functions are different, unless we take the limit $g \to 0$ in the Witten theory. Such a behaviour is in agreement with the energy regime in which the BS and Witten theories share the same observables.

Chapter 5

Quantum properties of topological Yang-Mills theories I: Ward identities and renormalizability

The so-called *Algebraic Renormalization* [98] provides a systematic setup to construct the quantum extension of classical symmetries, which allows to prove if the theory is renormalizable (or not) to all orders in perturbation theory, without explicitly computing Feynman diagrams. The proof of renormalizability is accomplished via computation of cohomological classes, defined by the Slavnov-Taylor identity, which contains all the information of the BRST transformations of the the model, under which the classical action is gauge invariant. Such an algebraic method applies to the perturbative regime, built order by order in the loop expansion of the quantum action. It gives all allowed non-trivial counterterms and anomalies, accordingly to the set of symmetries of the theory. As the procedure is recursive, the results are automatically extended to all orders. We must say that the convergence of the perturbative series is not handled within this algebraic setup, meaning that the method requires the theory to be renormalizable by power counting. In few words, the *Algebraic Renormalization* does not provide the renormalization of the theory for a particular regularization, but its renormalizability to all orders, independently of the regularization scheme.

Our aim is to apply the algebraic BRST-renormalization techniques to study the quantum properties of the topological Baulieu-Singer theory which is based on an *off-shell* BRST gauge fixing of a metric independent action of Schwarz type, composed only of topological invariants, namely, the Pontryagin action in four dimensions. The BS theory represents the quantization of the Pontryagin action, which, in turn, represents the instanton sector of QCD vacuum. (The analysis of the symmetry structure of the Pontryagin action and its consequences could reveal some topological aspects of the QCD asymptotic behavior in the low energy limit, or, in general, of Yang-Mills theories following the effective action (2.65) in the presence of the θ-vacuum.) As discussed in the previous chapter, such a topological theory possesses the same observables — given by the Donaldson polynomials — of the Witten *on-shell* topological theory (for $g \to 0$) derived from a *twisted* version of the $N = 2$ super Yang-Mills action. Despite this correspondence in the weak coupling limit, it is not guaranteed that both theories have the same quantum behavior (the same β-function), as they have different cohomological properties. Moreover, the BS theory does not recover the $N = 2$ SYM observables in the strong limit.

The quantum stability to all orders of the BS theory was first worked out in [127], where the author applied algebraic BRST-renormalization techniques. In the latter, it was chosen the Landau gauge for the gauge field, and the same constraints in the BS gauges, (4.31) and (4.30), with $\rho = 0$. The result, in this particular gauge choice, is that the theory is renormalizable to all orders with seven independent renormalization parameters. The allowed counterterms found in [127] are in agreement with one-loop evaluations via background method due to Birmingham *et al.* [128], where the topological nature of the theory is preserved at the quantum level. The Birmingham *et al.* evaluation was worked out based on the Labastida-Pernici gauge fixing [129] which, in turn, has its origin in the Batalin-Vilkovisky algorithm [130]. Taking a particular configuration of auxiliary fields, the Labastida-Pernici gauge-fixing action, S_{LP}, can be written as a BRST-exact term,

$$S_{LP} = \delta_{BV} \mathrm{Tr}\{\frac{1}{4}\bar{\chi}_{\mu\nu}(F^{(+)}_{\mu\nu} + G_{\mu\nu}) + \frac{1}{2}\bar{\phi}D_\mu\psi_\mu + \bar{c}\partial_\mu A_\mu\} \ , \tag{5.1}$$

with $F^{(+)}$ defined in (3.35), and δ_{BV} the Batilin-Vilkovisky operator of gauge transformations that consist of an off-shell nilpotent BRST operator, $\delta^2_{BV} = 0$. The gauges in S_{LP} are the same as the ones employed in the algebraic analysis of [127] (taking $G_{\mu\nu} = 0$), whose action has the same off-shell structure of (5.1) as it is based on the Baulieu-Singer approach. So it its not surprising that the results of both methods are in agreement.

A similar BRST algebraic analysis was performed in [53], where the authors considered a subtle change in the gauge choice of the topological ghost ψ^a_μ, for which they also used the Landau gauge constraint,

$$\partial_\mu\psi^a_\mu = 0 \ , \tag{5.2}$$

instead of $D^{ab}_\mu\psi^b_\mu = 0$. In this type of Landau gauges, also known as self-dual Landau gauges, the topological action enjoys a new symmetry called *vector supersymmetry*, providing a new Ward identity to the BS theory, which reduces the number of independent renormalization parameters from seven to four. By investigating the topological Yang-Mills theories in self-dual Landau gauges, we discovered two extra symmetries [54]. One of them relates the topological ghost with the Faddeev-Popov one,

$$\delta\psi^a_\mu \Longleftrightarrow D^{ab}_\mu c^b \ . \tag{5.3}$$

Aftermath, applying the new Ward identities corresponding to these extra symmetries, we verified that the theory has, in fact, only one independent renormalization parameter. As a consequence of the vector supersymmetry, we proved that the gauge propagator and the vacuum polarization vanish to all orders in perturbation theory. Armed with this result, we were able to demonstrated that the theory is tree-level exact, in other words, that the n-points Green functions of the theory does not receive any radiative corrections at the quantum level, due to their vertex structure, and cohomological properties. The system of Z-factors, dependent on the remaining renormalization parameter, showed up an unusual ambiguity, which is absolutely consistent for a vanishing β-function, as we can directly infer from the absence of radiative corrections in self-dual Landau gauges.

5.1 Symmetries in self-dual and anti-self-dual Landau gauges

Working in the (anti-)self-dual Landau gauges ((A)SDL) amounts to considering the constraints [53]

$$\begin{aligned} \partial_\mu A^a_\mu &= 0 \ , & (5.4) \\ \partial_\mu \psi^a_\mu &= 0 \ , & (5.5) \\ F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu} &= 0 \ . & (5.6) \end{aligned}$$

Through the introduction of the three BRST doublets $(\bar{c}^a, b^a)$, $(\bar{\chi}^a_{\mu\nu}, B^a_{\mu\nu})$ and $(\bar{\phi}, \bar{\eta})$, described in eq. (4.28), the complete gauge-fixed topological action in the (A)SDL gauges takes the form

$$S[\Phi] = S_0[A] + S_{gf}[\Phi] \;, \tag{5.7}$$

for all fields $\Phi \equiv \{A, \psi, c, \phi, \bar{c}, b, \bar{\phi}, \bar{\eta}, \bar{\chi}, B\}$, where

$$\begin{aligned} S_{gf}\,[\Phi] &= s \int d^4z \left[\bar{c}^a \partial_\mu A^a_\mu + \frac{1}{2}\bar{\chi}^a_{\mu\nu}\left(F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu}\right) + \bar{\phi}^a \partial_\mu \psi^a_\mu\right] \\ &= \int d^4z \left[b^a \partial_\mu A^a_\mu + \frac{1}{2}B^a_{\mu\nu}\left(F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu}\right) + \left(\bar{\eta}^a - \bar{c}^a\right)\partial_\mu \psi^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b + \right. \\ &- \frac{1}{2} g f^{abc} \bar{\chi}^a_{\mu\nu} c^b \left(F^c_{\mu\nu} \pm \widetilde{F}^c_{\mu\nu}\right) - \bar{\chi}^a_{\mu\nu}\left(\delta_{\mu\alpha}\delta_{\nu\beta} \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\right) D^{ab}_\alpha \psi^b_\beta + \bar{\phi}^a \partial_\mu D^{ab}_\mu \phi^b + \\ &+ \left. g f^{abc} \bar{\phi}^a \partial_\mu \left(c^b \psi^c_\mu\right)\right] \;. \end{aligned} \tag{5.8}$$

Following the *Algebraic Renormalization* setup described in [98], the starting point for the quantum investigation is to write the Slavnov-Taylor identity in its local form. To this aim, we need to introduce external sources in order to control the non-linear nature of the BRST transformations (4.5), in the form of BRST doublets, two of them to be precise, given by[1]

$$\begin{aligned} s\tau^a_\mu &= \Omega^a_\mu \ , \qquad s\Omega^a_\mu \ = \ 0 \ , \\ sE^a &= L^a \ , \qquad sL^a \ = \ 0 \ . \end{aligned} \tag{5.11}$$

We shall see later that we need an extra doublet to control a non-linear bosonic symmetry of the full classical action, because of the non-linear transformation

$$\delta B^a_{\mu\nu} = f^{abc} c^b \bar{\chi}^c_{\mu\nu} \ , \tag{5.12}$$

so that the extra doublet is given by

$$s\Lambda^a_{\mu\nu} = K^a_{\mu\nu} \ , \quad sK^a_{\mu\nu} = 0 \ . \tag{5.13}$$

The corresponding quantum number of the external sources are displayed in Table 5.1 below. The respective external action, written as a BRST-exact contribution to control the non-linear transformations without changing the physical content,

[1]The non-linearity of a symmetry of the action, in which a generic field ϕ_i transforms as

$$\delta\phi_i = C_{ij_1\cdots j_n}\phi_{j_1}\cdots\phi_{j_n} \ , \tag{5.9}$$

would imply, for example, the variation of a given n-point Green function in the form

$$\delta\langle\phi_{l_1}(x_1)\cdots\phi_i(x_i)\cdots\phi_{j_n}(l_n)\rangle = \langle\phi_{l_1}(x_1)\cdots[C_{ij_1\cdots j_n}\phi_{j_1}(x_{j_1})\cdots\phi_{j_n}(x_{j_n})]\cdots\phi_{l_n}(x_n)\rangle \ , \tag{5.10}$$

showing that the composite operator $C_{ij_1\cdots j_n}\phi_{j_1}\cdots\phi_{j_n}$ is inserted as an unique entity, that needs to enter in the renormalization process. We then introduce external sources $\{Y_i, X_i\}$ to control the non-linearity, whereby $Y_i = C_{ij_1\cdots j_n}\phi_{j_1}\cdots\phi_{j_n}$, with X_i as its doublet pair, to be introduced into the action in the trivial part of BRST cohomology. We absorb the BRST transformation of the doublet $\{Y_i, X_i\}$ in the symmetry δ, written in its local form where $\delta\phi_i$ is replaced by $\frac{\delta S}{\delta Y_i}$, and after proving the quantum stability of the model, the physical limit is obtained by setting the external sources to zero, $\{Y_i, X_i\}|_{phys} \rightarrow 0$.

Source	τ	Ω	E	L	Λ	K
Dim	3	3	4	4	2	2
Ghost n^o	-2	-1	-3	-2	-1	0

Table 5.1: Quantum numbers of the external sources.

takes the form

$$
\begin{aligned}
S_{ext} &= s\int d^4z\left(\tau^a_\mu D^{ab}_\mu c^b + \frac{g}{2}f^{abc}E^a c^b c^c + g f^{abc}\Lambda^a_{\mu\nu}c^b\bar{\chi}^c_{\mu\nu}\right) \\
&= \int d^4z\Big[\Omega^a_\mu D^{ab}_\mu c^b + \frac{g}{2}f^{abc}L^a c^b c^c + g f^{abc}K^a_{\mu\nu}c^b\bar{\chi}^c_{\mu\nu} + \tau^a_\mu\left(D^{ab}_\mu\phi^b + g f^{abc}c^b\psi^c_\mu\right) \\
&+ g f^{abc}E^a c^b\phi^c + g f^{abc}\Lambda^a_{\mu\nu}c^b B^c_{\mu\nu} - g f^{abc}\Lambda^a_{\mu\nu}\phi^b\bar{\chi}^c_{\mu\nu} \\
&- \frac{g^2}{2}f^{abc}f^{bde}\Lambda^a_{\mu\nu}\bar{\chi}^c_{\mu\nu}c^d c^e\Big] \ . \qquad (5.14)
\end{aligned}
$$

Therefore, the full classical action we shall consider is

$$\Sigma[\Phi] = S_0[A] + S_{gf}[\Phi] + S_{ext}[\Phi] \ , \qquad (5.15)$$

where $S_0[A]$ is the Pontryagin action. The Slavnov-Taylor identity expresses the BRST invariance of the full action (5.15), so given by

$$\mathcal{S}(\Sigma) = 0 \ , \qquad (5.16)$$

where

$$
\begin{aligned}
\mathcal{S}(\Sigma) &= \int d^4z\left[\left(\psi^a_\mu - \frac{\delta\Sigma}{\delta\Omega^a_\mu}\right)\frac{\delta\Sigma}{\delta A^a_\mu} + \frac{\delta\Sigma}{\delta\tau^a_\mu}\frac{\delta\Sigma}{\delta\psi^a_\mu} + \left(\phi^a + \frac{\delta\Sigma}{\delta L^a}\right)\frac{\delta\Sigma}{\delta c^a} + \frac{\delta\Sigma}{\delta E^a}\frac{\delta\Sigma}{\delta\phi^a} + \right. \\
&+ \left. b^a\frac{\delta\Sigma}{\delta\bar{c}^a} + \bar{\eta}^a\frac{\delta\Sigma}{\delta\bar{\phi}^a} + B^a_{\mu\nu}\frac{\delta\Sigma}{\delta\bar{\chi}^a_{\mu\nu}} + \Omega^a_\mu\frac{\delta\Sigma}{\delta\tau^a_\mu} + L^a\frac{\delta\Sigma}{\delta E^a} + K^a_{\mu\nu}\frac{\delta\Sigma}{\delta\Lambda^a_{\mu\nu}}\right] \ . \qquad (5.17)
\end{aligned}
$$

In order to extend this symmetry to the quantum level, we must invoke the BRST invariance of the vacuum norm in the presence of an external source, $\langle 0|0\rangle_J$, in orders words, the BRST invariance of the partition functional, *i.e.*,

$$sZ[J] = 0 \,, \tag{5.18}$$

where

$$Z[J] = \langle 0|0\rangle_J = \int \mathcal{D}\Phi \, e^{-S-\int d^4x J_\sigma \Phi_\sigma} \,, \tag{5.19}$$

wherein

$$J_\sigma\Phi_\sigma = j^a_\mu A^a_\mu + j^a_b b^a + \omega^a_\mu \psi^a_\mu + \bar{\zeta}^a c^a + \zeta^a \bar{c}^a + j^a_\phi \phi^a + j^a_{\bar{\phi}} \bar{\phi}^a + j^a_{\bar{\eta}} \bar{\eta}^a + \omega^a_{\mu\nu} B^a_{\mu\nu} + \bar{\omega}^a_{\mu\nu} \bar{\chi}^a_{\mu\nu} \,, \tag{5.20}$$

being $J_\sigma \equiv \{j^a_\mu, j^a_b, \omega^a_\mu, \bar{\zeta}^a, \zeta^a, j^a_\phi, j^a_{\bar{\phi}}, j^a_{\bar{\eta}}, \omega^a_{\mu\nu}, \bar{\omega}^a_{\mu\nu}\}$ the classical sources coupled to the fields which, being classical, obey

$$sJ_\sigma = 0 \,. \tag{5.21}$$

Using equations (5.19)-(5.21), the BRST invariance (5.18) yields

$$\int d^4x \left(j^a_\mu s A^a_\mu - \omega^a_\mu s \psi^a_\mu - \bar{\zeta}^a s c^a - \zeta^a s \bar{c}^a + j^a_\phi s \phi^a + j^a_{\bar{\phi}} s \bar{\phi}^a - \bar{\omega}^a_{\mu\nu} s \bar{\chi}^a_{\mu\nu} \right) = 0 \,, \tag{5.22}$$

and finally, using the well-known equation in quantum field theory[1]

$$\frac{\delta\Gamma}{\delta\Phi_\sigma} = (-1)^\alpha J_\sigma \,, \tag{5.23}$$

where Γ is the quantum action, following the convention $\alpha = 0$ or 1 for fermionic

[1]As usual, we are using a short notation. The quantum fields Φ_σ here are the ones that obey the classical equations of motion, *i.e.*, $\Phi_\sigma \equiv \langle\Phi_\sigma\rangle_c$ — the expectation value of Φ_σ only taking into account the connected Feynman diagrams.

or bosonic fields, respectively, one obtains

$$\mathcal{S}(\Gamma) = 0 \ , \tag{5.24}$$

where

$$\begin{aligned}\mathcal{S}(\Gamma) &= \int d^4z \left[\left(\psi^a_\mu - \frac{\delta\Gamma}{\delta\Omega^a_\mu} \right) \frac{\delta\Gamma}{\delta A^a_\mu} + \frac{\delta\Gamma}{\delta\tau^a_\mu}\frac{\delta\Sigma}{\delta\psi^a_\mu} + \left(\phi^a + \frac{\delta\Gamma}{\delta L^a} \right) \frac{\delta\Gamma}{\delta c^a} + \frac{\delta\Gamma}{\delta E^a}\frac{\delta\Gamma}{\delta\phi^a} + \right. \\ &+ \left. b^a\frac{\delta\Gamma}{\delta\bar{c}^a} + \bar{\eta}^a\frac{\delta\Gamma}{\delta\bar{\phi}^a} + B^a_{\mu\nu}\frac{\delta\Gamma}{\delta\bar{\chi}^a_{\mu\nu}} + \Omega^a_\mu\frac{\delta\Gamma}{\delta\tau^a_\mu} + L^a\frac{\delta\Gamma}{\delta E^a} + K^a_{\mu\nu}\frac{\delta\Gamma}{\delta\Lambda^a_{\mu\nu}} \right] \ , \end{aligned} \tag{5.25}$$

which shows that the Slavnov-Taylor identity consists of a Ward identity automatically transferred to the quantum level. Such a behavior hides a general property of quantum extension of classical symmetry known as *principle of quantum action*, see [98], which states that exact (like the Slavnov-Taylor identity) or linear broken classical symmetries are also symmetries of the quantum action, in few words[1],

$$\delta_{sym}(\Sigma) = \triangle_{cl} \quad \text{implies} \quad \delta_{sym}(\Gamma) = 0 \ , \tag{5.26}$$

if the polynomial $\triangle_{cl}$ is at most linear in the fields, (in the case of exact symmetries, $\triangle_{cl} = 0$); whereby $\delta_{sym}(\Sigma)$ stands for transformations on the classical fields, and $\delta_{sym}(\Gamma)$, for transformations on $\langle\phi_\sigma\rangle_c$, see footnote on previous page.

Together with the Slavnov-Taylor identity, the full action possesses a rich set of Ward identities composed of the following symmetries:

[1]The quantum action principle (QAP) illustrated by equation (5.26) is not a trivial issue. It is the renormalized version of Schwinger action principle [131], and was worked out in [132; 133; 134], where it was proved that QAP is applicable for local, Lorentz invariant and power-counting renormalizable theories. For a systematic proof of QAP via algebraic analysis we strongly suggest [98].

(i) Ordinary Landau gauge fixing and Faddeev-Popov anti-ghost equation:

$$
\begin{aligned}
\frac{\delta\Sigma}{\delta b^a} &= \partial_\mu A^a_\mu \ , \\
\frac{\delta\Sigma}{\delta \bar{c}^a} - \partial_\mu \frac{\delta\Sigma}{\delta \Omega^a_\mu} &= -\partial_\mu \psi^a_\mu \ .
\end{aligned}
\tag{5.27}
$$

(ii) Topological Landau gauge fixing and bosonic anti-ghost equation:

$$
\begin{aligned}
\frac{\delta\Sigma}{\delta \bar{\eta}^a} &= \partial_\mu \psi^a_\mu \ , \\
\frac{\delta\Sigma}{\delta \bar{\phi}^a} - \partial_\mu \frac{\delta\Sigma}{\delta \tau^a_\mu} &= 0 \ .
\end{aligned}
\tag{5.28}
$$

(iii) Bosonic ghost equation:

$$
\mathcal{G}^a_\phi \Sigma = \Delta^a_\phi \ , \tag{5.29}
$$

where

$$
\begin{aligned}
\mathcal{G}^a_\phi &= \int d^4z \left(\frac{\delta}{\delta \phi^a} - g f^{abc} \bar{\phi}^b \frac{\delta}{\delta b^c} \right) \ , \\
\Delta^a_\phi &= g f^{abc} \int d^4z \left(\tau^b_\mu A^c_\mu + E^b c^c + \Lambda^b_{\mu\nu} \bar{\chi}^c_{\mu\nu} \right) \ .
\end{aligned}
\tag{5.30}
$$

(iv) Ordinary Faddeev-Popov ghost equation:

$$
\mathcal{G}^a_1 \Sigma = \Delta^a \ , \tag{5.31}
$$

where

$$
\begin{aligned}
\mathcal{G}^a_1 &= \int d^4z \left[\frac{\delta}{\delta c^a} + g f^{abc} \left(\bar{c}^b \frac{\delta}{\delta b^c} + \bar{\phi}^b \frac{\delta}{\delta \bar{\eta}^c} + \bar{\chi}^b_{\mu\nu} \frac{\delta}{\delta B^c_{\mu\nu}} + \Lambda^b_{\mu\nu} \frac{\delta}{\delta K^c_{\mu\nu}} \right) \right] \ , \\
\Delta^a &= g f^{abc} \int d^4z \left(E^b \phi^c - \Omega^b_\mu A^c_\mu - \tau^b_\mu \psi^c_\mu - L^b c^c + \Lambda^b_{\mu\nu} B^c_{\mu\nu} - K^b_{\mu\nu} \bar{\chi}^c_{\mu\nu} \right)
\end{aligned}
\tag{5.32}
$$

(v) Second Faddeev-Popov ghost equation:

$$\mathcal{G}_2^a \Sigma = \Delta^a \ , \tag{5.33}$$

where

$$\mathcal{G}_2^a = \int d^4z \left[\frac{\delta}{\delta c^a} - g f^{abc} \left(\bar{\phi}^b \frac{\delta}{\delta \bar{c}^c} + A_\mu^b \frac{\delta}{\delta \psi_\mu^c} + c^b \frac{\delta}{\delta \phi^c} - \bar{\eta}^b \frac{\delta}{\delta b^c} + E^b \frac{\delta}{\delta L^c} \right) \right] \ . \tag{5.34}$$

(vi) Vector supersymmetry[1]:

$$\mathcal{W}_\mu \Sigma = 0 \ , \tag{5.35}$$

where

$$\begin{aligned} \mathcal{W}_\mu \ = \ & \int d^4z \left[\partial_\mu A_\nu^a \frac{\delta}{\delta \psi_\nu^a} + \partial_\mu c^a \frac{\delta}{\delta \phi^a} + \partial_\mu \bar{\chi}_{\nu\alpha}^a \frac{\delta}{\delta B_{\nu\alpha}^a} + \partial_\mu \bar{\phi}^a \left(\frac{\delta}{\delta \bar{\eta}^a} + \frac{\delta}{\delta \bar{c}^a} \right) + \right. \\ + \ & \left. (\partial_\mu \bar{c}^a - \partial_\mu \bar{\eta}^a) \frac{\delta}{\delta b^a} + \partial_\mu \tau_\nu^a \frac{\delta}{\delta \Omega_\nu^a} + \partial_\mu E^a \frac{\delta}{\delta L^a} + \partial_\mu \Lambda_{\nu\alpha}^a \frac{\delta}{\delta K_{\nu\alpha}^a} \right] \ . \end{aligned} \tag{5.36}$$

(vii) Bosonic non-linear symmetry:

$$\mathcal{T}(\Sigma) = 0 \ , \tag{5.37}$$

where

$$\mathcal{T}(\Sigma) = \int d^4z \left[\frac{\delta\Sigma}{\delta\Omega_\mu^a} \frac{\delta\Sigma}{\delta\psi_\mu^a} - \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta\phi^a} - \frac{\delta\Sigma}{\delta K_{\mu\nu}^a} \frac{\delta\Sigma}{\delta B_{\mu\nu}^a} + (\bar{c}^a - \bar{\eta}^a) \left(\frac{\delta\Sigma}{\delta\bar{c}^a} + \frac{\delta\Sigma}{\delta\bar{\eta}^a} \right) \right] \ .$$

[1]Written in the form $\mathcal{W}_\mu = \sum_A \delta_\mu \Phi^A \frac{\delta}{\delta\Phi^A}$, the generators δ_μ and the BRST operator satisfy a supersymmetric algebra $\{s, \delta_\mu\} = \partial_\mu$.

(viii) Global ghost supersymmetry:

$$\mathcal{G}_3\Sigma = 0 \ , \tag{5.38}$$

where

$$\mathcal{G}_3 = \int d^4z \left[\bar{\phi}^a \left(\frac{\delta}{\delta\bar{\eta}^a} + \frac{\delta}{\delta\bar{c}^a}\right) - c^a\frac{\delta}{\delta\phi^a} + \tau^a_\mu\frac{\delta}{\delta\Omega^a_\mu} + 2E^a\frac{\delta}{\delta L^a} + \Lambda^a_{\mu\nu}\frac{\delta}{\delta K^a_{\mu\nu}}\right] \ . \tag{5.39}$$

The last two symmetries are the new ones introduced in [54]. The non-linear bosonic symmetry (vii) is precisely the one mentioned above, see eq. (5.3) which relates the FP and topological ghosts, as

$$\delta\psi^a_\mu = \frac{\delta\Sigma}{\delta\Omega^a_\mu} = D^{ab}_\mu c^b \ , \tag{5.40}$$

see eq. (5.38). The vector supersymmetry (vi) is a characteristic feature of topological theories in Landau gauges[1] [135], first introduced in the four-dimensional case in [53]. We remark that the Faddeev-Popov ghost equations (5.31) and (5.33) can be combined to obtain an exact global supersymmetry,

$$\Delta\mathcal{G}^a\Sigma = 0 \ , \tag{5.41}$$

where

$$\begin{aligned} \Delta\mathcal{G}^a \;&=\; \mathcal{G}^a_1 - \mathcal{G}^a_2 \;=\; \int d^4z \; f^{abc}\left[\left(\bar{c}^b - \bar{\eta}^b\right)\frac{\delta}{\delta b^c} + \bar{\phi}^b\left(\frac{\delta}{\delta\bar{\eta}^c} + \frac{\delta}{\delta\bar{c}^c}\right) + A^b_\mu\frac{\delta}{\delta\psi^c_\mu} + \right. \\ &+\; \left. \bar{\chi}^b_{\mu\nu}\frac{\delta}{\delta B^c_{\mu\nu}} + c^b\frac{\delta}{\delta\phi^c} + \Lambda^b_{\mu\nu}\frac{\delta}{\delta K^c_{\mu\nu}} + \tau^b_\mu\frac{\delta}{\delta\Omega^c_\mu} + E^b\frac{\delta}{\delta L^c}\right] \ . \end{aligned} \tag{5.42}$$

We observe the similarity of the equation (5.41) with the vector supersymmetry

[1] In the 3D Chern-Simons theory, this kind of symmetry is used to proved that the theory is completely finite, *i.e.*, it possesses vanishing β-function and anomalous dimensions.

(5.35). It is also worth mentioning that, even though the ghost number of the operator (5.42) is -1, resembling an anti-BRST symmetry, it is not a genuine anti-BRST symmetry[1].

5.2 Renormalizability: Anomalies and quantum stability

5.2.1 Most general counterterm

Loop expansion. In perturbation theory, the quantum action is expanded around the classical action, *i.e.*,

$$\Gamma = \sum_{n=0}^{\infty} \epsilon^n \Gamma^{(n)} \ , \tag{5.43}$$

whereby the perturbative parameter ϵ is naturally recognized as the Plank constant $\hbar$. $\Gamma^{(n)}$ represents the contribution of Feynman diagrams at n-loop order, being $\Gamma^{(0)} = \Sigma$ the classical action. At one-loop order,

$$\Gamma = \Sigma + \epsilon \Sigma^c \ , \tag{5.44}$$

where $\Gamma^{(1)} \equiv \Sigma^c$ is the most general counterterm at one-loop given by an local integrated polynomial in the fields (and their derivatives), parameters and sources, with mass dimension four and vanishing ghost number, which obey all Ward identities of the model. Replacing (5.44) into (5.24) one gets

$$S(\Gamma) = S(\Sigma) + \epsilon S_\Sigma \Sigma^c + \mathcal{O}(\epsilon^2) = 0 \ , \tag{5.45}$$

[1]See for instance [136] for the explicit anti-BRST symmetry in topological gauge theories.

where S_Σ is the linearized Slavnov-Taylor operator[1] given by

$$
\begin{aligned}
\mathcal{S}_\Sigma &= \int d^4z \left[\left(\psi^a_\mu - \frac{\delta\Sigma}{\delta\Omega^a_\mu} \right) \frac{\delta}{\delta A^a_\mu} - \frac{\delta\Sigma}{\delta A^a_\mu}\frac{\delta}{\delta\Omega^a_\mu} + \frac{\delta\Sigma}{\delta\tau^a_\mu}\frac{\delta}{\delta\psi^a_\mu} + \left(\Omega^a_\mu + \frac{\delta\Sigma}{\delta\psi^a_\mu} \right) \frac{\delta}{\delta\tau^a_\mu} + \right. \\
&+ \left(\phi^a + \frac{\delta\Sigma}{\delta L^a} \right) \frac{\delta}{\delta c^a} + \frac{\delta\Sigma}{\delta c^a}\frac{\delta}{\delta L^a} + \frac{\delta\Sigma}{\delta E^a}\frac{\delta}{\delta\phi^a} + \left(L^a + \frac{\delta\Sigma}{\delta\phi^a} \right) \frac{\delta}{\delta E^a} + \\
&+ \left. b^a \frac{\delta}{\delta\bar{c}^a} + \bar{\eta}^a \frac{\delta}{\delta\bar{\phi}^a} + B^a_{\mu\nu}\frac{\delta}{\delta\bar{\chi}^a_{\mu\nu}} + K^a_{\mu\nu}\frac{\delta}{\delta\Lambda^a_{\mu\nu}} \right] . \qquad (5.46)
\end{aligned}
$$

Therefore, from (C.1) and (C.21), we conclude that a non-linear symmetry is transferred to the counterterm in its linearized version,

$$
S_\Sigma \Sigma^c = 0 \, . \qquad (5.47)
$$

The linear ones are obviously directly transferred to Σ^c. Hence, following the principle of quantum action summarized in (5.26), which relates the classical and quantum worlds, imposing (C.1), (5.27), (5.28), (5.31), (5.33), (5.35), (5.37), and (C.9) to Γ we find that the most general counterterm that can be added to the

[1] We call S_Σ the *linearized* Slavnov-Taylor identity because it is a linear operator, *i.e.*, $S_\Sigma(A + B + \cdots + C) = S_\Sigma A + S_\Sigma B + \cdots + S_\Sigma C$. Note that the Slavnov-Taylor operator is not linear: $S(\Gamma) \neq S(\Sigma) + \epsilon S(\Sigma^c)$.

classical action must obey, together with (C.14),

$$\frac{\delta\Sigma^c}{\delta b^a} = 0 \,, \tag{5.48}$$

$$\frac{\delta\Sigma^c}{\delta\bar{c}^a} - \partial_\mu\frac{\delta\Sigma^c}{\delta\Omega^a_\mu} = 0 \,, \tag{5.49}$$

$$\frac{\delta\Sigma^c}{\delta\bar{\eta}^a} = 0 \,, \tag{5.50}$$

$$\frac{\delta\Sigma^c}{\delta\bar{\phi}^a} - \partial_\mu\frac{\delta\Sigma^c}{\delta\tau^a_\mu} = 0 \,, \tag{5.51}$$

$$\mathcal{G}^a_\phi\Sigma^c = 0 \,, \tag{5.52}$$

$$\mathcal{G}^a_1\Sigma^c = 0 \,, \tag{5.53}$$

$$\mathcal{G}^a_2\Sigma^c = 0 \,, \tag{5.54}$$

$$\mathcal{W}_\mu\Sigma^c = 0 \,, \tag{5.55}$$

$$\mathcal{T}_\Sigma\Sigma^c = 0 \,, \tag{5.56}$$

$$\mathcal{G}_3\Sigma^c = 0 \,, \tag{5.57}$$

where $\mathcal{T}_\Sigma$ is the linear version of the operator (5.37), given by

$$\begin{aligned}\mathcal{T}_\Sigma &= \int d^4z\left[\frac{\delta\Sigma}{\delta\Omega^a_\mu}\frac{\delta}{\delta\psi^a_\mu} - \frac{\delta\Sigma}{\delta\psi^a_\mu}\frac{\delta}{\delta\Omega^a_\mu} - \frac{\delta\Sigma}{\delta L^a}\frac{\delta}{\delta\phi^a} - \frac{\delta\Sigma}{\delta\phi^a}\frac{\delta}{\delta L^a} + \frac{\delta\Sigma}{\delta K^a_{\mu\nu}}\frac{\delta}{\delta B^a_{\mu\nu}} + \frac{\delta\Sigma}{\delta B^a_{\mu\nu}}\frac{\delta}{\delta K^a_{\mu\nu}}\right.\\ &+ \left.(\bar{c}^a - \bar{\eta}^a)\left(\frac{\delta}{\delta\bar{\eta}^a} + \frac{\delta}{\delta\bar{c}^a}\right)\right] \,.\end{aligned} \tag{5.58}$$

The operator $\mathcal{S}_\Sigma$ is nilpotent,

$$S_\Sigma S_\Sigma = 0 \,, \tag{5.59}$$

and defines a cohomology in the space of fields, such that the constraint (C.14) represents a cohomology problem for Σ^c. Moreover, there is no room for gauge anomalies in the Slavnov-Taylor identity as the new set of sources introduced to control the non-linearities are only composed of doublets, see (5.11) and (5.13).

It means that the redefinitions (4.42) is enough to recover the subspace with trivial cohomology, cf. (4.45), and then, due to the isomorphism between this subspace and the whole space [123], we automatically infer that the cohomology of the theory is trivial. Hence, the Slavnov-Taylor identity is anomaly-free and the solution of (C.14) is of the form

$$\Sigma^c = \mathcal{S}_\Sigma \Delta^{(-1)} \,, \tag{5.60}$$

where $\Delta^{(-1)}$ is an integrated local polynomial in the fields and sources and their derivatives bounded by dimension four, and with ghost number -1. In principle, without imposing the Ward identities, the most general counterterm is

$$\begin{aligned}
\Sigma^c \;=\;& S_\Sigma \int d^4x \,\{ c_1 \bar{\chi}^a_{\mu\nu}\partial_\mu A^a_\nu + c_2 f^{abc}\bar{\chi}^a_{\mu\nu}A^b_\mu A^c_\nu + c_3\bar{\phi}^a\partial_\mu\psi^a_\mu + c_4 f^{abc}\bar{\phi}^a A^b_\mu \psi^c_\mu \\
&+ c_5\bar{c}^a\partial_\mu A^a_\mu + c_6\tau^a_\mu\partial_\mu c^a + c_7 f^{abc}\tau^a_\mu A^b_\mu c^c + c_8 f^{abc}E^a c^b c^c + c_9 E^a\phi^a + c_{10}L^a c^a \\
&+ c_{11}\Omega^a_\mu A^a_\mu + c_{12}\tau^a_\mu\psi^a_\mu + c_{13}b^a\bar{c}^a + c_{14}b^a\bar{\eta}^a + c_{15}\bar{\eta}^a\partial_\mu A^a_\mu + c_{16}\partial_\mu\bar{\phi}^a\partial_\mu c^a \\
&+ c_{17}B^a_{\mu\nu}\bar{\chi}^a_{\mu\nu} + c_{18}f^{abc}c^a\bar{\eta}^b\bar{\eta}^c + c_{19}f^{abc}c^a\bar{c}^b\bar{c}^c + c_{20}f^{abc}\bar{\phi}^a\partial_\mu c^b A^c_\mu \\
&+ c_{21}\bar{\phi}^a c^a A^b_\mu A^b_\mu + c_{22}\bar{\phi}^a c^b A^a_\mu A^b_\mu + c_{23}f^{abc}\bar{\phi}^a c^b b^c + c_{24}f^{abc}\bar{\eta}^a\bar{c}^b c^c \\
&+ c_{25}f^{abc}\bar{\phi}^a\phi^b\bar{\eta}^c + c_{26}f^{abc}\bar{\phi}^a\phi^b\bar{c}^c + c_{27}\bar{\phi}^a\phi^a\bar{\phi}^b c^b + c_{28}\bar{\phi}^a\phi^b\bar{\phi}^a c^b + c_{29}\bar{c}^a\bar{\phi}^b c^a c^b \\
&+ c_{30}\bar{\eta}^a c^a\bar{\phi}^b c^b + c_{31}\bar{\phi}^a\partial_\mu\psi^a_\mu + c_{32}f^{abc}\bar{\phi}^a c^b\partial_\mu A^c_\mu + c_{33}f^{abc}c^a\bar{\chi}^b_{\mu\nu}\bar{\chi}^c_{\mu\nu} + c_{34}\Lambda^a_{\mu\nu}K^a_{\mu\nu} \\
&+ c_{35}f^{abc}\Lambda^a\bar{\chi}^b c^c + c_{36}f^{abc}\Lambda^a_{\mu\nu}A^b_\mu A^c_\nu + c_{37}\Lambda^a_{\mu\nu}\partial_\mu A^a_\nu + c_{38}\Lambda^a_{\mu\nu}B^a_{\mu\nu} \\
&+ c_{39}K^a_{\mu\nu}\bar{\chi}^a_{\mu\nu} \;\} \,,
\end{aligned} \tag{5.61}$$

where c_i are arbitrary constants. Applying S_Σ and using the equations of motion, the constraints (C.14)-(5.55) imply that the counterterm above takes the form

$$\begin{aligned}
\Sigma^c \;=\;& \mathcal{S}_\Sigma \int d^4z \,\{ a_1 \left[\left(\Omega^a_\mu - \partial_\mu\bar{c}^a\right) A^a_\mu + \left(\tau^a_\mu - \partial_\mu\bar{\phi}^a\right)\psi^a_\mu \right] + a_2(\tau^a_\mu - \partial_\mu\bar{\phi}^a)\partial_\mu c^a \\
&+ a_3\bar{\chi}^a_{\mu\nu}\partial_\mu A^a_\nu + a_4 f^{abc}\bar{\chi}^a_{\mu\nu}A^b_\mu A^c_\nu \} \,,
\end{aligned} \tag{5.62}$$

where a_1, a_2, a_3 and a_4 are arbitrary constant coefficients, to be calculated by Feynman diagrams. Although the introduction of the new sources $K_{\mu\nu}$ and $\Lambda^a_{\mu\nu}$ to control the non-linearity of the new symmetry $\mathcal{T}$, the counterterm is the same as the one found in [53], in the presence of the vector supersymmetry constraint (5.55). Now, applying the bosonic symmetry constraint (5.56), one can straightforwardly show that

$$a_1 = a_2 = 0 \ , \tag{5.63}$$

and that

$$a_4 = \frac{a_3}{2} \ . \tag{5.64}$$

Hence, the most general local counterterm obeying the symmetry content of the model is reduced to the simple form

$$\Sigma^c = \mathcal{S}_\Sigma \int d^4z \; a \; \bar{\chi}^a_{\mu\nu} F^a_{\mu\nu} \ , \tag{5.65}$$

where the parameter a_4 was renamed as a: the only renormalization parameter allowed by the Ward identities of the model. Explicitly, the counterterm (5.65) reads

$$\Sigma^c = a \int d^4z \, \{ B^a_{\mu\nu} F^a_{\mu\nu} - 2\bar{\chi}^a_{\mu\nu} D^{ab}_\mu \psi^b_\nu - g f^{abc} \bar{\chi}^a_{\mu\nu} c^b F^c_{\mu\nu} \} \ . \tag{5.66}$$

As pointed out in [127], the choice of Landau gauges forbids the presence of the counterterm $\mathrm{Tr}\,(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu})^2$. An isolated Yang-Mills term, $\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}$, is also not produced at the quantum level. Such a result proves that the minima of the effective action still correspond to instanton configurations, in other words, that the topological structure of the vacuum is not destroyed at the quantum level. This is in agreement with previous one-loop computations carried out in [128]. As mentioned before, this agreement is not surprising as the calculation performed in [128] was based on the Batalin-Vilkovisky algorithm [130], which coincides to the BS approach for a particular configuration of Batalin-Vilkovisky auxiliary fields.

5.2.2 Quantum stability

Once we have at our disposal the most general counterterm consistent with all Ward identities, we must verify if the counterterm can absorb the divergences arising in the evaluation of Feynman graphs. In other words, if the counterterm (5.66) can be consistently absorbed by the classical action (5.15) by means of the multiplicative redefinition of the fields, sources and parameters of the model. Therefore, starting from the equation (5.44), we must show that Γ at one-loop is of the form $\Sigma(\Phi_0, \mathcal{J}_0, g_0)$, where

$$\Sigma(\Phi_0, \mathcal{J}_0, g_0) = \Sigma(\Phi, \mathcal{J}, g) + \epsilon\Sigma^c(\Phi, \mathcal{J}, g) \ , \tag{5.67}$$

whereby

$$\begin{aligned} \Phi_0 &= Z_\Phi^{1/2}\Phi \ , \quad \Phi_0 = \{A^a_\mu, \psi^a_\mu, c^a, \bar{c}^a, \phi^a, \bar{\phi}^a, b^a, \bar{\eta}^a, \bar{\chi}^a_{\mu\nu}, B^a_{\mu\nu}\} \ , \\ \mathcal{J}_0 &= Z_{\mathcal{J}}\mathcal{J} \ , \quad \mathcal{J} = \{\tau^a_\mu, \Omega^a_\mu, E^a, L^a, \Lambda^a_{\mu\nu}, K^a_{\mu\nu}\} \ , \\ g_0 &= Z_g g \ . \end{aligned} \tag{5.68}$$

Due to the recursive nature of algebraic renormalization theory [98], to impose the validity of the Ward identities to Γ at one-loop is equivalent to impose their validity to Γ at all orders in perturbation theory[1]. Therefore, replacing the final

[1]It is not difficult to visualize the recursive property of the Algebraic Renormalization. If we would like to extend the renormalized one-loop action $\Sigma(\Phi_0, \mathcal{J}_0, g_0)$ to the two-loops order, we would start with

$$\Gamma_{2-loops} = \Gamma_{1-loop} + \epsilon^2\Sigma^c_{2-loops} \ . \tag{5.69}$$

As the structure of $\Gamma_{1-loop} \equiv \Sigma(\Phi_0, \mathcal{J}_0, g_0)$ is identical of the classical action one, the Ward identities are the same for the renormalized fields, parameters and sources, and the form of $\Sigma^c_{2-loops}$ will be the same as Σ^c, with the new coefficients corresponding to the two-loops counterterm (5.66) in the stability condition given by eq. (5.67), a direct and straightforward analysis shows that the model is quantum stable, as the resulting Z factors obey the following system of equations:

$$
\begin{aligned}
Z_A^{1/2} &= Z_b^{-1/2} = Z_g^{-1} \, , \\
Z_{\bar{c}}^{1/2} &= Z_{\bar{\eta}}^{1/2} = Z_{\psi}^{-1/2} = Z_{\Omega} = Z_c^{-1/2} \, , \\
Z_{\bar{\phi}}^{1/2} &= Z_{\phi}^{-1/2} = Z_{\tau} = Z_L = Z_g^{-1} Z_c^{-1} \, , \\
Z_E &= Z_g^{-2} Z_c^{-3/2} \, , \\
Z_K &= Z_g^{-1} Z_c^{-1/2} Z_{\bar{\chi}}^{-1/2} \, , \\
Z_{\Lambda} &= Z_g^{-2} Z_c^{-1} Z_{\bar{\chi}}^{-1/2} \, , \\
Z_B^{1/2} Z_A^{1/2} &= Z_{\bar{\chi}}^{1/2} Z_c^{1/2} = 1 + \epsilon a \, .
\end{aligned} \tag{5.70}
$$

The results (5.70) are self consistent and show that the model is renormalizable to all orders in perturbation theory.

It is worth mentioning again that the Ward identities (C.1)-(C.9) hold at all orders with the classical action Σ replaced by the 1PI generating functional Γ. In addition, we would like to emphasize that the result (5.66) is a direct consequence of the absence of anomalies in the Slavnov-Taylor identity. The anomalous Slavnov-Taylor identity would give $S_{\Sigma}\Sigma^c = \triangle^{(1)}$, being $\triangle^{(1)}$ a local polynomial with ghost number 1; but the cohomology of the linearized BRST operator vanishes, which automatically restricts the most general counterterm of the theory to the trivial part of the cohomology. As a consequence of this triviality, the cohomology vanishes in any ghost number sector.

Feynman diagrams of the model, which shows that $\Sigma^c_{2-loops}$ can be absorbed into Γ_{1-loop}, proving the renormalizability at two-loops order. From two- to three-loops order, the process is identical, and so to all orders.

5.3 Consequences of the Ward identities for the two-point functions

In this section we provide some strong consequences of the Ward identities in terms of the two-point functions of the theory. Specifically, we compute exact properties[1] of the propagators and 1PI two-point functions. The conventions and notation here employed can be found in the Appendix A. Needless to say, since the theory is renormalizable to all orders in perturbation theory, the Ward identities are valid for the quantum action Γ and not only for the classical one Σ, as it was proved in the previous section.

First of all, we evoke the discrete Faddeev-Popov symmetry (dFPs) to recall that all two-point functions carrying a non-vanishing ghost-number vanish, namely,

$$\Gamma_{(\Phi^A\Phi^B)}(p) = \langle\Phi^A\Phi^B\rangle(p) = 0 \;\; \forall \;\; g_A + g_B \neq 0 \; . \tag{5.71}$$

Second, from Lorentz covariance it is easy to infer that we must have, for the (anti-)self-dual fields,

$$\langle b^a B^b_{\mu\nu}\rangle(p) = 0 \; , \tag{5.72}$$

$$\langle c^a \bar{\chi}^b_{\mu\nu}\rangle(p) = 0 \; , \tag{5.73}$$

and

$$\Gamma^{ab}_{(bB)\mu\nu}(p) = 0 \; , \tag{5.74}$$

$$\Gamma^{ab}_{(c\bar{\chi})\mu\nu}(p) = 0 \; . \tag{5.75}$$

[1]By exact we mean valid to all orders in perturbation theory. In most cases, this means tree-level exact, *i.e.*, all radiative corrections vanish.

5.3.1 1PI two-point functions

Since the Ward identities are written for the 1PI generating functional, it is easier to start with the 1PI two-point functions. All 1PI two-point functions obtained in this subsection are displayed in Table 5.2.

5.3.1.1 Consequences of the Landau gauge fixings

The ordinary Landau gauge fixing (5.27), in terms of the quantum action, is given by

$$\frac{\delta\Gamma}{\delta b^a(x)} = \partial^x_\mu A^a_\mu(x) \ , \tag{5.76}$$

where ∂^x_μ stands for the spacetime derivative with respect to the coordinates of the point x_μ. In the same way, the topological Landau gauge fixing (5.28) can be written as

$$\frac{\delta\Gamma}{\delta\bar{\eta}^a(x)} = \partial^x_\mu \psi^a_\mu(x) \ . \tag{5.77}$$

- *The bA mixed 1PI function.*

  To obtain the bA mixed 1PI function, we vary the equation (5.76) with respect to $A^b_\nu(y)$,

$$\frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta b^a(x)} = \delta^{ab}\partial^x_\nu\delta(x-y) \ . \tag{5.78}$$

  Hence,

$$\Gamma^{ab}_{(bA)\nu}(x,y) = \delta^{ab}\partial^x_\nu\delta(x-y) \ . \tag{5.79}$$

  Taking the Fourier transform of eq. (5.79) one obtains

$$\int \frac{d^4p}{(2\pi)^4}\Gamma^{ab}_{(bA)\mu}(p)e^{ip(x-y)} = \int \frac{d^4p}{(2\pi)^4}\delta^{ab} ip_\mu e^{ip(x-y)} \ . \tag{5.80}$$

Thus,

$$\Gamma^{ab}_{(bA)\mu}(p) = i\delta^{ab} p_\mu \ . \tag{5.81}$$

The mixed two-point vertex function (5.81) is tree-level exact, as expected from the relation $Z_b Z_A = 1$ in (5.70).

- *The bb 1PI function.*

In the same way, by varying (5.76) with respect to $b^b(y)$, one trivially finds

$$\Gamma^{ab}_{(bb)}(p) = 0 \ . \tag{5.82}$$

- *The $\bar{\eta}\psi$ mixed 1PI function.*

Now, varying the equation (5.77) with respect to $\psi^b_\nu(y)$ and Fourier transforming the resulting equation, one finds

$$\Gamma^{ab}_{(\bar{\eta}\psi)\mu}(p) = i\delta^{ab} p_\mu \ , \tag{5.83}$$

which is in accordance with the relation $Z_{\bar{\eta}} Z_\psi = 1$ in (5.70).

- *The $\bar{\eta}c$ mixed 1PI function.*

And, the variation of (5.77) with respect to $c^a(y)$ leads to

$$\Gamma^{ab}_{(\bar{\eta}c)}(p) = 0 \ . \tag{5.84}$$

5.3.1.2 Consequences of the vector supersymmetry

The vector supersymmetry (5.35), in terms of the 1PI generating functional, reads[1]

$$\int d^4z \left[\partial_\gamma A^c_\kappa \frac{\delta\Gamma}{\delta\psi^c_\kappa} + \partial_\gamma c^c \frac{\delta\Gamma}{\delta\phi^c} + \partial_\gamma \bar{\chi}^c_{\sigma\kappa} \frac{\delta\Gamma}{\delta B^c_{\sigma\kappa}} + \partial_\gamma \bar{\phi}^c \left(\frac{\delta\Gamma}{\delta\bar{\eta}^c} + \frac{\delta\Gamma}{\delta\bar{c}^c} \right) + \right.$$
$$\left. + \left(\partial_\gamma \bar{c}^c - \partial_\gamma \bar{\eta}^c\right) \frac{\delta\Gamma}{\delta b^c} + \ldots \right] = 0 \quad (5.85)$$

- *The BB 1PI function.*

  Varying (5.85) with respect to $B^b_{\alpha\beta}(y)$ and $\bar{\chi}^a_{\mu\nu}(x)$ we get

$$\int d^4z \left[\delta^{ac}\delta_{\mu\sigma}\delta_{\nu\kappa}\partial^z_\gamma \delta(z-x) \frac{\delta^2\Gamma}{\delta B^b_{\alpha\beta}(y)\delta B^c_{\sigma\kappa}(z)} + \ldots \right] = 0 \ . \tag{5.86}$$

  After integration over z, a Fourier transformation of (5.86) yields

$$p_\gamma \Gamma^{ab}_{(BB)\mu\nu\alpha\beta}(p) = 0 \ , \tag{5.87}$$

  which, by contraction with p_γ/p^2, simply reduces to

$$\Gamma^{ab}_{(BB)\mu\nu\alpha\beta}(p) = 0 \ . \tag{5.88}$$

- *The topological ghost and the BA 1PI functions.*

  In the same way, by varying with respect to $\bar{\chi}^a_{\alpha\beta}(x)$ and $A^b_\mu(y)$, one finds

$$-\int d^4z \left[\delta(z-y)\partial^z_\kappa \frac{\delta^2}{\delta\bar{\chi}^a_{\alpha\beta}(x)\delta\psi^b_\mu(z)} + \delta(z-x)\partial^z_\kappa \frac{\delta^2}{\delta A^b_\mu(y)\delta B^a_{\alpha\beta}(z)} + \ldots \right] = 0 \ . \tag{5.89}$$

[1]For simplicity, only the relevant terms are written in (5.85).

Hence,

$$-\partial_\kappa^y \Gamma^{ab}_{(\bar{\chi}\psi)\alpha\beta\mu}(x,y) + \partial_\kappa^x \Gamma^{ab}_{(BA)\alpha\beta\mu}(x,y) = 0 \ . \tag{5.90}$$

Fourier transforming this last equation (with attention to the point where the derivative is taken), one obtains

$$\Gamma^{ab}_{(\bar{\chi}\psi)\alpha\beta\mu}(p) = -\Gamma^{ab}_{(BA)\alpha\beta\mu}(p) \ . \tag{5.91}$$

The relation (5.91) is consistent with the relations (5.70) by means of $Z_B Z_A = Z_{\bar{\chi}} Z_\psi$. Moreover, it is easy to infer from the antisymmetry in α and β indices that they should be transverse,

$$\Gamma^{ab}_{(\bar{\chi}\psi)\alpha\beta\mu}(p) = -\Gamma^{ab}_{(BA)\alpha\beta\mu}(p) = X_1(p^2)\epsilon_{\alpha\beta\mu\nu}p_\nu + y(p^2)\left(\delta_{\alpha\mu}p_\beta - \delta_{\beta\mu}p_\alpha\right) \ , \tag{5.92}$$

where $X_1(p^2)$ and $y(p^2)$ are generic form factors.

- *The Faddeev-Popov and bosonic ghost 1PI functions.*

  Another consequence of the vector supersymmetry concerns the Faddeev-Popov ghost and the bosonic ghost 1PI two-point functions. By varying (5.85) with respect to $c^a(y)$ and $\bar{\phi}^b(x)$, one gets (the proof is very similar to the one displayed in the demonstration of (5.91))

$$\Gamma^{ab}_{(\bar{\phi}\phi)}(p) = \Gamma^{ab}_{(\bar{c}c)}(p) \ . \tag{5.93}$$

  where (5.84) was used. Expression (5.93) is in harmony with the relation $Z_{\bar{c}} Z_c = Z_{\bar{\phi}} Z_\phi$ in (5.70).

- *The $\bar{c}\psi$ mixed 1PI function.*

  In the same lines of (5.91) and (5.93), by varying (5.85) with respect to

$\phi^a(x)$ and $\psi^b_\mu(x)$, one can prove that

$$\Gamma^{ab}_{(\bar{c}\psi)\mu}(p) = -\Gamma^{ab}_{(\bar{\eta}\psi)\mu}(p) = -i\delta^{ab}p_\mu \ , \tag{5.94}$$

where (5.83) must be employed. The tree-level exactness (5.94) is in accordance with the relation $Z_{\bar{c}}Z_\psi = Z_{\bar{\eta}}Z_\psi = 1$ and the fact that $Z_\psi = Z_c$ and $Z_{\bar{\eta}} = Z_{\bar{c}}$, all coming from the relations (5.70).

- *The topological gluon 1PI function.*

  Now, we consider the topological gluon vacuum polarization $\Gamma^{ab}_{(AA)\mu\nu}(p)$. Remarkably, as can be verified in the App. B, it identically vanishes,

$$\Gamma^{ab}_{(AA)\mu\nu}(p) = 0 \ . \tag{5.95}$$

We will discuss this result in more details in Sec. 5.4.

$\downarrow \Phi^A$ \ $\Phi^B \rightarrow$	A^b_α	ψ^b_α	c^b	ϕ^b	$\bar{c}^b$	b^b	$\bar{\phi}^b$	$\bar{\eta}^b$	$\bar{\chi}^b_{\alpha\beta}$	$B^b_{\alpha\beta}$
A^a_μ	0	—	—	—	—	—	—	—	—	—
ψ^a_μ	0	0	—	—	—	—	—	—	—	—
c^a	0	0	0	—	—	—	—	—	—	—
ϕ^a	0	0	0	0	—	—	—	—	—	—
$\bar{c}^a$	0	$-i\delta^{ab}p_\alpha$	$\Gamma^{ab}_{(\bar{\phi}\phi)}$	0	0	—	—	—	—	—
b^a	$i\delta^{ab}p_\alpha$	0	0	0	0	0	—	—	—	—
$\bar{\phi}^a$	0	0	0	$\Gamma^{ab}_{(\bar{c}c)}$	0	0	0	—	—	—
$\bar{\eta}^a$	0	$i\delta^{ab}p_\alpha$	0	0	0	0	0	0	—	—
$\bar{\chi}^a_{\mu\nu}$	0	$-\Gamma^{ab}_{(BA)\mu\nu\alpha}$	0	0	0	0	0	0	0	—
$B^a_{\mu\nu}$	$-\Gamma^{ab}_{(\bar{\chi}\psi)\mu\nu\alpha}$	0	0	0	0	0	0	0	0	0

Table 5.2: Exact results for the two-point vertex functions $\Gamma^{AB}_{(\Phi\Phi)}(p)$. The traces — are redundancies since the table is (anti-)symmetric by the line-column exchange.

5.3.2 Propagators

Now we focus on the connected two-point functions. With this intent, we have to employ the Legendre transformation (A.3) in the Ward identities. All propagators obtained in this subsection are collected in Table 5.3.

5.3.2.1 Consequences of the Landau gauge fixings

The ordinary Landau gauge fixing equation (5.27), in terms of the connected Green functional, takes the form

$$-J^a_{(b)}(x) = \partial^x_\mu \frac{\delta W}{\delta J^a_{(A)\mu}(x)} \;, \tag{5.96}$$

while the topological gauge fixing equation (5.28) turns into

$$J^a_{(\bar\eta)}(x) = \partial^x_\mu \frac{\delta W}{\delta J^a_{(\psi)\mu}(x)} \;. \tag{5.97}$$

- *The bA mixed propagator.*

  Variation of equation (5.96) with respect to $J^b_{(b)}(y)$ leads to

$$\delta^{ab}\delta(x-y) = \partial^x_\mu \langle A^a_\mu(x) b^b(y)\rangle \;. \tag{5.98}$$

  This equation is easily solved in momentum space. Its Fourier transformation leads to

$$\delta^{ab}\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} = \partial^x_\mu \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \langle A^a_\mu b^b\rangle(p) \;, \tag{5.99}$$

  providing

$$\delta^{ab} = ip_\mu \langle A^a_\mu b^b\rangle(p) \;, \tag{5.100}$$

whose solution is

$$\langle b^a A^b_\mu \rangle(p) = i\delta^{ab}\frac{p_\mu}{p^2} \ . \tag{5.101}$$

This is in complete accordance with the relation $Z_b Z_A = 1$ in (5.70).

- *The BA mixed propagator.*

  The variation of equation (5.96) with respect to $J^b_{(B)\alpha\beta}(y)$ leads to the the transversality of $\langle B^a_{\alpha\beta} A^b_\mu \rangle(p)$, which is evident from the antisymmetry of its indices α and β. Hence, the BA propagator must be of the form

$$\langle B^a_{\alpha\beta} A^b_\mu \rangle(p) = B_1(p^2)\epsilon_{\alpha\beta\mu\nu}p_\nu + B_2(p^2)\left(\delta_{\alpha\mu}p_\beta - \delta_{\beta\mu}p_\alpha\right) \ , \tag{5.102}$$

  where $B_1(p^2)$ and $B_2(p^2)$ are generic form factors.

- *The $\bar{\eta}\psi$ mixed propagator.*

  Now, varying equation (5.97) with respect to $J^b_{(\psi)\nu}(y)$ and following the lines in the obtention of (5.101), we get

$$\langle \bar{\eta}^a \psi^b_\mu \rangle(p) = i\delta^{ab}\frac{p_\mu}{p^2} \ . \tag{5.103}$$

  The exact result (5.103) is consistent with $Z_{\bar{\eta}} Z_\psi = 1$ in (6.7).

- *The $\bar{c}\psi$ mixed propagator.*

  At last, by varying equation (5.97) with respect to $J^b_{(\bar{c})}(y)$, a transversality condition is gained (after Fourier transformation),

$$p_\mu \langle \bar{c}^a \psi^b_\mu \rangle(p) = 0 \ . \tag{5.104}$$

  However, from Lorentz covariance, the only possibility is that $\langle \bar{c}^a \psi^b_\mu \rangle(p) =$

$\delta^{ab}P(p^2)p_\mu$. Thus, inevitably, $P(p^2) = 0$, leading to

$$\langle \bar{c}^a \psi^b_\mu \rangle(p) = 0 \; . \tag{5.105}$$

5.3.2.2 Consequences of the vector supersymmetry

In terms of the connected Green functional the vector supersymmetry (5.35) reads

$$\int d^4z \left[\partial^z_\gamma \frac{\delta W}{\delta J^c_{(A)\kappa}(z)} J^c_{(\psi)\kappa}(z) - \partial^z_\gamma \frac{\delta W}{\delta J^c_{(c)}(z)} J^c_{(\phi)}(z) - \partial^z_\gamma \frac{\delta W}{\delta J^c_{(\bar{\chi})\kappa\sigma}(z)} J^c_{(B)\kappa\sigma}(z) + \right.$$
$$\left. + \; \partial^z_\gamma \frac{\delta W}{\delta J^c_{(\bar{\phi})}(z)} \left(J^c_{(\bar{\eta})}(z) + J^c_{(\bar{c})}(z) \right) - \partial^z_\gamma \left(\frac{\delta W}{\delta J^c_{(\bar{c})}(z)} - \frac{\delta W}{\delta J^c_{(\bar{n})}(z)} \right) J^c_{(b)}(z) + \ldots \right] = 0 \; . \tag{5.106}$$

- *The topological gluon propagator.*

  The topological gluon propagator is obtained by varying equation (5.106) with respect to $J^a_{(A)\mu}(x)$ and $J^a_{(\psi)\nu}(y)$,

$$\int d^4z \left[\partial^z_\gamma \frac{\delta^2 W}{\delta J^a_{(A)\mu}(x) \delta J^c_{(A)\kappa}(z)} \delta^{bc} \delta_{\nu\kappa} \delta(z-y) + \ldots \right] = 0 \; . \tag{5.107}$$

  Hence, after integration in z and a Fourier transformation, we get

$$p_\gamma \langle A^a_\mu A^b_\nu \rangle(p) = 0 \; . \tag{5.108}$$

  By contraction with p_γ/p^2, we obtain

$$\langle A^a_\mu A^b_\nu \rangle(p) = 0 \; . \tag{5.109}$$

  Thus, the topological gluon propagator vanishes just like the associated vacuum polarization (5.95). See Sec. 5.4 for extra discussions about this issue.

- *The Faddeev-Popov and bosonic ghost propagators.*

The relation between the Faddeev-Popov ghost propagator $\bar{c}c$ and the bosonic ghost propagator $\bar{\phi}\phi$ is obtained by varying equation (5.106) with respect to $J^a_{(\bar{c})}(x)$ and $J^b_{(\phi)}(y)$,

$$\int d^4z \left[-\partial^z_\gamma \frac{\delta^2 W}{\delta J^a_{(\bar{c})}(x) \delta J^c_{(c)}(z)} \delta^{cb} \delta(z-y) + \partial^z_\gamma \frac{\delta^2 W}{\delta J^b_{(\phi)}(y) \delta J^c_{(\bar{\phi})}(z)} \delta^{ca} \delta(z-x) + \ldots \right] = 0 \,, \tag{5.110}$$

which reduces to

$$\partial^y_\gamma \langle \bar{c}^a(x) c^b(y) \rangle + \partial^x_\gamma \langle \bar{\phi}^a(x) \phi^b(y) \rangle = 0 \,. \tag{5.111}$$

Thus, after a Fourier transformation, we get

$$\langle \bar{c}^a c^b \rangle(p) = \langle \bar{\phi}^a \phi^b \rangle(p) \,, \tag{5.112}$$

which confirms, once again, the relation $Z_{\bar{c}} Z_c = Z_{\bar{\phi}} Z_\phi$ in (5.70). We refer to Sec. 5.4 for the proof of the tree-level exactness of the ghost (Faddeev-Popov and bosonic) propagator.

- *The topological ghost and the mixed BA propagators.*

The topological ghost propagator $\langle \bar{\chi}\psi \rangle$ can be computed by varying (5.106) with respect to $J^a_{(\bar{\psi})\mu}(x)$ and $J^b_{(B)_{\alpha\beta}}(y)$,

$$\int d^4z \left[\partial^z_\gamma \frac{\delta^2 W}{\delta J^b_{(B)\alpha\beta}(y) \delta J^a_{(A)\mu}(z)} \delta(z-x) - \partial^z_\gamma \frac{\delta^2 W}{\delta J^a_{(\psi)\mu}(x) \delta J^b_{(\bar{\chi})\alpha\beta}(z)} \delta(z-y) + \ldots \right] = 0 \,, \tag{5.113}$$

which reduces to

$$\partial_\gamma^y \langle \bar{\chi}^b_{\alpha\beta}(y)\psi^a_\mu(x)\rangle - \partial^x_\gamma \langle A^a_\mu(x) B^b_{\alpha\beta}(y)\rangle = 0 \; . \tag{5.114}$$

After a Fourier transformation, we get

$$\langle \bar{\chi}^b_{\alpha\beta}\psi^a_\mu\rangle(p) = -\langle B^b_{\alpha\beta}A^a_\mu\rangle(p) \; . \tag{5.115}$$

The result (5.115) agrees with (5.91) and with $Z_{\bar{\chi}}Z_\psi = Z_B Z_A$ in (5.70).

- *The $\bar{\eta}c$ mixed propagator.*

  Following the same reasoning as before, we vary equation (5.106) with respect to $J^a_{(c)}(x)$ and $J^b_{(B)\alpha\beta}(y)$ and find that

$$\langle \bar{\eta}^a c^a\rangle(p) = \langle \bar{c}^a c^a\rangle(p) \; . \tag{5.116}$$

  This relation is consistent with $Z_{\bar{\eta}} = Z_{\bar{c}}$ in (5.70).

$\downarrow\ \Phi^A$ | $\Phi^B \rightarrow$	A^b_α	ψ^b_α	c^b	ϕ^b	$\bar{c}^b$	b^b	$\bar{\phi}^b$	$\bar{\eta}^b$	$\bar{\chi}^b_{\alpha\beta}$	$B^b_{\alpha\beta}$
A^a_μ	0	—	—	—	—	—	—	—	—	—
ψ^a_μ	0	0	—	—	—	—	—	—	—	—
c^a	0	0	0	—	—	—	—	—	—	—
ϕ^a	0	0	0	0	—	—	—	—	—	—
$\bar{c}^a$	0	0	$\langle\bar{\phi}^a\phi^b\rangle$	0	0	—	—	—	—	—
b^a	$i\delta^{ab}p_\alpha/p^2$	0	0	0	0	0	—	—	—	—
$\bar{\phi}^a$	0	0	0	$\langle\bar{c}^a c^b\rangle$	0	0	0	—	—	—
$\bar{\eta}^a$	0	$i\delta^{ab}p_\alpha/p^2$	$\langle\bar{c}^a c^b\rangle$	0	0	0	0	0	—	—
$\bar{\chi}^a_{\mu\nu}$	0	$-\langle B^a_{\mu\nu}A^b_\alpha\rangle$	0	0	0	0	0	0	0	—
$B^a_{\mu\nu}$	$-\langle\bar{\chi}^a_{\mu\nu}\psi^b_\alpha\rangle$	0	0	0	0	0	0	0	0	0

Table 5.3: Exact results for the propagators $\langle\Phi^A\Phi^B\rangle(p)$. The traces — are redundancies since the table is (anti-)symmetric by the line-column exchange.

5.4 Two-point function tree-level exactness

5.4.1 Few words about the topological gluon propagator

In the previous section, an exact proof of the vanishing of the gluon connected two-point function was worked out. In the present subsection, we compute the tree-level gluon propagator and show that its vanishing is very much related to the particular choice of (Landau-type) gauge we have employed. For this computation, we introduce two gauge parameters α and β through the following quadratic terms:

$$-\frac{\alpha}{2}\int d^4z\; b^a b^a \qquad \text{and} \qquad -\frac{\beta}{2}\int d^4z\; B^a_{\mu\nu}B^a_{\mu\nu}\,, \tag{5.117}$$

where the choice of signs was done in such a way that these gauge parameters are strictly non-negative. Hence, the terms that contribute to the tree-level topological gluon propagator are given by

$$\tilde{S} = \int d^4z \left[b^a\left(\partial_\mu A^a_\mu - \frac{\alpha}{2}b^a\right) + B^a_{\mu\nu}\left(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu} - \frac{\beta}{2}B^a_{\mu\nu}\right)\right]\,. \tag{5.118}$$

By integrating out the auxiliary fields (b, B), one obtains

$$\tilde{S} = \int d^4z \left[\frac{(\partial_\mu A^a_\mu)^2}{2\alpha} + \frac{(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu})^2}{2\beta}\right]\,. \tag{5.119}$$

Keeping just quadratic terms on A^a_μ leads to

$$\tilde{S}^{\text{quad}} = -\frac{1}{2\alpha}\int d^4z\; A^a_\mu\partial_\mu\partial_\nu A^a_\nu - \frac{2}{\beta}\int d^4z\left(A^a_\mu\partial^2 A^a_\mu - A^a_\mu\partial_\mu\partial_\nu A^a_\nu\right)\,, \tag{5.120}$$

which is expressed in momentum space as

$$\tilde{S}^{\text{quad}} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} A^a_\mu(p)\Delta^{ab}_{\mu\nu}A^b_\nu(-p)\,, \tag{5.121}$$

with

$$\Delta^{ab}_{\mu\nu} = \delta^{ab}\left[\frac{4}{\beta}p^2\delta_{\mu\nu} - \left(\frac{4}{\beta} - \frac{1}{\alpha}\right)p_\mu p_\nu\right]\,. \tag{5.122}$$

Consequently, the tree-level gluon propagator is

$$\langle A^a_\mu A^b_\nu\rangle_0(p) = \delta^{ab}\left[\frac{\beta}{4p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{\alpha}{p^2}\frac{p_\mu p_\nu}{p^2}\right]\,. \tag{5.123}$$

The gauge condition we have considered throughout this work corresponds to setting $\alpha = \beta = 0$. From eq. (5.123) it is clear that, for such a choice, the gluon propagator vanishes at the tree-level (and this property holds to all orders as proved in the last section). Therefore, this choice is extremely peculiar, since when writing the Feynman rules for this theory, every diagram with gluon lines vanishes. Nonetheless, one can easily see that with the appropriate choice of $\beta = 4$, the Yang-Mills term is recovered (see (5.118)). As it is well known, the presence of such term leads to deep relations between topological Yang-Mills theories quantized in a certain class of gauges and supersymmetric gauge theories, see [117].

5.4.2 Exactness of the Faddeev-Popov ghost two-point functions

In this subsection, we give a proof using Wick theorem that the Faddeev-Popov ghosts two-point function is tree-level exact. For this, we use the property defined by eq. (5.112). Hence, let us have a closer look at the $\langle\bar{\phi}^a(x)\phi^b(y)\rangle$. By definition,

$$\langle\bar{\phi}^a(x)\phi^b(y)\rangle = \int [\mathcal{D}\Phi]\,\bar{\phi}^a(x)\phi^b(y)\mathrm{e}^{-S_{gf}} = \int [\mathcal{D}\Phi]\,\bar{\phi}^a(x)\phi^b(y)\mathrm{e}^{-S_{int}}\mathrm{e}^{-S_{quad}}\,, \tag{5.124}$$

with Φ a shorthand notation for the complete set of fields of the theory (see App. A). The actions S_{quad} and S_{int} stand for the quadratic and interacting parts of S_{gf}, respectively. The interacting part of S_{gf} is schematically expressed as

$$S_{int} = \int d^4z\left[BAA + \bar{c}Ac + \bar{\chi}cA + \bar{\chi}cAA + \bar{\chi}A\psi + \bar{\phi}A\phi + \bar{\phi}c\psi\right]\,. \tag{5.125}$$

Therefore, eq. (5.124) is rewritten as

$$\begin{aligned}\langle\bar{\phi}^a(x)\phi^b(y)\rangle &= \int [\mathcal{D}\Phi]\,\bar{\phi}^a(x)\phi^b(y)\,\exp\left(-\int d^4z\,[BAA + \bar{c}Ac + \bar{\chi}cA + \bar{\chi}cAA +\right.\\ &+ \left.\bar{\chi}A\psi + \bar{\phi}A\phi + \bar{\phi}c\psi]\right)\mathrm{e}^{-S_{quad}}\,.\end{aligned} \tag{5.126}$$

As usual, one can expand the exponential for the interacting part, leading to

$$\begin{aligned}\langle\bar{\phi}^a(x)\phi^b(y)\rangle &= \langle\bar{\phi}^a(x)\phi^b(y)\rangle_0 - \int d^4z\langle\bar{\phi}^a(x)\phi^b(y)\,[BAA + \bar{c}Ac + \bar{\chi}cA + \bar{\chi}cAA +\\ &+ \bar{\chi}A\psi + \bar{\phi}A\phi + \bar{\phi}c\psi]_z\rangle_0 + \ldots\,.\end{aligned} \tag{5.127}$$

where $\langle\ldots\rangle_0$ means that the expectation value is taken with respect to the quadratic action. As it is apparent from Table 5.3, the only non-vanishing two-point function involving $(\bar{\phi},\phi)$ is $\langle\bar{\phi}\phi\rangle$. Therefore, we have to single out Wick contractions of ϕ with $\bar{\phi}$. Consequently, the first order correction to (5.124) is

$$\begin{aligned}&\int d^4z\langle\bar{\phi}^a(x)\phi^b(y)\left[BAA + \bar{c}Ac + \bar{\chi}cA + \bar{\chi}cAA + \bar{\chi}A\psi + \bar{\phi}A\phi + \bar{\phi}c\psi\right]_z\rangle_0 =\\ =&\int d^4z\langle\bar{\phi}^a(x)\phi^b(y)\left[\bar{\phi}A\phi + \bar{\phi}c\psi\right]_z\rangle_0 = 0\,,\end{aligned} \tag{5.128}$$

where we have kept just terms containing ϕ and $\bar{\phi}$ since the contraction with any other fields but those vanishes. Going to higher orders renders the insertion of $\bar{\phi}A\phi$ and $\bar{\phi}c\psi$ on integrated spacetime points. The analysis is divided in the following possibilities:

- We consider just $\bar{\phi}A\phi$ insertions. In this case, the number of $(\bar{\phi}, \phi)$ fields is even and is always possible to contract $(\bar{\phi}, \phi)$ in pairs. Nevertheless, for each factor $\bar{\phi}A\phi$ introduced, one also introduces an A field which must be contracted with some other field. In the interacting part, the only non-vanishing correlation function involving A is $\langle BA \rangle$. However, this introduces the term BAA containing two A fields and, at the end, one will have to contract A with some field different from B, which vanishes.

- We consider just $\bar{\phi}c\psi$ insertions. This leads to a mismatch on the pairing of $(\bar{\phi}, \phi)$ fields and gives zero automatically.

- We consider mixed insertions of $\bar{\phi}A\phi$ and $\bar{\phi}c\psi$. If the insertions are such that there is an odd number of $(\bar{\phi}, \phi)$ fields, then it gives zero. If not, one comes back to the first bullet.

The conclusion is that one ends up with the exact tree-level relation,

$$\langle \bar{c}^a(x) c^b(y) \rangle = \langle \bar{\phi}^a(x) \phi^b(y) \rangle = \langle \bar{\phi}^a(x) \phi^b(y) \rangle_0 \,. \tag{5.129}$$

Such an argument can be understood by computing the Feynman rules of the theory and noticing that there is no non-vanishing diagram except for the tree-level one for $\langle \bar{\phi}^a(x) \phi^b(y) \rangle$. It is important to emphasize that this is a consequence of the vanishing of the gluon propagator, a feature of the particular gauge choice used in this paper, as discussed in the previous subsection.

The explicit form of the tree-level Faddeev-Popov ghost propagator is easily computed from the gauge fixing action (6.2), providing

$$\langle \bar{c}^a c^b \rangle(p) = \langle \bar{\phi}^a \phi^b \rangle(p) = \delta^{ab} \frac{1}{p^2} \; . \tag{5.130}$$

For completeness, one can compute the 1PI two-point functions $\Gamma^{ab}_{(\bar{c}c)}(p)$ and $\Gamma^{ab}_{(\bar{\phi}\phi)}(p)$ from the identity

$$\sum_C \Gamma_{(\Phi_A \Phi_C)}(p) \langle \Phi_C \Phi_B \rangle(p) = -\delta_{AB} \; . \tag{5.131}$$

Choosing $\Phi_A = \bar{c}^a$ and $\Phi_B = \bar{c}^b$, one can straightforwardly find

$$\Gamma^{ab}_{(\bar{c}c)}(p) = \Gamma^{ab}_{(\bar{\phi}\phi)}(p) = \delta^{ab} p^2 \; , \tag{5.132}$$

where (5.93) was employed.

5.4.3 Exactness of the topological ghost two-point functions

As for the Faddeev-Popov ghosts, it is possible to prove that the topological ghosts $(\bar{\chi}, \psi)$ two-point function is tree-level exact. The proof goes in very strict analogy with the Faddeev-Popov ghosts case and, due to this, we will just mention the main points. To do it, we benefit from the relation (5.92) and compute $\langle B^b_{\alpha\beta}(x) A^a_\mu(y) \rangle$ instead. The only non-vanishing contracting involving the B field is with the gauge field A and vice-versa. Hence, looking at the form of the interaction action (5.125), one sees that the only insertions allowed are those with BAA. Therefore,

$$\begin{aligned}\langle B^b_{\alpha\beta}(x)A^a_\mu(y)\rangle &= \langle B^b_{\alpha\beta}(x)A^a_\mu(y)\rangle_0 - \int d^4z\langle B^b_{\alpha\beta}(x)A^a_\mu(y)(BAA)_z\rangle_0 + \\ &+ \frac{1}{2!}\int d^4zd^4w\langle B^b_{\alpha\beta}(x)A^a_\mu(y)(BAA)_z(BAA)_w\rangle_0 + \ldots \end{aligned} \quad (5.133)$$

As is easily seen in eq. (5.133), the number of A fields due to the insertions is always bigger than the number of B fields. Therefore, the gauge fields will have to be contracted with some other field rather than B, resulting in vanishing contributions. Again, this is a consequence of the simplifying properties of the gauge condition we have chosen. For the explicit form of the topological ghost tree-level propagator, we refer to [49].

In the same lines of the previous subsection, it is easy to show that the 1PI two-point functions $\Gamma^{ab}_{(\bar{\chi}\psi)\alpha\beta\mu}$ and $\Gamma^{ab}_{(BA)\alpha\beta\mu}$ are also tree-level exact. The proof follows by setting $\Phi_A = \bar{\chi}^a_{\alpha\beta}$ and $\Phi_A = \bar{\chi}^b_{\mu\nu}$ in (5.131) and employing the propagators derived in [49].

Henceforth, together with the results of the consequences of the Ward identities for the two-point functions, we conclude that all two-point functions of the present model are tree-level exact. Such a behavior suggests a general property of topological gauge theories. In particular, the vacuum polarization and the gauge field propagator vanish to all orders in perturbation theory, as a consequence of the vector supersymmetry in ASDL gauges. This fact will be pivotal to prove that, not only the two-point functions are tree-level exact, but any n-point Green function of the model in ASDL gauges also does not receive radiative corrections at the quantum level, thus explained by the topological off-shell BRST cohomology.

Chapter 6

Quantum properties of topological Yang-Mills theories II: Renormalization ambiguity and tree-level exactness

In this section we generalize the (A)SDL gauges by introducing two gauge parameters. The modification relies in altering the Landau gauge condition on the gauge field to the linear covariant gauges and the (anti-)self-duality condition of the field strength to a non-(anti-)self-dual one, see (6.1) below. The gauge condition for the topological ghost remains the Landau transverse condition. It turns out that this gauge is not generally renormalizable. Nevertheless, we show that if we consider the linear covariant gauges and the non-(anti-)self-dual gauge separately, these gauges are indeed renormalizable to all orders in perturbation theory. In both classes of gauges, the (A)SDL gauges is recovered by continuously setting the gauge parameters to zero. It is worth mentioning that the vector supersymmetry [53; 54] is not present in these new classes of gauges.

Beyond the renormalizability proof, we discuss the fact that the renormalization factors (the Z factors) display a kind of freedom in their solution. It seems to be that, in these classes of gauges, there is a universal property allowing two free Z factors. Such an ambiguity is not present in ordinary Yang-Mills theories. The origin of this freedom is also discussed and linked to the triviality of the BRST cohomology of topological Yang-Mills theories. Moreover, we use the gauge propagator as an example to show how some of the Z factors are irrelevant in the renormalization of such objects. Such an analysis will be useful to discuss later the β-function in topological gauge theories, and its relation with the gauge choices. In particular, the connection between the absence of radiative correction in the (A)SDL gauges and the vanishing of the β-function in the off-shell BS theory, accordingly to the Feynman diagrams structure in presence of the vector supersymmetry.

6.1 Generalized classes of renormalizable gauges

In order to generalize the (A)SDL gauges, for the gauge field we employ the linear covariant gauge condition; for the field strength, a non-(anti-)self-dual gauge condition is chosen and, for the topological ghost, we set the Landau gauge constraint, namely[1],

$$\begin{aligned} \partial_\mu A^a_\mu &= -\alpha b^a \,, \\ F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu} &= -\beta B^a_{\mu\nu} \,, \\ \partial_\mu \psi^a_\mu &= 0 \,, \end{aligned} \tag{6.1}$$

[1]It should be noted that α and β must be necessarily negative quantities. Otherwise, the Boltzmann factor would be with the wrong sign in the path integral.

where α and β are gauge parameters. We did not consider a similar modification of the transverse condition of the topological ghost ψ^a_μ, as it would not alter the classical behavior of the gauge propagator we are interested in. In this suitable generalization, the vector supersymmetry is recovered by setting α and β to zero. The complete gauge-fixing action in the gauge choices (5.4) takes the form

$$\begin{aligned}
S_{gf}(\alpha,\beta) &= s\int d^4z \left[\bar{c}^a\left(\partial_\mu A^a_\mu + \frac{\alpha}{2}b^a\right) + \frac{1}{2}\bar{\chi}^a_{\mu\nu}\left(F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu} + \frac{\beta}{2}B^a_{\mu\nu}\right) + \bar{\phi}^a\partial_\mu\psi^a_\mu\right] \\
&= \int d^4z \left[b^a(\partial_\mu A^a_\mu + \frac{\alpha}{2}b^a) + \frac{1}{2}B^a_{\mu\nu}\left(F^a_{\mu\nu} \pm \widetilde{F}^a_{\mu\nu} + \frac{\beta}{2}B^a_{\mu\nu}\right) + (\bar{\eta}^a - \bar{c}^a)\,\partial_\mu\psi^a_\mu\right. \\
&+ \bar{c}^a\partial_\mu D^{ab}_\mu c^b - \frac{1}{2}gf^{abc}\bar{\chi}^a_{\mu\nu}c^b\left(F^c_{\mu\nu} \pm \widetilde{F}^c_{\mu\nu}\right) - \bar{\chi}^a_{\mu\nu}\left(\delta_{\mu\alpha}\delta_{\nu\beta} \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\right)D^{ab}_\alpha\psi^b_\beta \\
&+ \left.\bar{\phi}^a\partial_\mu D^{ab}_\mu\phi^b + gf^{abc}\bar{\phi}^a\partial_\mu\left(c^b\psi^c_\mu\right)\right] . \qquad (6.2)
\end{aligned}$$

The full action is then

$$\Sigma(\alpha,\beta) = S_o[A] + S_{gf}(\alpha,\beta) + S_{ext} , \qquad (6.3)$$

being S_{ext} the same external action as in eq. (5.14).

By integrating out the auxiliary field $B^a_{\mu\nu}$, a Yang-Mills term is produced. It is worth noting that this is not a genuine Yang-Mills term because it is multiplied by a gauge parameter and is originated from a BRST variation, *i.e.*, it belongs to the trivial sector of the cohomology of s [48; 98]. The tree-level gauge propagator is easily computed,

$$\langle A^a_\mu A^b_\nu\rangle_0(p) = \delta^{ab}\left[\frac{\beta}{4p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{\alpha}{p^2}\frac{p_\mu p_\nu}{p^2}\right] , \qquad (6.4)$$

which is a pure gauge propagator, cf. (5.123). This is consistent with the fact that topological gauge theories carry only global physical degrees of freedom. In fact, the unphysical nature of the gauge field as a local object is even more appealing at the (A)SDL gauges, where $\alpha = \beta = 0$ and the gauge propagator (6.4) vanishes [53; 54]. This property, being a very peculiar result for this gauge choice, has a strong consequence: all connected n-point Green functions are tree-level exact, as we shall discuss in the next section.

The action (5.15) is not renormalizable in general. A long but straightforward computation leads to a counterterm that can not be absorbed by the classical action (5.15). For instance, a term of the form

$$gf^{abc}\frac{\delta\Sigma}{\delta\phi^a}c^b c^c \tag{6.5}$$

appears at the quantum level, which is not present in the original action $\Sigma(\alpha, \beta)$ — see the last paragraph in Appendix D. Nevertheless, there are three special cases of (6.3) in which all divergences can be absorbed: the case $\alpha = \beta = 0$ which is the (A)SDL gauges; the case $\alpha \neq 0$ and $\beta = 0$, called α-gauges, and the case $\alpha = 0$ and $\beta \neq 0$, called β-gauges. Let us start our discussion with the special case of the (A)SDL gauges ($\alpha = \beta = 0$), whose quantum properties were previously studied.

6.2 Renormalization ambiguity

In the case of the (A)SDL gauges, the action (6.3) reduces to $\Sigma_{(A)SDL} = \Sigma(\alpha = \beta = 0)$ given by (5.15). Due to the set of Ward identities in these gauges, the most general counterterm (5.66) can indeed be reabsorbed in the classical action $\Sigma(\alpha = \beta = 0)$ by means of multiplicative redefinition of the fields, sources and parameters according to (5.67) and (5.68), and the resulting Z factors obey the system of equations displayed in (5.70), with only one independent renormalization parameter. As carried out in previous sections, this system is self-consistent. However it is clearly undetermined because there are fifteen equations and seventeen fields, sources and parameters. It means that there are two free Z factors, characterizing an ambiguity in the renormalization of the theory. For instance, the system (5.70) in the way we have written it, can be completely fixed by suitably choosing Z_g and Z_c. We will return to this issue later on, and analyse the origin of such an ambiguity.

6.2.1 Quantum stability of α-gauges

Now, let us consider the case where $\beta = 0$ while keeping α arbitrary in the action (5.15), the α-gauges. The full action is now

$$\Sigma_\alpha = \Sigma|_{\beta=0} \ . \tag{6.6}$$

The proof of renormalizability is established in the Appendix C. It turns out that the most general counterterm is also given by (5.66). The α-gauges also show themselves to be stable by means of (5.70) supplemented by the renormalization factor of the gauge parameter α,

$$Z_\alpha^{1/2} = Z_g^{-1} \ . \tag{6.7}$$

In practice, this equation reveals the nature of the coupling constant. Its renormalization is directly associated to the renormalization of the gauge parameter α. We must keep in mind that both parameters were introduced at the same time by the gauge-fixing action, *i.e.*, in the trivial part of the BRST cohomology, which means that the coupling constant is also a non-physical gauge parameter of the model. In the end, we gain one more equation for the system of equation determining the Z factors but we also gain an extra Z factor, Z_α. Hence, the ambiguity remains.

6.2.2 Quantum stability of β-gauges

The third case we study is characterized by setting $\alpha = 0$ and maintaining an arbitrary β in the original action (5.15), the β-gauges. The full action is then

$$\Sigma_\beta = \Sigma|_{\alpha=0} \ . \tag{6.8}$$

This action is also renormalizable, as discussed in Appendix D, and the most general counterterm assumes the form (D.26). An interesting feature to be observed at this point (which also occurs at the (A)SDL and the α-gauges) is that the Faddeev-Popov term does not appear in the counterterm (D.26). This implies that $Z_g Z_A^{1/2} = 1$. Using this information in the terms $a_1 B\partial A$ and $a_2 gBAA$ of (D.26), one finds $a_2 = a_1/2$. Then, the counterterm (D.26) is simplified to

$$\begin{aligned}
\Sigma^c_\beta &= S_\Sigma \int d^4x \left(a\,\bar{\chi}^a_{\mu\nu} F^a_{\mu\nu} + \tilde{a}\beta\bar{\chi}^a_{\mu\nu} B^a_{\mu\nu}\right) \\
&= \int d^4x \left[a\left(B^a_{\mu\nu} F^a_{\mu\nu} - 2\bar{\chi}^a_{\mu\nu} D^{ab}_\mu \psi^b_\nu - g f^{abc} \bar{\chi}^a_{\mu\nu} c^b F^c_{\mu\nu}\right) + \frac{\tilde{a}}{2}\beta B^a_{\mu\nu} B^a_{\mu\nu}\right]
\end{aligned} \tag{6.9}$$

where we have renamed the renormalization constants as $a = a_1/2$ and $\tilde{a}/2 = a_4$. All relations between the Z factors can be straightforwardly found from (5.67), (5.68) and (6.9). The result preserves the system formed by (5.70), with the additional equation

$$Z_\beta Z_B = 1 + \epsilon\tilde{a} \ . \tag{6.10}$$

Again, an extra equation is gained together with an extra Z factor, Z_β. For this reason the ambiguity persists.

6.2.3 Discussing the Z factors system

In the previous section, we have discussed the algebraic renormalization properties in three classes of gauges, namely, the (A)SDL gauges, and the α- and β-gauges, respectively. In particular, the (A)SDL gauges can be obtained from the latter classes by continuous deformations, *i.e.*, $\alpha \to 0$ or $\beta \to 0$. In all cases the action is renormalizable to all orders in perturbation theory. However, the system of Z factors is, in all cases, undetermined. The number of equations n and the number of variables z (the Z factors) are related by $z = n + 2$ in all three cases. It seems that there is a kind of freedom in the choice of two of the Z factors. We will now discuss this ambiguity in more details.

6.2.3.1 Comparison with Yang-Mills theories

To understand more closely the origin of such ambiguities, we must observe that the set of symmetries in the gauges analyzed eliminates the kinetic term of the Faddeev-Popov ghost at the counterterms. Because of this, we get

$$Z_c Z_{\bar{c}} = 1 \; . \tag{6.11}$$

From the gauge-ghost vertex ($\bar{c}Ac$), which is also absent in the counterterm[1], and the relation (6.11), we achieve

$$Z_g Z_A^{1/2} = 1 \; . \tag{6.12}$$

The two relations (6.11) and (6.12) are decoupled, in other words, only by determining Z_c or $Z_{\bar{c}}$ we do not get any information about Z_g or Z_A. Nevertheless, the

[1] In Yang-Mills theories quantized at the Landau gauge this property is known as the *non-renormalization of the gluon-ghost vertex* [98]. The same result is obtained here for a more general class of gauges.

factor Z_A could be individually determined if the classical action had a kinetic term for the gauge field. In the usual Yang-Mills theory, where the term $F^a_{\mu\nu}F^a_{\mu\nu}$ is present, Z_A can be directly determined from the gauge field kinetic term. But in topological Yang-Mills theories there are no kinetic terms for the gauge field. By this fact, the determination of Z_A becomes impossible.

The same analysis we did for the Faddeev-Popov ghost terms can be performed for the bosonic ghost term, leading to

$$Z_{\bar{\phi}}Z_{\phi} = 1 \; . \tag{6.13}$$

From the $\bar{\phi}A\phi$ vertex we also obtain (6.12).

For any other interacting term including A, g also appears, making the combination gA or g^2A^2 to be irrelevant due to (6.12). Moreover, the mixed propagators encoding A also do not give any extra information. The analysis for the source terms also does not help (these terms always include an extra variable for each new relation between Zs.). Ultimately, one can infer that (6.11) and (6.12) are the main basic relations that could solve the puzzle. Essentially, we need two extra informations about the Z-factors which are not encoded in the system (5.70). It is not difficult to conclude that the absence of a Yang-Mills term in the original action is the origin of the ambiguity of the Z_A factor.

Another feature in the ordinary Yang-Mills theories (quantized in the Landau gauge) is that $Z_c = Z_{\bar{c}}$ which relies on the discrete symmetry

$$\begin{aligned} c^a &\longrightarrow \bar{c}^a \; , \\ \bar{c}^a &\longrightarrow -c^a \; . \end{aligned} \tag{6.14}$$

This condition, together with the Faddeev-Popov ghost kinetic term, are sufficient to determine Z_c and $Z_{\bar{c}}$. It is easy to see that the action (5.15) does not obey such a symmetry[1], which explains the second ambiguity. (In Witten quantization, such an ambiguity will not appear by this reasoning as the the Witten action contains discrete symmetries ensured by the time-reversal symmetry (6.14) in Landau gauge, together with

$$\begin{aligned}\phi &\rightarrow \bar{\phi}\,, \quad \bar{\phi} \rightarrow \phi\,,\\ \psi_\mu &\rightarrow \chi_\mu\,, \quad \chi_\mu \rightarrow \psi_\mu\,,\end{aligned} \tag{6.15}$$

whereby the components of χ_μ is defined as follows

$$\chi_0 \equiv \eta\,, \quad \chi_i \equiv \chi_{0i} = \frac{1}{2}\varepsilon_{ijk}\chi_{jk}\,, \tag{6.16}$$

implying a "particle-antiparticle" relationship between $\bar{c}$ and c, $\bar{\phi}$ and ϕ, and ψ_μ and χ_μ, as demonstrated in [49].)

In essence we can infer that the difference between YM theories and *on-shell* topological YM theories relies in their cohomology properties. The non-trivial character of YM cohomology enables extra equations to determine Z_A and Z_c. Moreover, a non-trivial cohomology implies on local physical degrees of freedom whose renormalization affect physical observables. Thus, a freedom in the choice of some renormalization factors could affect physical observables in catastrophic ways. On the other hand, the trivial nature of topological YM cohomology is associated with the fact that all local degrees of freedom are non-physical (see (6.4) for instance — the gauge field propagator is totally gauge dependent) and such kind of freedom in how some objects renormalize can be interpreted as a reflex of the cohomology triviality.

[1]It is instructive to observe that discrete symmetries between the other ghosts of topological Yang-Mills theories (ϕ^a and $\bar{\phi}^a$ and; ψ^a_μ and $\bar{\chi}^a_{\mu\nu}$) are also not present in (5.15).

6.2.3.2 Non-physical gauge field propagators

The ambiguity can also be understood by looking at the gauge field propagators. For instance, at the (A)SDL gauges, the gauge field propagator vanishes to all orders in perturbation theory [54]. We immediately find that we have a liberty to choose any Z_A we want: take $\langle A^a_\mu A^b_\nu\rangle_R$ as the dressed propagator and $\langle A^a_\mu A^b_\nu\rangle_0$ the bare one, thus,

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_R = Z_A\langle A^a_\mu(x)A^b_\nu(y)\rangle_0 = 0 \quad \Rightarrow \quad \langle A^a_\mu(x)A^b_\nu(y)\rangle_0 = 0, \tag{6.17}$$

independently of Z_A.

In the α-gauges we found that $Z_\alpha = Z_A$, see (5.70) and (6.7). The expression of the tree-level gluon propagator at the α-gauges is easily computed,

$$\langle A^a_\mu A^b_\nu\rangle_0(p) = \delta^{ab}\frac{\alpha}{p^2}\frac{p_\mu p_\nu}{p^2}\ . \tag{6.18}$$

Therefore, after the redefinitions of the fields and parameters and using (5.70) and (6.7), Z_A is canceled at both sides of (6.18). Again, we conclude that we have the liberty to choose any renormalization factor for the gauge field.

The β-gauges is no different from the previous cases. From (5.70) and (6.10) one obtains

$$Z_\beta = Z_A\left[1 + 2\epsilon\left(\tilde{a} - 2a\right)\right]\ . \tag{6.19}$$

Now, the tree-level gluon propagator takes the form

$$\langle A^a_\mu A^b_\nu\rangle_0(p) = \delta^{ab}\frac{\beta}{4p^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\ . \tag{6.20}$$

Once again, after the renormalizations, the factor Z_A is canceled at both sides of

(6.20).

From the gauge field propagator, the freedom in the choice of Z_A is clearly illustrated. As a consequence of the first equation in (5.70), *i.e.*, $Z_A = Z_g^{-\frac{1}{2}}$, this freedom is transmitted to the renormalization of the coupling parameter.

We may wonder if this renormalization ambiguity is not intrinsic to the topological YM theory, in other words, if there is an undiscovered Ward identity capable of defining the Z factors system[1], or an operation capable of recovering the Yang-Mills discrete symmetries without destroying the Ward identities, but this is not necessary in (A)SDL gauges. Despite the absence of discrete symmetries of the type (6.14), we will prove that the impositions

$$Z_c = Z_{\bar{c}} = 1 \quad \text{and} \quad Z_\phi = Z_{\bar{\phi}} = 1 \tag{6.21}$$

are consistent with the model in this particular case — and therefore, with a vanishing β-function, see (5.70) — due to the impossibility of closing loops in the Feynman diagrams. In any case, we could question if the assumption (6.21) being consistent with a model with a vanishing β-function is, in fact, a consequence of taking $Z_A = 1$ as a freedom of the theory; automatically, from (5.70), $Z_g = 1$ as well, and (6.21) is also obtained. But this specific choice, at a first moment, seems to be artificial for a generic gauge, as it would impose the tree-level exactness of the gauge propagator (and, consequently, of the FP and bosonic ghost ones), which is a particular consequence of the vector supersymmetry in (A)SDL gauges.

[1]We point out the difficult of finding such a Ward identity that could relate some Z factors. In fact, even if a new Ward identity eliminates the last renormalization parameter, a, see (5.66), the ambiguity will remain. It strongly suggests that the absence of discrete symmetries lie in the origin of the renormalization ambiguity.

6.3 Absence of radiative corrections

We will prove that all connected n-point Green functions of four-dimensional topological Yang-Mills theories in the Baulieu-Singer approach, quantized in the (anti-)self-dual Landau gauges are tree-level exact, *i.e.*, that the theory does not possess radiative corrections in this gauge choice, see [56], as a consequence of the topological off-shell BRST cohomology, and the Ward identities of the model, in particular, of the vector supersymmetry which ensures that the gauge field propagator vanishes to all orders in perturbation theory.

6.3.1 Feynman rules

In the following, we collect the Feynman rules derived from the full action in (A)SDL gauges (5.15). The relevant propagators are represented by[1]

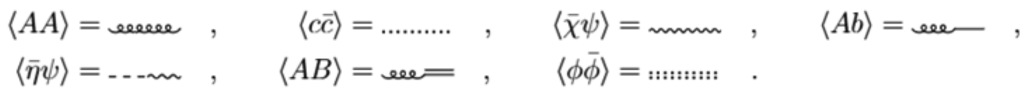

Figure 6.1: Propagators in (A)SDL gauges.

The relevant vertices are represented by:

In principle we do not have to include the gauge propagator in Fig. 6.1 — which is null — but this will be necessary to visualize the tree-level exactness of the theory, since such a propagator, as discussed later on, is required to close loops, leading to vanishing diagrams at the quantum level.

[1]From (6.2), a $\bar{c}\psi$ mixed propagator also seems to be relevant. However, this term can easily be eliminated by a trivial-Jacobian redefinition of the $\bar{\eta}$ field given by $\bar{\eta} \to \bar{\eta} + \bar{c}$.

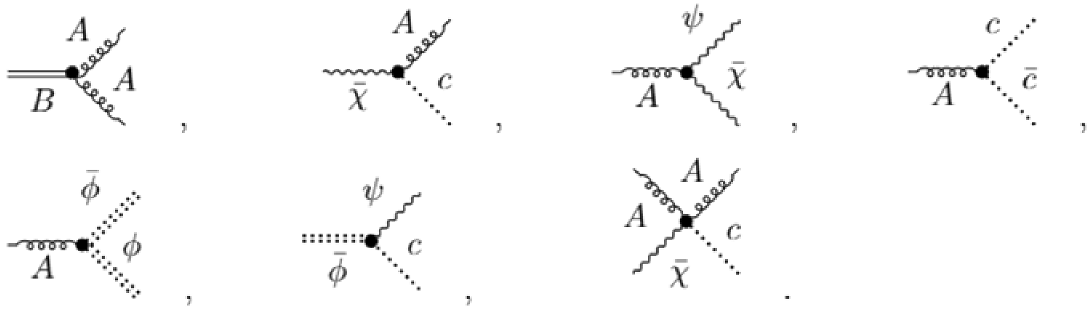

Figure 6.2: Vertices in (A)SDL gauges.

6.3.2 Feynman diagram structures and tree-level exactness

To show that the action (5.15) defines a theory free of radiative corrections, it is convenient to split the argumentation into propositions.

Proposition 1 *Any connected loop diagram containing an internal A-leg vanishes unless the branch generated by the A-leg ends up in external B- or b-legs.*

Proof. To prove this proposition, we must consider a combination of two facts: 1) $\langle AA\rangle = 0$ to all orders and 2) the gauge field only propagates through the non-vanishing mixed propagators $\langle BA\rangle$ and $\langle bA\rangle$. Hence, from an internal A-leg arising from an arbitrary vertex, denoted by a black dot, we only have two possibilities:

Figure 6.3: Internal gauge field propagation.

In the same way, the fields B and b only propagate through A. Graphically, we now have

Figure 6.4: Internal propagation of the B and b fields.

Nonetheless, the former is not at our disposal since there is no vertex containing b, *vide* Fig. 6.2. The latter, on the other hand, must be a BAA vertex since it is the only one containing B. Thus, an internal A-leg in any loop diagram will propagate to B and the latter will end up in a BAA vertex,

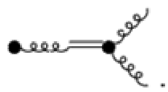

Figure 6.5: Propagation of the gauge field to the BAA vertex.

Applying the above reasoning for the two newly created A-legs, we end up with two more BAA vertices and four A-legs. Since the number of A-legs only increases, we can continue this process *ad infinitum* leading to a cascade effect of exponential proliferation of A-legs:

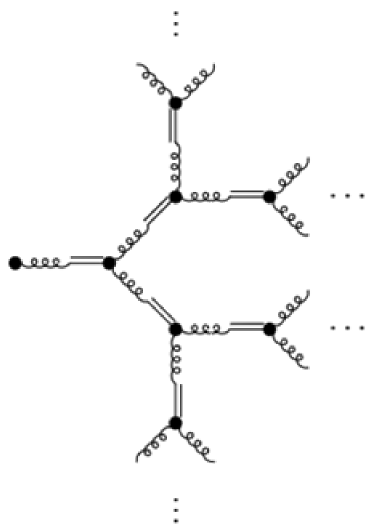

Figure 6.6: Cascade effect.

There are three possibilities here: 1) trying to close a loop in the diagram in Fig. 6.6 requires an $\langle AA \rangle$ internal propagator, which would result in a vanishing diagram; 2) to consider external A-legs, which also requires a $\langle AA \rangle$ propagator, resulting in a vanishing diagram and; 3) one could consider that all remaining A-legs end up in external B- or b-legs. QED.

We should note that all vertices, except one, present in (5.15) contain at least one A-leg, therefore the cascade effect always occur for these cases. The only exception is the vertex $\bar{\phi} c \psi$.

Corollary 1.1 *In a connected loop diagram, any branch arising from the vertex $\bar{\phi} c \psi$ results in a vanishing diagram unless this branch ends up in external B- or b-legs.*

Proof. Let us start with the vertex of interest, *i.e.*, $\bar{\phi} c \psi$. To construct a loop diagram from this three-vertex we have to propagate it to another vertex. The $\bar{\phi}$-leg could only propagate to the vertex $\bar{\phi} A \phi$ through $\langle \bar{\phi} \phi \rangle$; the c-leg only to $\bar{c} A c$ through $\langle \bar{c} c \rangle$ and; the ψ-leg to the vertexes $\bar{\chi} A \psi$, $\bar{\chi} c A$ or $\bar{\chi} c A A$ through $\langle \psi \bar{\chi} \rangle$ ($\langle \bar{\eta} \psi \rangle$ is not considered because there is no vertex containing $\bar{\eta}$). Graphically, the possibilities of completing the legs arising from this vertex are

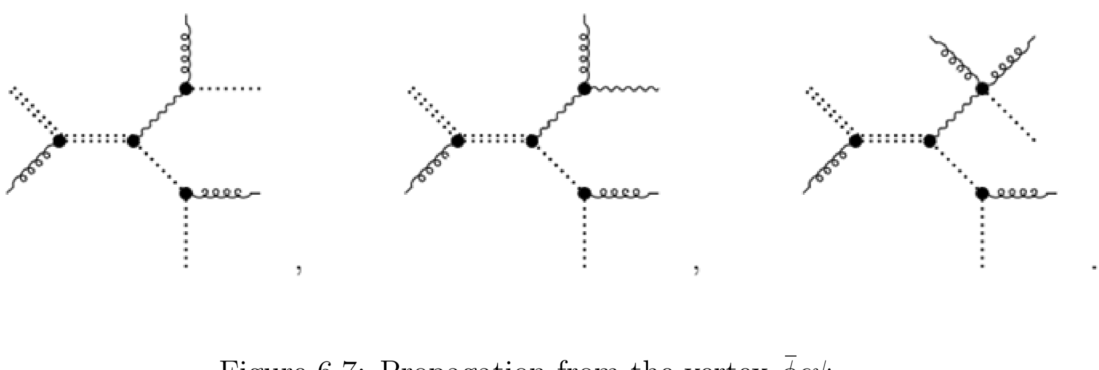

Figure 6.7: Propagation from the vertex $\bar{\phi} c \psi$.

But all possible branches contain at least one remaining A-leg. By evoking Proposition 1, the proof is completed. QED.

Corollary 1.2 *Any connected loop diagram containing a* $(\Phi_i \neq \{B, b\})$*-external leg vanishes.*

Proof. There are two steps toward this proof: 1) consider the external leg joined to a vertex containing an A field. In this case, A is an internal leg. Thus, Proposition 1 takes place and the graph either vanishes or generates a branch with external B- or b-legs and no loop can be constructed; 2) now, consider the external leg joined to a vertex not containing A, *i.e.* the vertex $\bar{\phi}c\psi$. The field $\bar{\phi}$ only propagates through $\langle\bar{\phi}\phi\rangle$, c through $\langle\bar{c}c\rangle$, and ψ only through $\langle\bar{\chi}\psi\rangle$ or $\langle\bar{\eta}\psi\rangle$. For this reason, it is impossible to propagate the vertex $\bar{\phi}c\psi$ to another vertex $\bar{\phi}c\psi$. In other words, from the vertex $\bar{\phi}c\psi$, we should necessarily propagate it to the vertexes containing an A field. Now, Corollary 1.1 takes place and the graph, again, either vanishes or generates a branch with external B- or b-legs and no loop can be constructed. QED.

Proposition 2 *Any connected* n*-point Green function composed of* B *and* b *fields of the form* $\langle B(x_1)B(x_2)...b(x_{n-1})b(x_n)\rangle$ *vanishes.*

Proof. Due to (4.28), and the fact that expectation values of any BRST-exact terms vanish. One can write these n-functions as BRST-exact correlators, namely

$$\langle BBB\ldots bb\rangle = \langle s\bar{\chi}BB\ldots bb\rangle = \langle s(\bar{\chi}BB\ldots bb)\rangle = 0\ , \tag{6.22}$$

and

$$\langle BBB\ldots bb\rangle = \langle BB\ldots s\bar{c}b\rangle = \langle s(BBB\ldots \bar{c}bb)\rangle = 0\ , \tag{6.23}$$

which vanish due to BRST-invariance. QED.

Proposition 3 *All connected n-point Green functions are tree-level exact.*

Proof. Let us take a connected loop diagram with n external legs with arbitrary fields Φ_i. From Corollary 1.2, if there is at least one field different from B or b, the graph either vanishes or is a tree-level graph. Then, there remains the possibility of a graph with n external legs formed by B or b fields. In this case Proposition 8.65 takes over and the Green function $\langle BB \ldots bb \rangle$ vanishes, meaning that this Green function is zero and receive no radiative corrections. Hence, all connected n-point Green functions are tree-level exact. QED.

In a few words, we conclude that *all connected n-point Green functions of four-dimensional topological gauge theories quantized in the (anti-)self-dual Landau gauges are tree-level exact.* This means that, in this gauge, the theory remains "classical" because there are no radiative corrections to be considered. This is a very interesting, yet subtle, result. The subtlety lives on the fact that the theory is not finite (so far, there is a non-trivial counterterm to be included in order to absorb the divergences of the theory, cf. eq. (5.66)) but the divergences are canceled out due to the vanishing of the gauge propagator which is always needed in order to close a loop diagram or due to the BRST symmetry.

6.4 β-functions in topological gauge theories

As the the topological Baulieu-Singer theory does not receive quantum corrections in (A)SDL gauges, we conclude that there is no running of the coupling constant, *i.e.*, that the β-function vanishes in this gauge,

$$Z_g = 1 \quad \text{or} \quad \beta_g^{(A)SDL} = 0 \;, \tag{6.24}$$

in absolute agreement with the system of Z-factors displayed in (5.70). The vanishing of the β-function in this case implies that $Z_A = 1$, and that $Z_c = Z_{\bar{c}} = 1$ and $Z_\phi = Z_{\bar{\phi}} = 1$, despite the absence of discrete symmetries between c and $\bar{c}$, and $\bar{\phi}$ and ϕ, which could enforce such relations from the beginning.

This result is completely different of the twisted $N = 2$ SYM one, which possesses a non-vanishing β-function proportional to g^3, see (3.77), as computed in [49] at one-loop, and proved to all orders in [47]. . We conclude that the *off-shell* Baulieu-Singer approach and twisted $N = 2$ SYM only possess equivalent β-functions if we take $g \to 0$ in the $N = 2$ side. This is in complete agreement with the fact that the observables of the *off-shell* Baulieu-Singer theory are only identical to the *on-shell* Witten ones in the weak coupling limit of the twisted $N = 2$, given by the Donaldson polynomials. The BS theory does not have the power to reproduce the $N = 2$ observables in the strong limit. The difference between the BS and Witten actions does not belong to the trivial part of the BRST cohomology. It proves that Brooks *et al.* BRST construction [49], which exactly recovers the Witten action, represents a distinct quantization scheme, where the complete BRST transformation cannot be reduced to a doublet subspace with trivial cohomology like in the BS approach (in the beginning of Sec. 4.2 we have discussed this point. We have shown, for instance, that in Witten quantization $(\bar{\phi}, \eta)$ is not a BRST doublet). This serves to elucidate the different behavior of the β-function of each theory, unless we go to the regime $g \to 0$ in the $N = 2$ side, where the observables of each theory are identical.

The most intriguing result is the one obtained by Birmingham *et al.* in [128], where the Batalin-Vilkovisky algorithm [130] had been employed. It configures a similar quantization to BS approach. As mentioned before, for a particular configuration of Batalin-Vilkovisky auxiliary fields, this scheme is identical to the one worked out in [127], *i.e.*, for the BS approach in the gauge $D^{ab}_\mu \psi^b_\mu = 0$ (the other gauges are the same as the (A)SDL ones). The cohomological properties of both approaches are identical. This gauge choice for the topological ghost, *i.e.*, with the covariant derivative instead of the ordinary one, breaks the vector supersymmetry, and consequently the gauge propagator does not vanishes to all orders anymore. This allows for quantum corrections, and for the possibility of a non-vanishing β-function. In fact, Birmingham *et al.* computed the one-loop correction for $\mathrm{Tr}\,(F \pm \widetilde{F})^2$, and proved that the correction possesses the same value of the ordinary Yang-Mills one, corresponding then to a non-vanishing β-function. Accordingly to the cohomology of the model, that protects the original topological structure of the classical action, they found that it is $\mathrm{Tr}\,(F \pm \widetilde{F})^2$ rather than $\mathrm{Tr}\,F^2$ which is renormalized. In this way, the minima of the effective action preserves the instanton configuration at the quantum level, (the same occurs for the BS approach in the β-gauges, as demonstrated by the counterterm (6.9).)

On the other hand, Brooks *et al.* [49] claimed that only a counterterm for $\mathrm{Tr}\,F^2$ was required. We must discard from the beginning the existence of a gauge anomaly in the BV (or BS) approaches in order to explain such a discrepancy, since it is forbidden in these models due to the trivial BRST cohomology, that makes it impossible to build an appropriate term that satisfies the Weiss-Zumino consistency conditions [128], cf. equation (4.39). The correct explanation must be based on the fact that BV (or BS) theory are quantically distinct of Brooks *et al.* construction (or Witten theory), as their methods are based on different BRST quantization schemes, with different cohomological properties.

The discrepancy between the β-functions of Birmingham *et al.* and the BS approach in (A)SDL gauges follows the same argumentation. It cannot be attributed to a gauge anomaly. However, the cohomological nature of both are similar, and we face the apparently contradictory explanation of attributing the discrepancy to gauge artefacts. In non-topological Yang-Mills theories, the β-function is an on-shell gauge-invariant physical quantity. Nonetheless, in gauge-fixed BRST topological theories of BS type, *the coupling constant is a non-physical gauge parameter*, introduced in the trivial part of the cohomology, together with the gauge-fixing action. In these terms, it is not contradictory that the β-function is gauge dependent as it computes the running of a non-physical gauge parameter. We must observe that the physical observables of the theory, the Donaldson invariants, naturally do not depend on the gauge coupling. There is no n-point local Green function that depends on g, but only global observables which are characterized in function of the target manifold, and that only depend on the spacetime global structure. So that there is no inconsistency that the observables of this kind of theory, described by topological invariants, *i.e.*, exact numbers, do not depend on the coupling constant, and consequently on its running, being g only a gauge parameter, and β_g, an unobservable gauge-dependent quantity.

Chapter 7

Gribov problem in Yang-Mills theories: Overview

The Gribov copies are ambiguities present in the Yang-Mills theories in which double counts of equivalent field configurations are not eliminated by the usual Faddeev-Popov gauge-fixing procedure. Such ambiguities, originally proposed by Vladimir Gribov in 1978 [42], brought light to the problem of color confinement in non-Abelian theories. The method for eliminating these ambiguities modifies the infrared (IR) behavior of the theory from the introduction of a restriction on the Feynman path integral whose integration over the fields configurations is now limited to a given region — the first Gribov region [137; 138], for which the Faddeev-Popov determinants are positive — where the copies are avoided. In the Abelian theories, like Quantum Electrodynamics, these copies or ambiguities are not relevant, as the copy equation only possesses trivial solutions in the thermodynamic limit.

The usual method for introducing the condition that promotes the elimination of gauge copies is accomplished by the Gribov-Zwanziger action [58; 139; 140]. It is a non-perturbative method imposed to all orders of perturbation theory.

In any case the imposition of this condition is only capable of eliminating the infinitesimal copies contained in the region of low energies. Moreover, the copies do not affect the ultraviolet region, preserving the asymptotic freedom. As a result this imposition generates non-local interacting terms beyond a quadratic term for the gluon in the IR which originates a mass parameter in the gluon propagator — related to the mass gap problem. In the presence of scalar and gluon condensates, the so-called Refined Gribov-Zwanziger (RGZ) action provides a gluon propagator in harmony with lattice simulations [59].

7.1 Faddeev-Popov gauge-fixing procedure

The starting point of Faddeev-Popov quantization is the functional generalization of the ordinary delta function of a real-valued and continuously differentiable function, $f(x)$, which is given by the expression

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \tag{7.1}$$

being x_i the roots of $f(x)$, $f(x_i) = 0$, and $|f'(x)|$ the Jacobian, where we have assumed that $f' \neq 0$ everywhere. By integrating (7.1), one obtain the following expression for the unit:

$$\frac{1}{\sum_i \frac{1}{|f'(x_i)|}} \int dx\, \delta(f(x)) = 1 \ . \tag{7.2}$$

In order to obtain a similar structure of the one used in the Yang-Mills case, it is useful to construct a two-dimensional toy model in polar coordinates $(\vec{r}, \theta)$, see [141], from which we can rewrite (7.2) in a gauge orbit, assuming that the system in invariant under a rotation ϕ, in the form

$$|\frac{\partial \mathcal{F}(\vec{r},\theta,\phi)}{\partial \phi}|_{\mathcal{F}(\vec{r},\theta,\phi)=0} \int d\phi\, \delta(\mathcal{F}(\vec{r},\theta,\phi)) = 1 \;, \tag{7.3}$$

where in $\mathcal{F}(\vec{r},\theta,\phi)$ denotes the function that intersects each orbit, characterized by a gauge transformation $\theta \to \theta(\phi)$, being the angle ϕ the *gauge parameter* of the symmetry. To obtain the expression above, we also consider that $\mathcal{F}$ intersects each orbit only once (for this reason we eliminated the sum over the roots, as we assumed only one root $\phi_i = \phi$).

The trick to perform the path integral over only one representative of each gauge orbit consists of introducing the unit (7.3) in the partition function

$$Z = N \int_0^{2\pi} d\theta \int_0^{\infty} rdr\, e^{-S(r)} \;, \tag{7.4}$$

where N is the normalization factor, and $S(r)$ the action invariant under rotations, resulting in

$$Z = N \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} rdr\, \triangle_{\mathcal{F}}(r)\delta(\mathcal{F}(\vec{r},\theta,\phi))\, e^{-S(r)} \;, \tag{7.5}$$

where we call

$$\triangle_{\mathcal{F}}(r) \equiv |\frac{\partial \mathcal{F}(\vec{r},\theta,\phi)}{\partial \phi}|_{\mathcal{F}(\vec{r},\theta,\phi)=0} \;, \tag{7.6}$$

as the Jacobian is taken with respect to the gauge parameter ϕ, and only depends on r. Now we can take the inverse transformation $\theta(\phi) \to \theta$ to eliminate the ϕ dependence in $\delta(\mathcal{F}(\vec{r},\theta,\phi))$. As the action is invariant under ϕ-rotations, it remains with the same argument, and we get

$$Z = N \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \int_0^{\infty} rdr\, \triangle_{\mathcal{F}}(r)\delta(\mathcal{F}(\vec{r},\theta))\, e^{-S(r)} \;. \tag{7.7}$$

As we can see, the dependence on ϕ was completely eliminated, so that we are able to perform the integration over ϕ, which only gives a *volume factor* 2π that can be absorbed in the normalization factor. In the end,

$$Z = N' \int_0^{2\pi} d\theta \int_0^{\infty} r dr \, \triangle_{\mathcal{F}} (r) \delta(\mathcal{F}(\vec{r}, \theta)) \, e^{-S(r)} \; , \tag{7.8}$$

with $N' = 2\pi N$. We conclude that *the insertion of the unit yields a partition function whose integration is evaluated over only one representative of each gauge orbit, independently of the gauge parameter, up to a volume factor that can be absorbed in the normalization factor.*

Yang-Mills case. The functional generalization of the unit (7.3) for a system with $N^2 - 1$ colors and infinite spacetime coordinates is given by

$$\triangle_{\mathcal{F}} \int \mathcal{D}U \, \delta(\mathcal{F}(A^U)) = 1 \; , \tag{7.9}$$

wherein we are using the notation

$$\delta(\mathcal{F}(A^U)) \equiv \prod_x \prod_a \delta(\mathcal{F}^a(A^U_\mu(x))) \; , \quad \text{and} \quad \mathcal{D}U \equiv \prod_x \prod_a d\theta^a(x) \; , \tag{7.10}$$

being $\theta^a(x)$ the local gauge parameters of the non-Abelian symmetry $U = e^{-igT^a\theta^a(x)}$, $U \in SU(N)$, and A^U_μ the gauge transformed field, cf. eq. (2.6),

$$A^U_\mu = U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U) U^\dagger \; , \tag{7.11}$$

which defines a *gauge orbit* of fields, *i.e.*, a class of gauge field configurations that only differ by a gauge transformation, that represents the same physics according to the gauge invariance of the Yang-Mills action under $SU(N)$ transformations. Moreover, as it is a multivariable system, the Jacobian is given by the absolute value of the Faddeev-Popov determinant,

$$\triangle_{\mathcal{F}}(A) = |\det \mathcal{M}^{ab}(x,y)| \ , \tag{7.12}$$

where

$$\mathcal{M}^{ab}(x,y) \equiv \frac{\delta \mathcal{F}^a(A^U_\mu(x))}{\delta\theta^b(y)}|_{\mathcal{F}(A^U)=0} \ . \tag{7.13}$$

Therefore, inserting the unit (7.9) in the Yang-Mills partition function, we get

$$Z_{YM} = \mathcal{N}\int \mathcal{D}U \int \mathcal{D}A \, \triangle_{\mathcal{F}}\,(A)\delta(\mathcal{F}(A^U))e^{-S_{YM}} \ . \tag{7.14}$$

Similar to the two-dimensional toy model, we perform an *inverse* gauge transformation to relate A^U_μ to A_μ, which can be done by taking the complex conjugate of (7.11) and isolating A_μ, in such a way that A^U_μ back to A_μ via

$$UA^U_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger = A_\mu \ . \tag{7.15}$$

As S_{YM} and the determinant $\triangle_{\mathcal{F}}$ are invariant under the gauge transformation (7.11), one obtains

$$Z_{YM} = \int \mathcal{D}U \int \mathcal{D}A \, \triangle_{\mathcal{F}}\,(A)\delta(\mathcal{F}(A))e^{-S_{YM}} \ . \tag{7.16}$$

Then we can separately integrate over the gauge group U, as it was factored,

$$Z_{YM} = \mathcal{N}V\int \mathcal{D}A \, \triangle_{\mathcal{F}}\,(A)\delta(\mathcal{F}(A))e^{-S_{YM}} \ , \tag{7.17}$$

where we denote the gauge group *volume* (the Haar measure of U) by $V \equiv \int \mathcal{D}U$, that can be absorbed by the normalization factor, as it is only a number independent of the gauge field.

For small $\theta^a(x)$, and the gauge condition

$$F^a(A_\mu(x)) = \partial_\mu A^a_\mu(x) - B^a_\mu(x) , \quad (7.18)$$

being $B_\mu(x)$ an auxiliary field, we automatically get the Faddeev-Popov determinant

$$\mathcal{M}^{ab}(x,y) = \frac{\delta \mathcal{F}^a(A_\mu(x))}{\delta {A^U}^c_\mu(z)} \frac{\delta {A^U}^c_\mu(z)}{\delta \theta^b(y)} |_{\mathcal{F}(A^U)=0} = -\partial_\mu D^{ab}_\mu \delta(x-y)|_{\mathcal{F}(A_\mu)=0} . \quad (7.19)$$

The condition $\mathcal{F}(A_\mu) = 0$ is naturally implemented by the $\delta(\partial_\mu A_\mu - B_\mu)$ that appear together with the determinant. Finally, the Yang-Mills partition function becomes

$$Z_{YM} = \mathcal{N} \int \mathcal{D}A |\det[-\partial_\mu D^{ab}_\mu \delta(x-y)]| \delta(\partial_\mu A_\mu - B_\mu) e^{-S_{YM}} , \quad (7.20)$$

where we absorbed V into $\mathcal{N}$. Using then the determinant identity for Grasmann variables $(\bar{c}^a, c^a)$,

$$\det \mathcal{M}^{ab}(x,y) = \int \mathcal{D}\bar{c}\mathcal{D}c \, \exp\{\bar{c}^a(x)\mathcal{M}^{ab}(x,y)c^b(y)\} , \quad (7.21)$$

and multiplying Z by the Gaussian factor

$$\int \mathcal{D}B \exp\{\frac{1}{2\alpha} \int d^4x B^2\} , \quad (7.22)$$

where α is the width of the Gaussian distribution, one finally obtains

$$Z_{YM} = \mathcal{N} \int \mathcal{D}A e^{-(S_{YM}+S_{gf})} , \quad (7.23)$$

where by S_{gf} is the well-known gauge-fixing action given by

$$S_{gf} = \int d^4x \left(\bar{c}^a \partial_\mu D^{ab}_\mu c^b - \frac{1}{2\alpha} (\partial_\mu A^a_\mu)^2 \right) . \qquad (7.24)$$

The Grasmann fields $\bar{c}^a$ and c^a are the famous Faddeev-Popov ghosts [57], which are anti-commuting scalar fields. Such fields violate the spin-statistics theorem, meaning that they are non-physical, i.e., they never appear in the physical spectrum of the theory as they possess negative norm and never attain a probabilistic interpretation. In other words, they are never observed in Nature. Their influence, however, are felt in virtual processes at the quantum level, in which $\bar{c}$ and c appear in loop diagrams, without being scattered in the end of the interaction. These ghost fields are exactly the ones predicted by R. Feynman to recover the unitarity of Yang-Mills theories [142]. In practice, the Faddeev-Popov quantization is a proof that the introduction of ghost fields is intimately related to the evaluation of the Feynman path integral by taking only one representative of each gauge orbit (regardless the Gribov copies).

The quantization of a field theory via Feynman path integral is based on the presupposition that we must sum over all field configurations according to all paths (in the field space) that can be constructed between the final and initial states, after a scattering process. Before studying the Gribov problem, we must remark that we made fragile assumptions in order to reproduce the ghosts predicted by Feynman. Firstly, we have used the Grasmannian identity (7.21) to generate the exponential of the FP determinant, but such an identity is not exactly the one that appears in the Feynman path integral after inserting the unit, since is not $\det \mathcal{M}^{ab}$ that showed up, but its absolute value $|\det \mathcal{M}^{ab}|$, see (7.12) and (7.20). In fact we have assumed that the FP determinant is positive. Secondly, we considered the wrong assumption that $\mathcal{F}^a(A)$ intersects each gauge orbit only once. These issues are intimately related to the Gribov problem, which consists in how to eliminate a residual gauge ambiguity that is not fixed by the Faddeev-Popov gauge-fixing procedure (as we shall discuss, this ambiguity indeed exists in non-Abelian theories, and is known as Gribov copies), without destroying the Feynman presupposition, *i.e.*, without losing any physical information concerning the Feynman quantization via path integral, that should be performed over all possible *paths*, in other words, over all physical field configurations.

7.2 Definition of the Gribov region: Elimination of infinitesimal copies

As we know, for a Yang-Mills theory with $SU(N)$ symmetry, the gauge transformation on the gauge field A_μ which preserves the theory invariance under an element $U \in SU(N)$ is defined by $A_\mu \longrightarrow A^U_\mu$, where A^U_μ is given by (7.11). When we try to fix the ambiguity using, for example, the Landau gauge (which is obtained by taking $\alpha \to 0$ in the gauge-fixing action (7.24)),

$$\partial_\mu A^a_\mu = 0\,, \tag{7.25}$$

one can prove that the imposition (7.25) is not enough to avoid double counting of equivalent field configurations in the Feynman path integral. Summarizing, in non-Abelian theories the FP quantization does not select only one representative of each gauge orbit in the Feynman path integral.

The so-called Gribov copies, first introduced by V. Gribov in [42], result from the fact that the *copy equation*

$$\partial_\mu A_\mu = \partial_\mu A^U_\mu \tag{7.26}$$

has nontrivial solutions in the Yang-Mills theory. In this case, A_μ and A^U_μ are called *copies*. For infinitesimal transformations $U = 1 - \alpha$ ($U^\dagger = 1 + \alpha$) with $\alpha = \alpha^a T^a$, in first order eq. (7.26) yields

$$-\partial_\mu(\partial_\mu \alpha + ig[\alpha, A_\mu]) = 0 \,, \tag{7.27}$$

or, by recognizing the covariant derivative in adjoint representation,

$$-\partial_\mu D_\mu \alpha = 0 \,. \tag{7.28}$$

This equation could be seen as an eigenvalue equation for the operator $-\partial_\mu D_\mu \equiv -\partial D$, where α is the zero mode of the operator. We must note that this operator is exactly the Faddeev-Popov ghost one. As $-\partial_\mu D_\mu$ is Hermitian, its eigenvalues are real. Form eq. (7.27) one observes that the copy equation can be seen as a Schrodinger equation with A_μ playing the role of the potential. For values of A_μ sufficiently small, the eigenvalues of the FP operator will be positive, as $-\partial^2$ only has positive eigenvalues[1]. As A_μ increases, it will attains a zero mode (7.28). Then, as A_μ increases further, it will become negative. This signal changing behavior will repeat over and over again every time the FP operator reaches a zero mode. The boundaries in which the FP operator has zero eigenvalues are called Gribov horizons. (See Figure 7.1 below, cf. [143].)

One of the famous Gribov solutions in his original paper is called the Gribov pendulum — a didactic explanation of these formal solutions can be found in [143]. Besides the formal solutions of (7.28), Gribov also proved that, in the infinitesimal case, for every eigenvalue of the operator ∂D, denoted by $\omega^a(A)$ below,

$$-\partial_\mu D^{ab}_\mu(A)\alpha^b = \omega^a(A)\alpha^a \,, \tag{7.29}$$

[1] In Abelian theories, such as QED, $-\partial^2$ is the "FP operator", and the copy equation only possesses trivial solutions in the thermodynamic limit, meaning that the Gribov copies are inoffensive in this case.

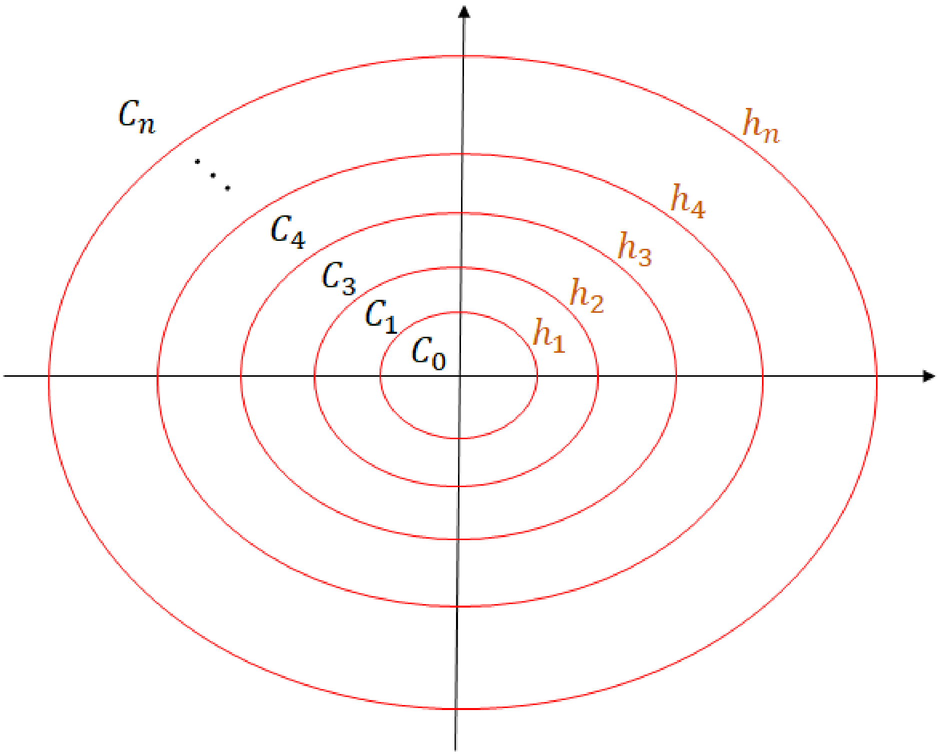

Figure 7.1: Functional field space divided into regions. Inside regions C_0, C_2, $\cdots$, C_{2N}, the eigenvalues of the FP ghost operator are positive. Inside C_1, $\cdots$, C_{2N+1}, negative. The regions are separated by lines h_n, which represents the Gribov horizons in which the FP operator has a renormalizable zero mode.

if ω^a is a solution, then $-\omega^a$ is also an eigenvalue of $-\partial D$. In order to avoid the infinitesimal copies, Gribov proposed to restrict the path integral domain to the region Ω defined by

$$\Omega = \{A^a_\mu;\ \partial_\mu A_\mu = 0,\ \mathcal{M}^{ab} > 0\}\,, \tag{7.30}$$

wherein $\mathcal{M}^{ab}$ is the FP operator $-\partial D$, so that the condition $\mathcal{M}^{ab} > 0$ implies

$$\int d^Dx \int d^Dy\, \varphi^a(x)\mathcal{M}^{ab}(x,y)\varphi^b(y) \,>\, 0\,, \tag{7.31}$$

for all well-behaved function $\varphi^a(x)$. This condition restricts the theory to the region inside the first Gribov horizon, denoted by C_0 in Fig. 7.1 above, in which all eigenvalues are positive. As for every positive eigenvalues there is a correspondent negative copy, this restriction does destroy the Feynman presupposition. In this region, the path integral is still performed over all physical field configurations (all possible paths), where only residual infinitesimal gauge ambiguities were eliminated. Moreover, as all eigenvalues of $\mathcal{M}^{ab}$ are positive in this region, the Grasmannian identity for the Faddeev-Popov determinant is well defined for its absolute value. This region inside the first Gribov horizon h_1 (see Fig. 7.1), known simply by *Gribov horizon*, is called *Gribov region*, and the implementation of the restriction to the Gribov region Ω is accomplished by the introduction of a step-function $\Theta(-\partial D)$ in the Feynman path integral, that leads to the well-known *no-pole condition*, as in this region the FP operator never reaches a zero mode, whose exponentiation will originate the Gribov horizon function. In 1991, Dell'Antonio and Zwanziger showed that all gauge orbit passes inside the Gribov region at least once [144]. As we shall see, the no-pole condition only affects the infrared regime of the theory, in which the coupling constant could not be treated as a perturbative parameter, proving that the asymptotic freedom in the high energy limit is preserved after introducing the Gribov horizon.

7.3 No-pole condition via Gribov semi-classical method

The main result of introducing the restriction of the Feynamn path integral domain to the Gribov region is a modified gluon propagator, due to the emergence of a massive parameter for the gauge field. In his seminal paper, Gribov implemented the no-pole condition for the Faddeev-Popov ghost propagator, *i.e.*, $-\partial D > 0$, by applying a semi-classical method in the limit of small A_μ. Perturbatively, in ordinary Yang-Mills theory, one obtains the one-loop improved FP ghost propagator

$$\langle \bar{c}_a(p) c_b(k) \rangle = \delta(p+k)\delta_{ab}\mathcal{G}(k^2) \tag{7.32}$$

with

$$\mathcal{G}(k^2) = \frac{1}{k^2}\frac{1}{(1 - \frac{11g^2N}{48\pi^2}\ln\frac{\Lambda^2}{k^2})^{\frac{9}{44}}} \ , \tag{7.33}$$

being Λ the UV cutoff. The expression above shows that the one-loop FP propagator has two poles, at

$$k^2 = 0 \quad \text{and} \quad k^2 = \Lambda^2 \exp\left(-\frac{1}{g^2}\frac{48\pi^2}{11N}\right) \ . \tag{7.34}$$

As we can immediately observe, for large k^2 the theory belongs to the Gribov region Ω, as the denominator of $\langle \bar{c}c \rangle \equiv \frac{1}{-\partial D}$ is positive in this region. However, for small k^2 in the order of $k^2 < \Lambda^2 \exp\left(-\frac{1}{g^2}\frac{48\pi^2}{11N}\right)$ we left the Gribov region, as the eigenvalues of the FP operator are not positive anymore. This analysis indicates that the only compatible poles to the Gribov region are the ones of the type $k^2 = 0$, as k^2 is always positive and for $k^2 \to 0$, it reaches the first Gribov horizon, where the FP operator finds a zero mode.

A correct implementation of a restriction to the Gribov region must eliminate the second pole of the FP propagator displayed in (7.34). The no-pole condition $-\partial D(A) > 0$ represents a *constraint* to the gauge field A_μ. Following the semi-classical Gribov method, in order to find this constraint at one-loop order, we treat A_μ as an external field, and then compute the Feynman diagrams for

$$\mathcal{G}^{ab}(k^2, A) = \delta^{ab}\mathcal{G}(k^2, A) \ , \tag{7.35}$$

see (7.32), up to the second order in A_μ, leaving the integration over A_μ in the path integral to be done in a second moment. The corresponding Feynman diagrams for $\mathcal{G}^{ab}(k^2)$ with external gauge fields are

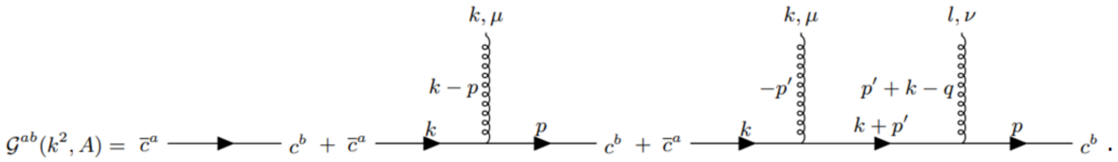

Figure 7.2: Ghost propagator with external gauge fields up to one-loop order.

The Feynman rule for the vertex $\bar{c}^a \partial_\mu A_\mu^k c^b$ is given by $ik_\mu f^{akb}$, where the incoming momentum k_μ stems from $\bar{c}$. These diagrams represent, in d dimensions, the three integrals below

$$
\begin{aligned}
I_1 &= \delta^{ab}(2\pi)^d \delta(k-q)\frac{1}{k^2} \;, && (7.36)\\
I_2 &= g\frac{1}{k^2}\frac{1}{p^2} f^{akb} i p_\mu A_\mu^k (k-p) \;, && (7.37)\\
I_3 &= g^2 \int \frac{d^d p'}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(p'+k)^2}\frac{1}{p^2} f^{akc} i\left(p'+k_\mu\right) A_\mu^k(-p') f^{c\ell b} i q_\nu A_\nu^\ell (p'+k-q) \;, && (7.38)
\end{aligned}
$$

As it is known [42; 143], we must disregard I_2. Due to the vertex and propagator structure of the gauge-fixed Yang-Mills action, there is no way to close loops from the second diagram after integrating over the gauge field. Replacing (7.36) and (7.38) into $\mathcal{G}(k^2, A)$ (8.44) yields

$$
\mathcal{G}(k^2, A) = \frac{1}{k^2} + \frac{Ng^2}{k^4\,(N^2-1)\,V}\int \frac{d^d q}{(2\pi)^d} A_\mu^a(-q) A_\nu^a \frac{(k-q)_\mu q_\nu}{(k-q)^2} \;, \qquad (7.39)
$$

therefore, from the definition

$$
\mathcal{G}(k^2, A) = \frac{1}{k^2}(1+\sigma(k, A)) \qquad (7.40)
$$

being $\sigma(k,A)$ the quantum corrections for the ghost propagator, one gets

$$\sigma(k,A) = \frac{Ng^2}{k^2\,(N^2-1)\,V}\int\frac{d^dq}{(2\pi)^d}A^a_\mu(-q)A^a_\nu(q)\frac{(k-q)_\mu q_\nu}{(k-q)^2}\ , \tag{7.41}$$

wherein V is the infinite volume factor. As we are working in the Landau gauge, $q_\mu A_\mu(q) =$, and $A^l_\mu A^l_\nu$ is transverse, *i.e.*,

$$A^l_\mu(q)A^l_\nu(-q) = \frac{1}{d-1}A^a_\lambda(q)A^a_\lambda(-q)\mathcal{P}_{\mu\nu} \quad \text{with} \quad \mathcal{P}_{\mu\nu}(q) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\ , \tag{7.42}$$

and $\sigma(k,A)$ can be rewritten in the form

$$\sigma(k,A) = \frac{Ng^2}{(d-1)(N^2-1)V}\frac{k_\mu k_\nu}{k^2}\int\frac{d^dq}{(2\pi)^d}A^a_\lambda(-q)A^a_\lambda(q)\frac{1}{(k-q)^2}\mathcal{P}_{\mu\nu}\ . \tag{7.43}$$

For small $\sigma(k^2,A)$, the Born approximation may be employed,

$$\mathcal{G}(k^2,A) \sim \frac{1}{k^2}\frac{1}{1-\sigma(k,A)}\ , \tag{7.44}$$

whereby the no-pole condition that corresponds to the restriction of the domain to the Gribov region reads

$$\sigma(k,A) < 1\ . \tag{7.45}$$

As $\sigma(k,A)$ decreases for increasing k^2, assuming that $A^a_\lambda(k)A^a_\lambda(-k)$ is positive (see [143]), the condition above is equivalent to imposing

$$\sigma(0,A) < 1\ , \tag{7.46}$$

where, taking the limit $k^2 \to 0$ in (7.43),

$$\sigma(0,A) = \frac{g^2N}{4V(N^2-1)}\int\frac{d^4q}{(2\pi)^4}\frac{A^a_\lambda(q)A^a_\lambda(-q)}{q^2}\ , \tag{7.47}$$

which defines the form factor $V(\Omega)$ as the theta function[1]

$$V(\Omega) = \Theta\left(1 - \sigma(0, A)\right) \ , \tag{7.48}$$

or, using the Heaviside expression,

$$V(\Omega) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{d\xi^2}{2\pi i\xi^2} e^{\xi^2(1-\sigma(0,A))} \ . \tag{7.49}$$

We should then introduce this factor into the path integral in order to implement the elimination of infinitesimal gauge copies, thus restricting the Feynman path integral domain to the Gribov region.

7.3.1 Modified gluon propagator in the presence of Gribov horizon

By restricting the Feynamn path integral domain to the Gribov region, we introduce the form factor (7.49) into the Yang-Mills partition function, so that

$$Z[J] = \mathcal{N} \int \frac{d\xi^2}{2\pi i\xi^2} \int DA\mathcal{D}\bar{c}\mathcal{D}c \, e^{\xi^2[1-\sigma(0,A)]} \exp\{-(S_{YM} + S_{gf} + \int d^dx \, J_i\Phi_i)\} \ , \tag{7.50}$$

wherein $\Phi_i \equiv \{A, \bar{c}, c\}$, being $J_i \equiv \{J^{(\bar{c})}, J^{(c)}, J_\mu\}$ their respective external sources. As we are interested in the free gluon propagator, we will take only the gluon quadratic part of the action and disregard the integration over $\bar{c}$ and c, thus

$$Z_A^{quad}[J] = \mathcal{N} \int \frac{d\xi^2}{2\pi i\xi^2} \int DAe^{\xi^2[1-\sigma(0,A)]} \exp\{-(S_{YM}^{quad} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \int d^dx \, J_\mu A_\mu)\} \ . \tag{7.51}$$

[1] $\Theta(x) = 1$ if $x > 0$, $\Theta(x) = 0$ if $x < 0$.

Applying the Fourier transform and using the expression (7.47) for $(0, A)$, the gluon propagator in momenta space reads

$$\langle A^a_\mu(k) A^b_\nu(p)\rangle = \mathcal{N}\delta(k+p)\int \frac{d\xi^2}{2\pi i\xi^2} e^{\xi^2} (\det K^{ab}_{\mu\nu})^{-\frac{1}{2}} (K^{ab}_{\mu\nu})^{-1} \,, \tag{7.52}$$

wherein

$$K^{ab}_{\mu\nu}(k,\xi^2) = \delta^{ab}\left[\xi^2 \frac{2Ng^2}{Vd(N^2-1)}\delta_{\mu\nu}\frac{1}{k^2} + \delta_{\mu\nu}k^2 + \left(\frac{1}{\alpha}-1\right)k_\mu k_\nu\right] \,. \tag{7.53}$$

A standard calculation of the determinant of $K^{ab}_{\mu\nu}$ yields

$$(\det K^{ab}_{\mu\nu})^{-\frac{1}{2}} = \exp\left[-\frac{d-1}{2}(N^2-1)V\int\frac{d^dq}{(2\pi)^d}\ln\left(q^2 + \frac{2\xi^2 Ng^2}{dV(N^2-1)}\frac{1}{q^2}\right)\right] \,, \tag{7.54}$$

then, replacing (7.53) and (7.54) into (7.52), one obtains

$$\langle A^a_\mu(k) A^b_\nu(p)\rangle = \mathcal{N}\delta(k+p)\int \frac{d\xi^2}{2\pi i} e^{f(\xi^2)} (K^{ab}_{\mu\nu})^{-1} \,, \tag{7.55}$$

with

$$f(\xi^2) = \xi^2 - \ln\xi^2 - \frac{d-1}{2}(N^2-1)V\int\frac{d^dq}{(2\pi)^d}\ln\left(q^2 + \frac{2\xi^2 Ng^2}{dV(N^2-1)}\frac{1}{q^2}\right) \tag{7.56}$$

Assuming that $K^{ab}_{\mu\nu}(k,\xi^2)$ does not oscillate too much, we are able to apply the method of steepest descent method in order to compute the integral over ξ^2,

$$\langle A^a_\mu(k) A^b_\nu(p)\rangle = \frac{\mathcal{N}}{2\pi i}\delta(k+p) e^{f(\xi^2_0)} (K^{ab}_{\mu\nu})^{-1}(k,\xi^2_0) \,, \tag{7.57}$$

whereby ξ^2_0 is the minimum of $f(\xi^2)$,

$$f'(\xi^2)|_{\xi^2=\xi^2_0} = 0 \,, \tag{7.58}$$

which gives

$$1 = \frac{1}{\xi_0^2} + \frac{d-1}{d} N g^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^4 + \gamma^4} \,, \tag{7.59}$$

where one defines the Gribov massive parameter γ by

$$\gamma^4 = \frac{2\xi^2 N g^2}{dV(N^2-1)} \,. \tag{7.60}$$

As $\xi_0^2 \sim V$, for a finite γ, we can neglect $\frac{1}{\xi_0^2}$ in (7.59) to obtain the so-called *gap equation*

$$1 = \frac{d-1}{d} N g^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^4 + \gamma^4} \,, \tag{7.61}$$

which fixes the infrared parameter $\gamma^2 \sim \Lambda^2$, $\Lambda \sim \mu e^{-\frac{1}{\xi_0^2 g^2(\mu)}}$, being μ the energy scale. With this result, to compute the tree-level gluon propagator in the presence of Gribov horizon, our task is reduced to the calculation of the inverse of $K^{ab}_{\mu\nu}$, see (7.57), by setting $\xi^2 = \xi_0^2$ accordingly to the relation (7.60). Hence, by taking $\alpha = 0$ in the end, and absorbing $\frac{e^{f(\xi_0^2)}}{2\pi i}$ in the normalization factor, one gets the modified transverse gluon propagator in the presence of a massive parameter,

$$\langle A^a_\mu(k) A^b_\nu(p) \rangle = \delta^{ab} \delta(p+k) \frac{k^2}{k^4 + \gamma^4} P_{\mu\nu}(k) \,. \tag{7.62}$$

In the UV limit $\gamma^4 \to 0$, for small g^2, we recover the ordinary gluon propagator. The Gribov correction is strong in the infrared limit, which shows that the Gribov horizon computes non-perturbative effects. From (7.62), we immediately note that the Gribov horizon originates a mass gap. Differently of standard perturbation theory, in which the gluon propagator diverges at the origin, the gluon propagator goes to zero at zero momentum — see figure below, extracted from [145]. Moreover, as the Gribov parameter originates two complex non-physical poles, $p^2 = \pm i\gamma^2$, it allows for a interpretation concerning confinement phases, where the gluon excitations disappear of the physical spectrum of the theory. In

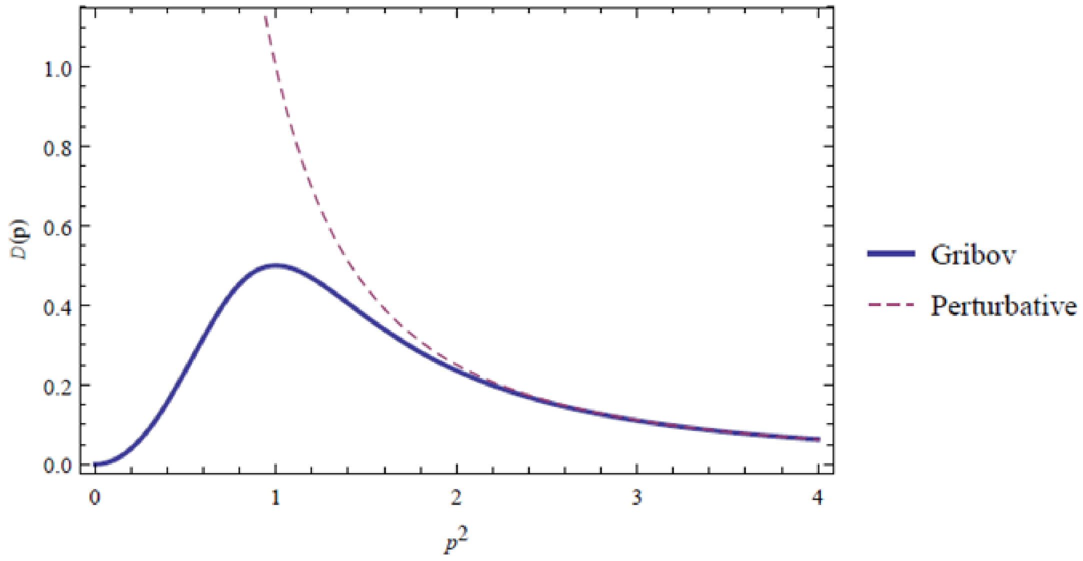

Figure 7.3: Form factor of gluon propagator. $\langle A_\mu A_\nu\rangle(p) = D(p)\mathcal{P}_{\mu\nu}$, where $D(p) = \frac{1}{p^2}$ in standard perturbation theory, and $D(p) = \frac{p^2}{p^4+\gamma^4}$ in the presence of Gribov horizon.

the paper [146], the authors analyze the correspondence between the dynamical mass scale introduced by the Gribov horizon and the Polyakov loop.

7.3.2 Enhanced Faddeev-Popov ghost propagator

After calculating the gluon propagator, we are able to determine the one-loop ghost propagator by integrating the gauge field,

$$\langle \bar{c}^a(p)c^b(k)\rangle = \delta^{ab}\delta(p+k)\frac{1}{k^2}\frac{1}{1-\langle\sigma(k,A)\rangle_{1PI}}\,, \tag{7.63}$$

whereby $\langle\sigma(k,A)\rangle_{1PI} \equiv \sigma(k)$ represents the expression (7.43) after connecting the gluon legs of the one-loop 1PI diagrams, see Fig. 7.2,

$$\sigma(k) = \frac{Ng^2}{(d-1)(N^2-1)V}\frac{k_\mu k_\nu}{k^2}\int\frac{d^dq}{(2\pi)^d}\langle A^a_\lambda(-q)A^a_\lambda(q)\rangle\frac{1}{(k-q)^2}\mathcal{P}_{\mu\nu}\,, \tag{7.64}$$

therefore, replacing (7.62) for $a = b$ and $\mu = \nu$ in expression above,

$$\sigma(k) = Ng^2 \frac{k_\mu k_\nu}{k^2} \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{q^4 + \gamma^4} \frac{1}{(k-q)^2} \mathcal{P}_{\mu\nu} \,. \tag{7.65}$$

As the detailed computation of (7.65) can be found in [141; 145], we will not reproduce it. The idea is to insert the gap equation identity (7.61) into the equation above, and provide a perturbative expansion for small momentum. The final expression of the ghost propagator for $k^2 \approx 0$ in the infrared regime is

$$\mathcal{G}^{ab}(k^2) = \delta^{ab} \frac{1}{k^4} \frac{d^2 + 2d}{d^2 - 3d + 2} \frac{1}{Ng^2 I_\gamma} \,, \tag{7.66}$$

wherein

$$I_\gamma = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q^4 + \gamma^4)} \,. \tag{7.67}$$

In the four-dimensional case, $d = 4$, and we get

$$I_\gamma^{d=4} = \delta^{ab} \frac{1}{k^4} \frac{128\pi^2\gamma^2}{Ng^2} \,. \tag{7.68}$$

This result is known as the *enhancement of FP ghost propagator*, with the absence of the second pole described in (7.34), as it was expected due to the implementation of the no-pole condition. The Gribov form factor for the gluon propagator, with the vanishing of $\langle AA \rangle$ at the origin, and the ghost enhancement predicted by Gribov copies is in agreement with old[1] lattice data [147; 148; 149; 150; 151], serving as an evidence of existence of the Gribov horizon in Yang-Mills theories.

[1] In order to obtain the recent data, we must introduce two-dimensional condensates. See the topic “RGZ theory” on page 160.

7.4 Gribov-Zwanziger theory: A generalization to all orders

In his original paper, Gribov developed a semi-classical method to implement the no-pole condition at one-loop order, in the limit of small A_μ — as described in previous section. In [58], D. Zwanziger generalized the restriction to Gribov region to all orders, *i.e.*, not only to small gluon field oscillations. The task is to find the lowest eigenvalue of the Faddeev-Popov operator, $\omega_{lowest}(A)$, and then introduce the theta function $\Theta(\omega_{lowest}(A))$ in the Feynman path integral, imposing

$$\omega_{lowest}(A) \geq 0 \ . \tag{7.69}$$

If we impose a restriction in which the lowest eigenvalue of the FP operator has to be positive, then all FP eigenvalues will be positive, and the theory will be restricted to the Gribov region.

Working out the eigenvalue equation for the Faddeev-Popov operator, see (7.29), by applying a degenerate perturbation theory following the decomposition

$$\mathcal{M}^{ab} = \mathcal{M}_0^{ab} + \mathcal{M}_1^{ab} = -\partial^2\delta_{ab} + gf^{abc}A_\mu^c\partial_\mu \ , \tag{7.70}$$

whereby $\mathcal{M}_0^{ab} \equiv -\partial^2\delta_{ab}$ is taken as the unperturbed operator, and $\mathcal{M}_1^{ab} \equiv f^{abc}A_\mu^c\partial_\mu$, the perturbation one, D. Zwanziger found that the condition

$$dV(N^2-1) - g^2 \int d^dx \int d^dy f_{bal} A_\mu^a(x)[\mathcal{M}^{-1}(A)]^{lm}\delta(x-y) f_{bkm} A_\mu^k(y) > 0 \tag{7.71}$$

in Landau gauge, is sufficient to impose (7.69). (For a pedagogical demonstration of (7.71), see [141].) The function

$$h(A) = g^2 \int d^dx \int d^dy f_{bal} A_\mu^a(x)[\mathcal{M}^{-1}(A)]^{lm}\delta(x-y) f_{bkm} A_\mu^k(y) \tag{7.72}$$

is the so-called *Gribov-Zwanziger horizon function.* As the positivity of $\omega_{lowest}(A)$ is implemented by the condition above, the restriction to the Gribov region is implemented by introducing the theta function $\Theta(dV(N^2)-h(A))$ in the Feynamn path integral, which yields, using the Heaviside expression,

$$Z_{GZ} = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \int \frac{d\gamma^*}{2\pi i\gamma^*} e^{\gamma^*(d(N^2-1)-h(A))} e^{-S_{YM}-S_gf} \ . \tag{7.73}$$

In the thermodynamic limit, $V \to \infty$, and the Gribov region is concentrated in its boundary[1] (we will discuss the geometric interpretation of this statement in details in Section ??). In this case, the Θ-function can be replaced by the δ-function, which means that we can eliminate the factor γ^* in the denominator above. Then we apply the saddle point approximation for the integration over γ^*,

$$Z = \int \frac{d\gamma^*}{2\pi i} e^{-v(\gamma^*)} \approx e^{-v(\gamma^*_0)} \ , \tag{7.74}$$

wherein

$$v(\gamma^*) = -\ln \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c e^{\gamma^*(dV(N^2-1)-h(A))-S_{YM}-S_gf} \ , \tag{7.75}$$

being γ^* determined by $v'(\gamma^*_0) = 0$, which yields

$$dV(N^2-1) = \frac{\int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\, h(A) e^{-\gamma^* h(A)-S_{YM}-S_{gf}}}{\int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\, e^{-\gamma^* h(A)-S_{YM}-S_{gf}}} \equiv \langle h(A)\rangle_{\gamma^*} \ . \tag{7.76}$$

From equations (7.74) and (7.75) we conclude that

$$Z_{GZ} = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\, e^{-S_{GZ}} \ , \tag{7.77}$$

[1]The same operation was done in eq. (7.59) when we neglected the term $\frac{1}{\xi_0^2}$ in order to obtain a finite massive parameter γ^4.

where S_{GZ} is the *Gribov-Zwanziger action* given by

$$S_{GZ} = S_{YM} + S_{gf} + \gamma^4 h(A) - \gamma^4 V d(N^2 - 1) \,, \qquad (7.78)$$

where we call $\gamma_0^* = \gamma^4$ — the parameter fixed by the Zwanziger's *gap equation* (7.76) — which is exactly the massive parameter obtained in the semi-classical Gribov method. It is easy to see that, in the lowest order, $\mathcal{M}^{-1} = \frac{1}{-\partial^2(1-\frac{gfA\partial}{\partial^2})} = \frac{1}{-\partial^2} + \mathcal{O}(gA)$, and then, using the relation $f^{abc}f^{dbc} = N\delta^{cd}$, up to the order g^2 one gets

$$h(A) = g^2 N \int d^d x A^a_\mu(x) \frac{1}{\partial^2} A^a(x) = g^2 N \int \frac{d^d p}{(2\pi)^4} A^a_\mu(p) \frac{1}{p^2} A^a(-p) \,, \qquad (7.79)$$

proving that the Zwanziger generalization recovers the Gribov horizon in the lowest order. The equivalence between both methods to all orders, according to their respective gap equations that fix the value of γ^4 via distinct ways, is not a trivial issue, however it has been worked out in literature, and therefore proved that Zwanziger and Gribov procedures are indeed equivalent to all orders, cf. [152; 153].

7.4.1 Local Gribov-Zwanziger action

The horizon function is well defined. As the eigenvalues of $\mathcal{M}^{ab}$ are restricted to be positive, the determinant of $\mathcal{M}^{ab}$ is always positive and $(\mathcal{M}^{-1})^{ab}$ do exist, since it never changes the sign, *i.e.*, never finds a zero mode inside the Gribov region. But the inverse of the Faddeev-Popov operator is non-local, and we will encounter serious difficulties in applying the standard Feynman rules to construct the loop diagrams for a non-local action. The localization of GZ action is achieved by the introduction of two pairs of auxiliary fields: one bosonic, $(\bar{\varphi}^{ab}, \varphi^{ab})$, and other fermionic, $(\bar{\omega}^{ab}, \omega^{ab})$. The local GZ action has the form

$$\begin{aligned} S^L_{GZ} &= S_{YM} + S_{gf} - \int d^dx \left(\bar{\varphi}^{ac} \mathcal{M}^{ab} \varphi^{bc} - \bar{\omega}^{ac} \mathcal{M}^{ab} \omega^{bc} \right) \\ &+ \gamma^2 \int d^dx g f^{abc} A^a_\mu (\varphi^{bc}_\mu + \bar{\varphi}^{bc}) - \int d^dx \gamma^4 d(N^2-1) , \end{aligned} \tag{7.80}$$

with the corresponding partition function

$$Z^L_{GZ} = \int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c \mathcal{D}\bar{\varphi} \mathcal{D}\varphi \mathcal{D}\bar{\omega} \mathcal{D}\omega \, e^{-S^L_{GZ}} , \tag{7.81}$$

so that, when the auxiliary fields $(\bar{\varphi}, \varphi, \bar{\omega}, \omega)$ are integrated out, we recover the original non-local GZ action. (For the integration over $\bar{\varphi}^{ab}_\mu$ and φ^{ab}_μ, $\gamma^2 g f^{abc} A^c_\mu \equiv J^{ab}_\mu$ works as a common external source for both fields.)

It should be a bosonic pair and a fermionic one, in order to absorb the internal determinant produced by the integration over the bosonic pair. With this, we can define the BRST transformations of the auxiliary fields as doublets pairs,

$$s\varphi^{ab}_\mu = \omega^{ab}_\mu , \quad s\omega^{ab}_\mu = 0 , \tag{7.82}$$

$$s\bar{\varphi}^{ab}_\mu = \bar{\omega}^{ab}_\mu , \quad s\bar{\omega}^{ab}_\mu = 0 , \tag{7.83}$$

which ensures that the physical content of the theory is preserved. With respect to the quantum numbers, all auxiliary fields has mass dimension 1, and ghost numbers $\{\bar{\varphi}, \varphi, \bar{\omega}, \omega\} = \{0, 0, -1, 1\}$, respectively.

In this local formalism, the gap equation that fixes the massive parameter is given by

$$\frac{\partial \Gamma_0}{\partial \gamma^2} = 0 , \tag{7.84}$$

which yields

$$\langle g f^{abc} A^a_\mu (\varphi^{bc}_\mu + \bar{\varphi}^{bc}) \rangle = 2\gamma^2 d(N^2-1) , \tag{7.85}$$

where

$$e^{-V\Gamma_0} = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}\bar{\varphi}\mathcal{D}\varphi\mathcal{D}\bar{\omega}\mathcal{D}\omega \; e^{-S^L_{GZ}} \; . \tag{7.86}$$

By computing (7.85) to leading order, we recover the Gribov gap equation obtained via semi-classical method (7.61). This is basically the proof of the leading order equivalence between the Gribov semi-classical method and the Zwanziger one in the local formalism. The all order proof was worked out in [153], as mentioned in previous section.

RGZ theory. In order to obtain a Gribov-Zwanziger theory harmony with lattice results, we must introduce two-dimensional condensates of the type $\langle A^2_\mu \rangle$, giving rise to the so-called *Refined Gribov-Zwanziger (RGZ) theory.* In the presence of condensates, the enhanced ghost propagator possesses only one pole in k^2, being proportional to $\frac{1}{k^2}$, instead of $\frac{1}{k^4}$. Besides that, the RGZ infrared gluon propagator has the form

$$\langle A^a_\mu(p) A^b_\nu(-p) \rangle = \delta^{ab} \frac{p^2 + M^2}{p^4 + (m^2 + M^2)p^2 + M^2 m^2 + 2g^2 N\gamma^4} \mathcal{P}_{\mu\nu} \tag{7.87}$$

whose internal structure

$$\frac{p^2 + a}{p^4 + bp^2 + c} \quad \text{with} \quad \{a, b, c\} \equiv \text{constants} \tag{7.88}$$

is in complete harmony with lattice QCD simulations, as pointed out by the authors of [59]. In eq. (7.87), M are the mass of A^2, and m, the mass of $\langle \bar{\omega}\omega \rangle$ and $\langle \bar{\varphi}\varphi \rangle$. As theses masses are dynamically generated in GZ theory, the idea is to construct an effective theory by introducing quadratic terms for these fields, without breaking the BRST symmetry. For the GZ theory in Landau gauge, $\langle A^2_\mu \rangle$ is on-shell BRST invariant. With respect to the topological BRST transformations, $\langle A^2_\mu \rangle$ is not BRST invariant, even on-shell, and we have no physical motivation to introduce such a condensate. In general, these masses are not dynamically generated in the topological case. As we shall demonstrate later on, the topological Yang-Mills theory restricted to the Gribov region via introduction of the GZ horizon function in the local formalism does not receive radiative corrections in the same way. Moreover, the introduction of the condensates does not modify the gap equation, which will be used to prove that the Gribov copies are inoffensive in topological YM theory. For all these reasons, we will not consider the introduction of two-dimensional condensates.

7.4.2 *Soft* breaking of BRST symmetry and the physical meaning of Gribov massive parameter

The introduction of the Gribov-Zwanziger horizon in the action explicitly breaks the BRST symmetry. This a very unwanted result, as the BRST symmetry is necessary to prove the unitarity, to ensure the renormalizability to all orders, and to define the physical gauge-invariant observables of the theory [154; 155; 156]. This breaking however brought to light the physical meaning of the infrared γ parameter, and its intrinsic non-perturbative character. After performing a transformation with trivial Jacobian on one of the auxiliary fields, one can prove that the BRST breaking is proportional to γ^2, in other words, the BRST symmetry is restored in the perturbative regime. One says that the BRST symmetry is only broken in a *soft* way, cf. [156; 157; 158; 159].

In order to prove such a statement, we must note that the Local GZ partition function allows for the redefinition of the auxiliary field ${}^{ab}_{\mu}$ according to the non-local shift

$$\omega^{ab}_{\mu} \rightarrow \omega^{ab}_{\mu} + g f^{dlm} \int d^d y [M^{-1}]^{ad} \partial_\nu (\varphi^{mb}_{\mu} D^{le}_{\nu} c^e) \; . \tag{7.89}$$

Again, as the eigenvalues of $\mathcal{M}^{ab}$ are positive, this shift is well defined. This transformation is admitted as it possesses trivial Jacobian, which means that the perturbative quantum results are preserved. From the shift (7.89), one gets

$$S^L_{GZ}|_{shifted} = S^L_{GZ} - \int d^4x g f^{adl} \bar{\omega}^{ac}_\mu \partial_\nu (\varphi^{lc}_\mu D^{de}_\nu c^e) \ , \tag{7.90}$$

which, under an ordinary Yang-Mills BRST transformation,

$$sS^L_{GZ}|_{shifted} = \gamma^2 g f^{abc} \int d^dx \left(A^a_\mu \omega^{bc}_\mu - D^{ad}_\mu c^d (\varphi + \bar{\varphi})^{bc}_\mu \right) \equiv \triangle_{\gamma^2} \ . \tag{7.91}$$

Henceforth, in the perturbative UV regime $\triangle^2_\gamma \propto \gamma^2 \to 0$, and $S^L_{GZ}|_{shifted}$ is BRST invariant. This result is an evidence that the Gibov copies is a non-perturbative method, whose effects due to the Gribov horizon only appear in the infrared regime. In the UV limit, the BRST structure of the standard Faddeev-Popov quantization is protected.

The consequence of the *soft* BRST breaking (7.91) is that the γ parameter cannot be interpreted as a gauge parameter, but it is a physical parameter that does belong to the trivial part of BRST cohomology. We conclude that, by eliminating the infinitesimal copies, a physical element of Yang-Mills theory in the infrared regime is brought to light. The algebraic proof of such a statement was first described in [59], and is, in fact, very simple. Taking the derivative of $S^L_{GZ}|_{shifted}$ with respect to γ^2, and then acting with the BRST operator, one gets

$$s\frac{\partial S^L_{GZ}|_{shifted}}{\partial \gamma^2} = \frac{\triangle_{\gamma^2}}{\gamma^2} \neq 0 \ , \tag{7.92}$$

which implies

$$\frac{\partial S^L_{GZ}|_{shifted}}{\partial \gamma^2} \neq s(something) \ , \tag{7.93}$$

since $s^2 = 0$, proving that the γ-dependent term of the local GZ action cannot be written as a BRST-exact term, *i.e.*, γ^2 cannot be introduced as a gauge parameter.

Moreover, the local Gribov-Zwanziger action is renormalizable [160]. The algebraic proof of its renormalizability to all orders was worked out in [161; 162]. In few words, the Gribov-Zwanziger action allows for the introduction of a physical infrared mass parameter in a renormalizable way, only by eliminating infinitesimal gauge copies in the Feynman path integral that are not fixed by the usual Faddeev-Popov gauge-fixing procedure.

7.5 Fundamental Modular Region

The Gribov region can be alternatively defined as the relative minima of the functional

$$||A^U||^2 = \mathrm{Tr}\int d^dx A^U_\mu A^U_\mu \ . \tag{7.94}$$

in other words, for each gauge orbit, to remain inside the Gribov region we must select the *path* that minimizes A^2. To prove this statement, if $||A^U||^2$ is an extremum, varying it with respect to a gauge transformation must vanish, therefore

$$\begin{aligned} \delta||A^U||^2 &= \delta[\frac{1}{2}\int d^dx A^a_\mu(x)A^a_\mu(x)] = \int d^dx[\delta A^a_\mu(x)]A^a_\mu(x) \\ &= -\int d^dx[D^{ab}_\mu\alpha^b(x)]A^a_\mu(x) = \int d^dx\alpha^a(x)\partial_\mu A^a_\mu(x) = 0 \ , \end{aligned} \tag{7.95}$$

which yields $\partial_\mu A^a_\mu = 0$, for an arbitrary gauge parameter $\alpha^a(x)$. Secondly, to be a minimum,

$$\delta^2||A^U||^2 > 0 \ , \tag{7.96}$$

which gives

$$-\int d^dx[\partial_\mu\alpha^a(x)]\delta A^a_\mu(x) = \int d^dx\alpha^a(x)(-\partial_\mu D^{ab}_\mu)\alpha^b(x) > 0 \ , \tag{7.97}$$

which implies $\partial_\mu D^{ab}_\mu > 0$ that is exactly the Gribov imposition to eliminate the infinitesimal copies, proving that the Gribov region, in which the Faddeev-Popov operator is positive definite, is indeed defined by gauge orbits that minimize A^2.

The Gribov region Ω enjoys the following properties: (i) it is convex [163], *i.e.*, if we would like to go from a field A^1_μ to a field A^2_μ, being both within the Gribov region, we would never cross the Gribov horizon. To prove this property, we must note that $\mathcal{M}^{ab}(A)$ is a linear operator in A_μ. Because of that, the gluon field

$$A_\mu(t) = (1-t)A^1_\mu + tA^2_\mu \quad \text{with} \quad t \in [0,1] \tag{7.98}$$

always belongs to the Gribov region, since

$$\mathcal{M}^{ab}(A_\mu(t)) = (1-t)\mathcal{M}^{ab}(A^1_\mu) + t\mathcal{M}^{ab}(A^2_\mu) > 0\ , \tag{7.99}$$

because $\mathcal{M}^{ab}(A^1_\mu) > 0$ and $\mathcal{M}^{ab}(A^2_\mu) > 0$, as we have previously admitted that $A^1_\mu, A^2_\mu \in \Omega$; and $t \in [0,1]$, *i.e.*, $(1-t)$ and $t \geq 0$. Thus, varying $t = 0 \to t = 1$, we go from A^1_μ to A^2_μ without going out of Ω, in other words, all paths between gluon fields inside Ω never cross a *cavity*, which proves that Ω is convex.

(ii) It is bounded in every direction. The task is to prove that, if $A_\mu \in \Omega$, then λA_μ will cross the Gribov horizon for λ large enough. The proof of this property is done as follows: firstly, we must note that the operator

$$\widetilde{\mathcal{M}}^{ab} = f^{abc}\partial_\mu A^c_\mu \tag{7.100}$$

is traceless, thus the sum of the eigenvalues of $\widetilde{\mathcal{M}}^{ab}$ is zero, and consequently at least one of its eigenvalues, ω, have to be negative. Therefore, as $\mathcal{M}^{ab} = -\partial^2\delta^{ab} + \widetilde{\mathcal{M}}^{ab}$ is linear, $\mathcal{M}^{ab}(\lambda A_\mu) = \lambda\mathcal{M}^{ab}(A_\mu)$ has the same eigenvector of

$\mathcal{M}^{ab}(A_\mu)$, denoted by $\phi^a(x)$, and we will find in a given A_μ-direction

$$\int dxdy\phi^a(x)\mathcal{M}^{ab}(\lambda A_\mu)(x,y)\phi^b(y) = \int dx\phi^a(x)(-\partial^2)\phi^a(x) + \lambda\omega \ , \qquad (7.101)$$

so that, as $\omega < 0$ do exist, for a negative ω, eq. (7.101) will become negative for very large positive λ, and $\mathcal{M}^{ab}(\lambda A)$ will not be positive definite anymore, proving that Ω is bounded in every direction. Precisely, in [164], the authors have proved that the Gribov region is contained in an ellipsoid (we will back to to the geometric interpretation of Ω in the topological case, and its ellipsoid structure will be discussed in details).

Following these two properties, we are tempted to think that the definition of the Gribov region being composed of gauge orbits that minimize A^2 is well defined, and that all gauge copies of Gribov type are avoided inside Ω. Unfortunately this is not the case. As the Fo operator possesses zero modes: $\exists\ \theta^a(x)$ such that $\mathcal{M}^{ab}\theta^b = 0$, the condition (7.96) is inconclusive to determine that the gauge orbit minimizes A^2 near the boundary of Ω. In this case, to prove that $||A||^2$ is a minimum we must consider the extra condition

$$\delta^3||A||^2 = 0 \ . \qquad (7.102)$$

However,

$$\delta^3||A||^2 = gf_{abc}\int dx\partial_\mu\theta^a(x)\theta^b(x)D^{cd}_\mu(x)\theta^d(x) \ , \qquad (7.103)$$

which is not zero for an arbitrary θ^a. We conclude that we have extra Gribov copies for gluon fields on the boundary of $\Omega \equiv \partial\Omega$, whose gauge orbits do not minimize A^2.

The fundamental modular region (FMR), usually denoted by Λ, is a region inside the Gribov region, $\Lambda \subset \Omega$, for field configurations near the origin. In the

FMR, the gluon field closest to the origin is selected to be the representative of the gauge orbit. For the domain restricted to Λ, one says that the path integral is done in the *minimal Landau gauge*. In [165], a numerical study of the FMR was performed, indicating the existence of extra copies near $\partial\Omega$. A local formalism for the implementation of FMR in the Feynman path integral is still lacking. In any case, it has been argued that the degenerate minima of FMR do not play any role as they have zero measure, see for instance [60; 141].

We would like to emphasize that the existence of extra gauge copies on the boundary of the Gribov region, and their possible consequences, does not invalidate the non-perturbative Gribov effects. The elimination of infinitesimal copies via introduction of the Gribov horizon preserves the physical content of the Feynman path integral, which is still integrated over all field configurations. From the beginning, by the way, we are just eliminating *infinitesimal* copies. It is not known how to deal with non-infinitesimal ones. Nevertheless, the introduction of the Gribov horizon, by which only gauge copies are eliminated, reveals an infrared massive parameter that cannot be written as a BRST-exact term, being therefore a physical parameter of the theory, and our aim is to analyze the possibility of generating such a physical parameter in the infrared of topological Yang-Mills theories, by introducing the Gribov horizon, or its analogous, in self-dual Landau gauges.

Chapter 8

Infinitesimal Gribov copies in gauge-fixed topological Yang-Mills theories

In this C=chapter we study the Gribov problem in four-dimensional topological Yang-Mills theories following the *off-shell* Baulieu-Singer approach in the self-dual Landau gauges. As standard gauge-fixed Yang-Mills theories suffer from the gauge copy (Gribov) ambiguity, one might wonder if and how this has repercussions for this analysis. The resolution of the small (infinitesimal) gauge copies, in general, affects the dynamics of the underlying theory. In particular, treating the Gribov problem for the standard Landau gauge condition in non-topological Yang-Mills theories strongly affects the dynamics of the theory in the infrared, as discussed in the previous chapter. Although the topological BS theory is investigated with the same gauge condition, the effects of the copies turn out to be completely different. In other words: in both cases, the copies are there, but the effects are very different. As suggested by the tree-level exactness of the topological model in this gauge choice, as demosntrated in Section 6.3, the Gribov copies are shown to be inoffensive at the quantum level.

To be more precise, following Gribov, we discuss the path integral restriction to the Gribov horizon. The associated gap equation, which fixes the so-called Gribov parameter, is however shown to only possess a trivial solution, making the restriction obsolete. We relate this to the absence of radiative corrections in both gauge and ghost sectors. We give further evidence by employing the renormalization group which shows that, for this kind of non-Abelian topological model, the gap equation indeed forbids the introduction of a massive Gribov parameter.

8.1 Equivalence between the topological BRST and Faddeev-Popov constructions

Following the *off-shell* Baulieu-Singer approach, the three gauge ambiguities (4.2)-(4.4) of the Pontryagin action $S_0[A]$, described in Chapter 4, are fixed in the (anti-)self-dual Landau gauges (which amounts to considering the gauge constraints (5.4)-(5.6)) via BRST quantization by introducing the gauge-fixing action $S_{gf}[\Phi]$ given by eq. (5.8). For computational convenience, we will consider the gauge-fixing action $S_gf(\alpha,\beta)$ given by (6.2), which corresponds to S_{gf} with two extra trivial BRST terms, namely,

$$S_{gf}(\alpha,\beta) = S_{gf} + \int d^4z \left(\frac{\alpha}{2}b^a b^a + \frac{\beta}{4}B^a_{\mu\nu}B^a_{\mu\nu}\right) \; ; \tag{8.1}$$

it is understood that, at the end, the limits $\beta \to 0$, $\alpha \to 0$ must be taken to recover the (anti-)self-dual Landau gauges. We relied on the standard BRST quantization lore here [98; 166], but it can be easily checked that upon integration over the various multipliers/auxiliary fields, the gauge fixing conditions are retrieved under the form of appropriate δ-functions and corresponding Jacobians, representing the "unities" of the textbook Faddeev-Popov quantization procedure, at least for the here considered self-dual Landau gauges. The proof of the latter statement may be conducted as follows.

The starting action is $S_{gf}(\alpha,\beta)$ while the corresponding generating functional reads

$$Z_{gf} = \int \mathcal{D}\Phi e^{-S_{gf}(\alpha,\beta)} \;, \tag{8.2}$$

where $\mathcal{D}\Phi = \mathcal{D}A\mathcal{D}\bar{\chi}\mathcal{D}\psi\mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}\bar{\phi}\mathcal{D}\phi\mathcal{D}\bar{\eta}\mathcal{D}b\mathcal{D}B$. Our aim is to show that (8.2) is equivalent to

$$Z_{FP} = \int \mathcal{D}A\mathcal{D}\psi \det(D_\pm)\delta(\partial A)\delta(F_\pm)\delta(\partial\psi) \;, \tag{8.3}$$

where $F_\pm = F \pm \tilde{F}$ and $\frac{1}{2}D_\pm \equiv \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta} \pm \epsilon_{\mu\nu\alpha\beta}\right)D^{ab}_\alpha$. Integration over the auxiliary fields b and B leads to

$$\begin{aligned} Z_{gf} &= \int [\mathcal{D}\Phi\mathcal{D}b\mathcal{D}B] \exp\{-\int d^4x \left[-\frac{1}{2\alpha}(\partial A)^2 - \frac{1}{4\beta}F^2_\pm\right] - \int d^4x \left[(\bar{\eta}^a - \bar{c}^a)\,\partial_\mu\psi^a_\mu\right. \\ &+ \bar{c}^a\partial_\mu D^{ab}_\mu c^b - \frac{1}{2}gf^{abc}\bar{\chi}^a_{\mu\nu}c^b\left(F^c_{\mu\nu} \pm \widetilde{F}^c_{\mu\nu}\right) - \bar{\chi}^a_{\mu\nu}\left(\delta_{\mu\alpha}\delta_{\nu\beta} \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\right)D^{ab}_\alpha\psi^b_\beta \\ &+ \left.\bar{\phi}^a\partial_\mu D^{ab}_\mu\phi^b + gf^{abc}\bar{\phi}^a\partial_\mu\left(c^b\psi^c_\mu\right)\right]\} \;. \end{aligned} \tag{8.4}$$

Some inconvenient terms can be eliminated by the following shifts:

$$\begin{aligned} \bar{\eta}^a &\longmapsto \bar{\eta}^a + \bar{c}^a \;, \\ \phi^b &\longmapsto \phi^b - gf^{cde}(\partial_\nu D^{bc}_\nu)^{-1}\partial_\mu\left(c^d\psi^e_\mu\right) \;, \\ \bar{c}^a &\longmapsto \bar{c}^a - \frac{1}{2}gf^{cde}\bar{\chi}^d_{\mu\nu}(F_\pm)^e_{\mu\nu}(\partial_\nu D^{ca}_\nu)^{-1} \end{aligned} \tag{8.5}$$

These transformations are valid perturbatively since $-\partial D > 0$ due to the absence of radiative corrections demonstrated in Section 6.3, cf. [54; 56]. Notice also that these shifts generate a trivial Jacobian. Hence,

$$\begin{aligned} Z_{gf} &= \int [\mathcal{D}\Phi\mathcal{D}b\mathcal{D}B] \exp\left\{-\int d^4x \left[-\frac{1}{2\alpha}(\partial A)^2 - \frac{1}{4\beta}F_\pm^2\right] - \int d^4x \left[\bar{\eta}^a\partial_\mu\psi_\mu^a + \bar{c}^a\partial_\mu D_\mu^{ab}c^b \right.\right. \\ &- \left.\left. \bar{\chi}_{\mu\nu}^a \left(\delta_{\mu\alpha}\delta_{\nu\beta} \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\right) D_\alpha^{ab}\psi_\beta^b + \bar{\phi}^a\partial_\mu D_\mu^{ab}\phi^b\right]\right\} . \end{aligned} \tag{8.6}$$

Integration over the Faddeev-Popov and bosonic ghosts and the corresponding anti-ghosts leads to cancelling contributions,

$$\begin{aligned} Z_{gf} &= \int \mathcal{D}A\mathcal{D}\bar{\eta}\mathcal{D}\bar{\chi}\mathcal{D}\psi \ \exp\left\{-\int d^4x \left[-\frac{1}{2\alpha}(\partial A)^2 - \frac{1}{4\beta}F_\pm^2\right] - \int d^4x \left[\bar{\eta}^a\partial_\mu\psi_\mu^a \right.\right. \\ &- \left.\left. \bar{\chi}_{\mu\nu}^a \left(\delta_{\mu\alpha}\delta_{\nu\beta} \pm \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\right) D_\alpha^{ab}\psi_\beta^b\right]\right\} . \end{aligned} \tag{8.7}$$

Integration over $\bar{\eta}$ and $\bar{\chi}$ (see for instance [167]) subsequently leads to

$$Z_{gf} = \int \mathcal{D}A\mathcal{D}\psi \ \exp\left\{-\int d^4x \left[-\frac{1}{2\alpha}(\partial A)^2 - \frac{1}{4\beta}F_\pm^2\right]\right\} \delta(\partial\psi)\delta(D_\pm\psi) . \tag{8.8}$$

From the usual manipulations, the α-term reproduces the usual delta for ∂A and the β-term a delta for $F_\pm$,

$$Z_{gf} = \int \mathcal{D}A\mathcal{D}\psi \ \delta(\partial A)\delta(F_\pm)\delta(\partial\psi)\delta(D_\pm\psi) . \tag{8.9}$$

This expression shows that the gauge is fixed as we intended.

An alternative and perhaps more insightful computation can be performed as follows. The field $\bar{\chi}$ is anti-symmetric and (anti-)self-dual. So we can fully anti-symmetrize it. Moreover, we can already use the other constraint $\delta(\partial\psi)$ to replace ψ with ψ^T (transverse) in the term $\bar{\chi}D_\pm\psi^T$. The field $\bar{\chi}$, as an (anti-)self-dual tensor field, contains 3 degrees of freedom, just as the ψ^T. Let us denote it by $\bar{\chi}_{ind}$. Hence, we can say we have six Grassmann independent variables, which allows to schematically rewrite (see [49]) $\bar{\chi}D_\pm\psi^T \equiv (\bar{\chi}_{ind}, \psi^T) * M * (\bar{\chi}_{ind}, \psi^T)$. This matrix operator M is six-dimensional and so is the Grassmann vector $(\bar{\chi}_{ind}, \psi^T)$. Eventually, integration over the six-dimensional Grassmann vector leads to

$$Z_{gf} = \int \mathcal{D}A\mathcal{D}\psi \; \delta(\partial A)\delta(F_{\pm})\delta(\partial\psi)\text{Pfaff}(M) \; , \tag{8.10}$$

where Pfaff stands for the Pfaffian. From the general relation $\text{Pfaff}(M) = \det^{1/2}(M) = \det D_{\pm}$, we finally get

$$Z_{gf} = \int \mathcal{D}A\mathcal{D}\psi \; \delta(\partial A)\delta(F_{\pm})\delta(\partial\psi)\det(D_{\pm}) \; . \tag{8.11}$$

Of course, it is possible to cross from (8.9) to (8.11) by evaluating the last δ-constraint, keeping in mind the other constraints and the calculus rules to deal with Grassmann Jacobians [167]. The BRST method is however more convenient and more general than the Faddeev-Popov procedure. Indeed, not every gauge fixing needs to be of the "unity type", a famous example being the non-linear gauges of the Baulieu–Thierry-Mieg type [168].

The equivalence between the BRST and FP gauge-fixing procedures serves to illustrate that the expected degrees of freedom are naturally recovered. From (8.8), particularly, the extra constraint $\delta(D_{\pm}\psi)$ selects the correct physical spectrum, i.e., the instanton modes. It could be different keeping in mind the fact that the Witten and BS theories share the same global observables. From the action $S_{gf}(\alpha, \beta)$, the $\bar{\chi}$ equation of motion gives

$$\Theta^{ab}_{\mu\nu\beta}\psi^b_\beta = 0 \; , \tag{8.12}$$

whereby

$$\Theta^{ab}_{\mu\nu\beta} \equiv (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta} \pm \epsilon_{\mu\nu\alpha\beta})D^{ab}_\alpha \; , \tag{8.13}$$

while the $\bar{\eta}$ equation of motion gives

$$\partial_\mu \psi_\mu^a = 0 \; . \tag{8.14}$$

The two last equations are precisely the two equations concerning the infinitesimal instanton moduli. We obtain here the same situation as present in the Witten version of the theory, see [102]; the only difference is the gauge choice. If we want to reproduce exactly the Witten equations, we should use the gauge constraint $D_\mu^{ab}\psi_\mu^a = 0$ for the topological ghost, instead of the Landau one. As this is a gauge condition anyhow, physics should not depend on it. The reason to prefer the Landau gauge is the associated larger symmetry content, in particular the vector supersymmetry, as it was originally noticed in [53]. Anyway, the relation is the same, that is, $n = d(\mathcal{M})$ the number of solutions at the instanton moduli space of the equations (8.15)-(8.16). Indeed, for instanton solutions in the vicinity of A_μ^a, that is, $A_\mu^a + \delta A_\mu^a$, we get from (5.6) the condition

$$\Theta_{\mu\nu\beta}^{ab}\delta A_\beta^b = 0 \; , \tag{8.15}$$

while the Landau gauge imposes

$$\partial_\mu \delta A_\mu^a = 0 \; . \tag{8.16}$$

Here, $d(\mathcal{M})$ is the dimension of the moduli space $\mathcal{M}$[1].

As we shall discuss later, the aforementioned tree-level exactness persists when the Gribov gauge fixing ambiguity is dealt with à la Gribov-Zwanziger [42; 160; 169], thereby indicating that the Gribov copies are inoffensive for this type of

[1]For a thorough analysis of $d(\mathcal{M})$ and its relation with the first Pontryagin number of the bundle E ($p_1(E)$), Euler characteristic ($\chi(M)$) and signature ($\sigma(M)$) of the manifold M, according to the gauge group, see [114]. For the $SU(2)$ group, for instance, $d(\mathcal{M}) = 8p_1 - \frac{3}{2}(\chi + \sigma)$.

topological theory. This then also shows that the algebraic setup of [124] remains valid, even when Gribov copies are taken into account.

8.2 Gauge ambiguities and copy equations

To write down the conditions for the existence of Gribov copies, i.e., the possibility of having multiple solutions to the gauge fixing constraints, we start with the gauge field. Let A'^a_μ differ from A^a_μ — which satisfies the Landau gauge condition, by assumption — by a pure infinitesimal gauge transformation, i.e., $A'^a_\mu = A^a_\mu + \delta A^a_\mu$; the gauge transformed field will be a copy of A^a_μ if the following is satisfied

$$\partial_\mu A'^a_\mu = 0 \ , \tag{8.17}$$

which amounts to

$$\partial_\mu D^{ab}_\mu \omega^b + \partial_\mu \alpha^a_\mu = 0 \ . \tag{8.18}$$

Notice that, by virtue of the condition (5.5), the second term in the above equation actually drops out, but we will keep it for now, so that at later stage, it will become clear why the condition (5.5) is such a convenient one.

Similarly, the gauge condition (5.5) features infinitesimal copies if

$$\partial_\mu D^{ab}_\mu \lambda^b = 0 \ . \tag{8.19}$$

In the current context, there is the possibility for the field strength gauge condition $F^a_{\mu\nu}$ to have copies as well. This is a novelty introduced by the topological model, insofar as, in the usual Yang-Mills theory, $F^a_{\mu\nu}$ is completely defined by the first constraint on A^a_μ (5.4), while in the topological case there is another independent gauge ambiguity involving A^a_μ, which is reflected in the behavior of the field strength that also transforms as a gauge field (4.4), as we discussed above. From the (anti-)self-dual gauge fixing (5.6), the new condition is obtained as follows

$$F'^a_{\mu\nu} \pm \tilde{F}'^a_{\mu\nu} = F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu} \;, \tag{8.20}$$

so that a copy is possible when

$$D^{ab}_{[\mu}\alpha^b_{\nu]} \pm \epsilon_{\mu\nu\alpha\beta}D^{ab}_{\alpha}\alpha^b_{\beta} = 0 \;. \tag{8.21}$$

In summary, the conditions for the existence of infinitesimal Gribov copies for the three local gauge parameters of the model are

$$\partial_\mu D^{ab}_\mu \omega^b + \partial_\mu \alpha^a_\mu = 0 \;, \tag{8.22}$$

$$\partial_\mu D^{ab}_\mu \lambda^b = 0 \;, \tag{8.23}$$

$$D^{ab}_{[\mu}\alpha^b_{\nu]} \pm \epsilon_{\mu\nu\alpha\beta}D^{ab}_{\alpha}\alpha^b_{\beta} = 0 \;. \tag{8.24}$$

We must verify if the system of equations (8.22)-(8.24) allows for (normalizable) zero modes. If we set $\alpha_\mu = 0$, the third equation trivializes, while the first two reduce to

$$\partial_\mu D^{ab}_\mu \omega^b = 0 \;, \tag{8.25}$$

$$\partial_\mu D^{ab}_\mu \lambda^b = 0 \;, \tag{8.26}$$

which shows that there is a sector for a particular configuration of the gauge parameters in which the usual Gribov copies are present. Indeed, these two copies equations are identical to the one which characterizes the infinitesimal Gribov problem in Yang-Mills theories in the Landau gauge [42; 143; 144; 160; 164; 169].

Analyzing the third equation separately, we can easily check that this equation also allows for zero modes. For $h(x) \in G$, we know that $h^{-1}\partial_\mu h$ belongs to the Lie algebra defined by the gauge group G, i.e., $h^{-1}\partial_\mu h(x) = [h^{-1}\partial_\mu h]^a(x)T^a$ where $[h^{-1}\partial_\mu h]^a$ is a scalar function for each μ (and a) and T^a are the generators of the Lie algebra. Moreover, it is well-known that for a pure gauge configuration

$$F_{\mu\nu}(h^{-1}\partial h) = 0 \, , \tag{8.27}$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a$. So if we set $\alpha^a_\mu = D^{ab}_\mu[h^{-1}\partial h]^b$, by using

$$[D_\mu, D_\nu] = F_{\mu\nu} \, , \tag{8.28}$$

we will get in both terms of (8.24) the expression (8.27), which shows in a simple way that (8.24) admits zero modes as well.

In the following, we discuss the relevance of these copies in view of the instanton properties of the moduli space and develop a strategy to eliminate them from the path integration.

8.3 Elimination of the infinitesimal copies

In order to eliminate the ambiguities related to the infinitesimal Gribov copies, we can start by eliminating the Gribov copies present in the sector $\alpha^a_\mu = 0$. For that, according to equations (8.25) and (8.26), we shall implement the usual Gribov-Zwanziger restriction to the region Ω defined by eq. (7.30), in which one imposes that the real eigenvalues of the Hermitian operator $-\partial_\mu D^{ab}_\mu \equiv -\partial D$ are positive. At its boundary, $\partial\Omega$, the FP operator acquires its first vanishing eigenvalues. This imposition eliminates the infinitesimal copies generated by the first two equations, viz. (8.22) and (8.23).

In the case with $\alpha^a_\mu \neq 0$ we can decompose α^a_μ according to the Helmholtz decomposition [122]. Since we are working in flat Euclidean space, for $\alpha^a_\mu(x)$

fields sufficiently smooth[1] that fall off as $\frac{1}{r}$ or faster at infinity, we may rely on a generalization of the Helmholtz theorem by which we can write the four-vector $\alpha^a_\mu(x)$ as

$$\begin{aligned}\alpha^a_\mu(x) &= -\partial_\mu\left[\int_{V'_4}\frac{\partial'_\nu\alpha^a_\nu(x')}{4\pi^2R^2(x,x')}d^4x' - \oint_{\Sigma'}\frac{\alpha^a_\nu(x')n'_\nu}{4\pi^2R^2(x,x')}d\Sigma'\right]\\ &- \partial_\beta\left[\int_{V'_4}\frac{\partial'_\beta\alpha^a_\mu(x') - \partial'_\mu\alpha^a_\beta(x')}{4\pi^2R^2(x,x')}d^4x' + \oint_{\Sigma'}\frac{\alpha^a_\beta(x')n'_\mu - \alpha^a_\mu(x')n'_\beta}{4\pi^2R^2(x,x')}d\Sigma'\right],\end{aligned} \tag{8.29}$$

with $R^2(x,x') = |x-x'|^2$, and n'_μ is the four-vector outward unit normal of the three-surface Σ' which encloses the four-volume V'_4, Σ' itself being sufficiently smooth. Thus, eliminating the surface integrals for vanishing fields on the boundary according to the conditions above, we conclude that we can split $\alpha^a_\mu(x)$ into its longitudinal and transverse parts in the form

$$\alpha^a_\mu = \partial_\mu\phi^a + \partial_\beta T^a_{\beta\mu}\ , \tag{8.30}$$

where ϕ^a is a scalar field, and $T^a_{\beta\mu}$ is an antisymmetric tensor given, respectively, by

$$\phi^a = -\int_{V'_4}\frac{\partial'_\nu\alpha^a_\nu(x')}{4\pi^2R^2(x,x')}d^4x'\ , \tag{8.31}$$

and

$$T^a_{\beta\mu} = -\int_{V'_4}\frac{\partial'_\beta\alpha^a_\mu(x') - \partial'_\mu\alpha^a_\beta(x')}{4\pi^2R^2(x,x')}d^4x'\ . \tag{8.32}$$

The divergence of the second term in (8.30) vanishes. Therefore,

$$\partial_\mu\alpha^a_\mu = \partial^2\phi^a, \quad \text{where} \quad \partial_\mu\partial_\mu \equiv \partial^2. \tag{8.33}$$

Returning to the copy equation (8.22), in principle if one chooses e.g. $\phi =$

[1]Here the term "sufficiently smooth" means functions that are at least C^2, i.e., twice continuously differentiable functions on the closure of the four-dimensional volume V_4.

$-\frac{\partial_\mu}{\partial^2}D_\mu\omega$, then this equation (8.22) has a solution. This would imply, in general, that all Gribov copies that exist in Yang-Mills theories are removed, but it is logically possible to generate new ones with a non-vanishing topological shift. But now comes the fact that so far, we did not use yet the second gauge condition (5.5). Doing so, the gauge condition for α_μ (or ψ_μ) demands that it must be transverse, which allows just for trivial ϕ (i.e., ψ_μ must be transverse). Thence, the usual Gribov restriction also eliminates the copies related to the gauge transformation of the topological parameter.

It remains to deal with eq. (8.24), the third copy equation. At a first glance, the condition $-\partial D > 0$ does not tell anything about the instantons. We could think about an analogous procedure to eliminate the copies arising from the third equation (8.24). Rewriting eq. (8.24) as

$$i\Theta^{ab}_{\mu\nu\beta}\alpha^b_\beta = 0 \;, \tag{8.34}$$

with $\Theta^{ab}_{\mu\nu\beta}$ defined in eq. (8.13), we could employ the extra Gribov-like restriction $i\Theta^{ab}_{\mu\nu\beta} \equiv i\Theta > 0$, i.e., we would impose positive eigenvalues for the Hermitian operator $i\Theta$. However, let us now motivate why this third restriction is not necessary.

Firstly, we recall that Witten noted that the partition function Z of his topological theory is independent of changes of the coupling constant g^2 (as long as $g^2 \neq 0$). He used this liberty to compute the observables in the weak coupling limit, $g^2 \to 0$, from which he obtained the Donaldson polynomials. The evaluation of Z in the weak coupling limit means that the theory is dominated by the classical minima. These minima correspond to the (anti-)instanton configurations $F^a_{\mu\nu} = \pm\widetilde{F}^a_{\mu\nu}$[1]. Once it was proven that the observables of the Witten

[1]See Sections 2.1.1 and 3.1.2.

and Baulieu-Singer theories are the same[1], we should then consider the instanton characterization not as a gauge fixing condition, but as a physical requirement in order to obtain the correct degrees of freedom that correspond to the description of all global observables. This was also stressed in [40]: condition (5.6) does not completely fix the gauge, on purpose, to be left with the finite set of degrees of freedom describing the instantons, the latter being exactly the kernel of (5.6). In fact, the (bosonic) "zero modes of the 3rd kind" will be exactly cancelled in computations against fermionic zero modes, related to the $\bar{\chi}$-equation of motion, see again [40]. Precisely, the Atiyah-Singer index theorem [170] counts the number of solutions of (8.15) and (8.16), which gives the correct dimension of the instanton moduli space, in complete harmony with instanton conformal properties[2] [171; 172]. In this sense, the structure of (8.15) and (8.16), and therefore of (8.34), are protected by the Atiyah-Singer theorem and its direct correspondence with the conformal properties of instanton configurations, indicating that no extra physical restrictions on the eigenvalues of $i\Theta$ need to be introduced.

However, one might question whether the restriction of the gauge fields to the Gribov region does not hamper the fact that we wish to "preserve" the instantons, as just motivated. In the case of the simplest $SU(2)$ instanton, we can provide an affirmative answer to this, inspired by the observations of [173]. Indeed, in this case the instanton field with winding number 1 is given by the expression

$$A^{(\mathrm{i})a}_{\ \ \mu} = \frac{1}{g}\frac{2}{r^2+\lambda^2} r_\nu \zeta^a_{\nu\mu} \,, \tag{8.35}$$

[1]See Section 4.2.1.

[2]If we take, for instance, the BPST instanton (2.33), it possesses 4 parameters for each translation X_μ, since the instanton is located in R^4; 1 scale size λ, and 3 global parameters associated to the $SU(2)$ gauge transformation, as the instanton is embedded in the gauge group. 8 parameters in total. As discussed in [171], the parameter λ arises from broken conformal invariance. The Atiyah-Singer theorem, in Euclidean space, gives the dimension of the moduli space $d(\mathcal{M}) = 4kN$, where k is the winding number. For the $SU(2)$ BPST instanton, $k = 1$ and $N = 2$, so that $d(\mathcal{M}) = 8$, as it should be.

see eq. (2.33), where we just call $(x-X)_\nu \equiv r_\nu$; λ denotes the "size" of the instanton, as discussed in Section 2, while the real constant antisymmetric matrices ζ^a are the 't Hooft tensors defined in (2.34), that obey the algebra

$$\begin{aligned} \left[\zeta^a, \zeta^b\right] &= 2f^{abc}\zeta^c \,, \\ \{\zeta^a, \zeta^b\} &= -\delta^{ab} \,. \end{aligned} \tag{8.36}$$

As we can see,

$$\partial_\mu A_\mu^{a\,(i)} = 0 \,, \tag{8.37}$$

which means that the (regular) instanton field is transverse and in the Landau gauge. From the latter transversality of the instanton field, the eigenvalue equation for the FP operator,

$$\mathcal{M}^{ab}(A^{(i)})\phi^a = -\omega^2\phi^a, \tag{8.38}$$

takes the form

$$\partial^2\phi^a + f^{abc}\frac{2}{r^2+\lambda^2}r_\mu\zeta^a_{\nu\mu}\partial_\nu\phi^c = -\omega^2\phi^a. \tag{8.39}$$

We immediately notice that this instanton has three trivial constant zero-modes. The other zero modes (thus giving $\omega = 0$) of eq. (8.39) were explicitly constructed in [173]. This means that the instanton belongs to the Gribov horizon $\partial\Omega$.

There is no strict proof that all instantons (with higher winding number) belong to the first Gribov region, but to the best of our knowledge, in the cases investigated in literature, topological Yang-Mills solutions (instanton, monopole, vortex) always belong to it — see again [173], or [174]. Let us also refer to [175], where it was discussed that for instantons a whole family of Gribov copies does exist.

The consequence of such rich zero-mode spectrum to our problem is immediate. If we consider the Gribov restriction, $-\partial D > 0$, for a generic gauge field in order to eliminate the Gribov copies in the first two copies equations, (8.22) and (8.23), the instantons belongs to the boundary of the first Gribov region, $\partial\Omega$ (where $-\partial D$ becomes zero) and are as such not eliminated from the game. One notes this property by the fact that the instantons are transverse, and the spectrum of the FP operator evaluated for an instanton displays zero modes. From the point of view of gauge copies under $-\partial D \geq 0$, the gauge fields obeying the (anti-)self dual condition $F = \pm\widetilde{F}$ are well-defined. The solutions to $F = \pm\widetilde{F}$ are elements of $\partial\Omega$.

Summing up, the only requirement to eliminate all (infinitesimal) gauge ambiguities is then the introduction of the Gribov horizon as it commonly done for usual Yang-Mills theories[1]. Then it remains to prove in the following section that also this restriction to the standard Gribov horizon eventually becomes trivial at the dynamical level.

8.4 Gribov gap equation and its triviality

We have mentioned that the tree-level exactness of the topological theory in the (A)SDL gauges, demonstrated in Section 6.3 cf. [56], suggests that the Gribov copies present in our model should be inoffensive. Due to the absence of radiative corrections, the tree-level propagator of the FP ghost field in momentum space obtained from the total action (5.7),

$$\langle \bar{c}^a c^b \rangle_0 (p) = \delta^{ab} \frac{1}{p^2} \ , \tag{8.40}$$

[1]Although all points discussed here indicate a similar behavior for a generic $SU(N)$ instanton field with an arbitrary winding number, a possible analytical treatment of such instantons will not be considered in the thesis.

will be valid to all orders in perturbation theory. From the expression above, one sees that the FP operator will be positive definite at the quantum level, consistent with the inverse of the FP propagator being positive, i.e., we are inside the first Gribov region, in such a way that the Gribov restriction to the path integral seems to be redundant. The origin of such behavior is the impossibility of closing loops in Feynman diagrams, as due to the vertex structures, at least one gauge field propagator is required to close loops, but $\langle A^a_\mu(x)A^b_\nu(y)\rangle = 0$ to all orders for this gauge choice [55; 56]. We point out that the same argument holds for the analysis of the third Gribov equation (8.24) and the propagator $\langle\bar{\chi}^a_{\mu\nu}\psi^b_\alpha\rangle_0(p)$.

Originally, the no-pole condition was achieved by treating the gauge field as an external source. Its quantum properties must be computed when the gauge field is integrated over. If we admit the Gribov copies to play a role in this case, we should consider that the introduction of the term that implements the restriction to the Gribov region might allow for radiative corrections, e.g. from a non-vanishing gauge propagator arising from the extra Gribov term (a metric dependent term) in the action. This might perturb the original cohomology arguments and, consequently, compromise the global properties of the topological theory at certain energy scale, this through the elimination of Gribov ambiguities. Taking into account the reasons discussed above, such behavior is highly unexpected. We will now show this in detail, first at one loop, afterwards we will generalize to all orders.

8.4.1 No-pole condition at one-loop

As discussed, all infinitesimal Gribov copies in the topological theory in (A)SDL gauges for the $SU(2)$ instanton are eliminated through the implementation of the restriction to the well-known Gribov region denoted by Ω (7.30), commonly performed in usual Yang-Mills theories in Landau gauge. Following the Gribov approach, this restriction is achieved via the introduction of a form factor $V(\Omega)$ in the generating function $Z[J]$, in such a way that the integration domain is limited by Ω. The original generating functional

$$Z_o[J] = \mathcal{N} \int \mathcal{D}\Phi \, e^{-S - \int d^4x J\Phi} \,, \tag{8.41}$$

is restricted to

$$Z[J] = \mathcal{N} \int_{\Omega} \mathcal{D}\Phi \, e^{-S - \int d^4x J\Phi} = \mathcal{N} \int \mathcal{D}\Phi V(\Omega) \, e^{-S - \int d^4x J\Phi} \,, \tag{8.42}$$

where $\mathcal{N} = Z[0]^{-1}$ is the normalization factor, $\mathcal{D}\Phi$ denotes the integration measure for all fields, i.e., $\mathcal{D}\Phi = \mathcal{D}A\mathcal{D}\psi\mathcal{D}c\mathcal{D}\phi\mathcal{D}\bar{c}\mathcal{D}b\mathcal{D}\bar{\phi}\mathcal{D}\bar{\eta}\mathcal{D}\bar{\chi}\mathcal{D}B$, while $J\Phi = J_i\Phi_i$ denotes the coupling of each field Φ_i with its respective external source J_i.

In the Yang-Mills theory, the form factor $V(\Omega)$ is obtained from the no-pole condition for the FP propagator, since the imposition $\mathcal{M}^{ab} > 0$ is equivalent to forbidding the existence of poles in the FP propagator [42]. In the topological case, see action (5.8), the operator $\mathcal{M}^{ab} = -\partial_\mu D_\mu^{ab}$ appears twice: in the FP ghost-anti-ghost quadratic term (treating A_μ^a as an external source), $\bar{c}\partial Dc$, as usual, but also in the bosonic ghost-anti-ghost term, $\bar{\phi}\partial D\phi$. By applying the Gribov semi-classical method we shall see that, at one-loop order, the no-pole condition in the topological theory takes the same form as for the standard Yang-Mills case.

For this purpose, we have only to analyze the vertices present in the total action (5.7), and apply the Feynman rules for the diagrams up to the order g^2, once we are considering the one-loop order. We should then verify which diagrams could be constructed with an incoming $\bar{c}$-leg ($\bar{\phi}$-leg), and an outgoing c-leg (ϕ-leg), whereby the gauge fields work as external sources. Let us start with the FP ghost propagator.

(i) *FP ghost propagator.* Using the following notation for the ghost propagator at one-loop with A as an external source, see eq.'s (7.32) and (7.40),

$$\langle \bar{c}^a(k)c^b(p)\rangle = \delta(p+k)\mathcal{G}^{ab}(k^2,A) = \delta(p+k)\delta^{ab}\frac{1}{k^2}\left[1+\sigma(k,A)\right], \tag{8.43}$$

our aim is to calculate $\sigma(k,A)$, which represents the loop correction to the tree-level part $1/k^2$. Firstly, we must note that the FP anti-ghost, $\bar{c}$, only propagates to c and ψ through the propagators $\langle\bar{c}c\rangle_0$ and $\langle\bar{c}\psi\rangle_0$ at the tree-level, respectively. Therefore, if we start with an incoming $\bar{c}$, we can propagate it to the vertices (a) $\bar{\phi}c\psi$, (b) $\bar{\chi}\partial A\psi$, (c) $\bar{\chi}cA$, (d) $\bar{\chi}cAA$, or (e) $\bar{c}Ac$. The first one does not produce external A-legs. If we propagate $\bar{c}$ to the vertex (b) through $\langle\bar{c}\psi\rangle_0$, we will get an external A-leg, and an internal $\bar{\chi}$-leg. Since $\bar{\chi}$ only propagates to ψ through $\langle\bar{\chi}\psi\rangle_0$, we could only connect at one-loop order the vertex (b) to another vertex $\bar{\chi}\partial A\psi$, producing one more time an external A-leg, and an internal $\bar{\chi}$, in such a way that we cannot generate an outgoing c. For the vertices (c) and (d), we fall back to the same situation: we generate external A-legs, but always accompanied by the internal $\bar{\chi}$-leg that never propagates to c in the end. We conclude that the only possibility to get an outgoing c from $\bar{c}$ with only external A-legs is to construct the diagram by using the vertex (e)[1]. Namely, for

$$\mathcal{G}(k^2,A) = \frac{1}{N^2-1}\delta^{ab}G(k^2,A)^{ab}\,, \tag{8.44}$$

we construct the diagrams displayed in Figure 7.2 in Section 7.3, which proves that the possible diagrams are reduced to the same ones of the standard Yang-Mills theory. We conclude that the no-pole condition for the FP ghost propagator in this topological model gives the same result of the one found for Yang-Mills theory, *i.e.*, the same volume factor $V(\Omega)$ described in (7.48) according to (7.47).

[1]We remark that the whole argument can be made easier by a redefinition $\bar{\eta}\mapsto\bar{\eta}+\bar{c}$ in the action (5.7) in order to eliminate the $\bar{\eta}\psi$ mixing term.

We should then introduce this factor into the path integral in order to implement the elimination of the gauge copies. We must do the same procedure to eliminate possible copies in the bosonic ghost propagator, but as we will see now, the no-pole condition (7.46) for the bosonic ghost is valid for both, the FP and bosonic ghosts.

(ii) *Bosonic ghost propagator*. The proof of the last statement is immediate. The bosonic anti-ghost field $\bar{\phi}$ only propagates to ϕ through $\langle\bar{\phi}\phi\rangle_0$, thus an incoming $\bar{\phi}$, we can only connect to the vertex $\bar{\phi}A\phi$. Aftermath, the construction of the Feynman diagrams up to g^2 order with A fields as external sources takes the same form of the FP case, see Figure 7.2, only replacing $\bar{c}$ by $\bar{\phi}$, and c by ϕ. The Feynman rules are exactly the same, consequently the no-pole condition for the bosonic ghost generates the same expression for $\sigma(k, A)$, and the condition (7.46) is valid for the FP and bosonic ghost sectors.

In a few words, although the complex structure of the total action (5.7), in which there are two ghosts sectors to implement the no-pole condition, for the FP ghost sector and the bosonic one, the elimination of all Gribov copies in the topological Yang-Mills in the (A)SDL gauges for $SU(2)$ instantons is achieved by introducing in the path integral a form factor $V(\Omega)$ (7.48) which is identical to the one obtained in the usual Yang-Mills theory in the Landau gauge.

8.4.2 Gap equation at one-loop

From (5.7), (8.42) and (**??**), the generating functional for the first Gribov region takes the form

$$Z = N \int \mathcal{D}A^a_\mu \mathcal{D}\Phi' \int \frac{d\xi^2}{2\pi\xi^2 i} \exp\{\xi^2 - S - \xi^2\sigma(0, A)\}. \qquad (8.45)$$

in which Φ' denotes all fields except the gauge field. The effective potential, Γ, is defined as usual by

$$e^{-\Gamma} = e^{-V\varepsilon} = Z \ , \qquad (8.46)$$

where ε represents the vacuum energy.

In order to calculate Γ at one-loop order, $\Gamma^{(1)} = V\varepsilon^{(1)}$, we must select only the quadratic part of the total action S (here $\sigma(0, A)$ is already quadratic as it was only calculated up to one-loop order), namely,

$$e^{-V\varepsilon^{(1)}} = Z_{quad} \ , \qquad (8.47)$$

whereby, using (7.47),

$$e^{-\Gamma^{(1)}} = \int \mathcal{D}A^a_\mu \mathcal{D}\Phi' e^{-S_{\rm quad}[\Phi]}. \qquad (8.48)$$

After integrating out the auxiliary fields b^a, $B^a_{\mu\nu}$, and all other fields except A^a_μ, we get the quadratic action for the gauge field

$$S_{\rm quad}[A] = \int d^4p A^a_\mu(p) \left[\frac{4}{\beta}p^2\delta_{\mu\nu} - \left(\frac{4}{\beta} - \frac{1}{\alpha}\right)p_\mu p_\nu\right] A^a_\nu(-p) + {\rm rest} \ . \qquad (8.49)$$

Taking into account all quadratic terms,

$$Z_{quad} = N \int \mathcal{D}A^a_\mu \int \frac{d\xi^2}{2\pi\xi i} \exp\left\{\xi^2 - \ln\xi - \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A^a_\mu(k) Q_{\mu\nu}(k,\xi)\delta^{ab} A^b_\nu(-k) + {\rm rest}\right\} \ , \qquad (8.50)$$

wherein

$$Q_{\mu\nu}(k,\xi) = \left[\frac{4}{\beta}k^2 + \frac{\xi^2 g^2 N}{2V(N^2-1)k^2}\right]\delta_{\mu\nu} + \left(\frac{1}{\alpha} - \frac{4}{\beta}\right)k_\mu k_\nu \ . \qquad (8.51)$$

Therefore,

$$Z_{quad} = N \int \frac{d\xi}{2\pi i} e^{[f(\xi)+\text{rest}']} , \tag{8.52}$$

where,

$$f(\xi) = \xi^2 - \frac{1}{V} \ln \xi + \ln[(\det Q_{\mu\nu}\delta^{ab})^{-\frac{1}{2}}] = \xi^2 - \frac{1}{V} \ln \xi - \frac{1}{2} \ln \det[Q_{\mu\nu}\delta^{ab}] . \tag{8.53}$$

We also changed the variable $\xi^2 \to \xi^2 V$ to pull out explicitly the volume factor here, to make clear that the action is an extensive quantity ($\sim V$).

Calculating the determinant, we find

$$\ln \det[Q_{\mu\nu}\delta^{ab}] = (N^2-1)(d-1) \sum_k \ln \left(\frac{\beta A + 4k^4}{\beta k^2} \right) + (N^2-1) \sum_k \ln \left(\frac{k^2}{\alpha} + \frac{A}{k^2} \right) , \tag{8.54}$$

where

$$A = \frac{\xi^2 g^2 N}{2(N^2-1)}, \tag{8.55}$$

and k refers to momenta in Fourier space. Working out the last term of (8.54), we get

$$\sum_k \ln \left(\frac{k^2}{\alpha} + \frac{A}{k^2} \right) = \sum_k \ln \left(\frac{k^4}{\alpha} + A \right) - \sum_k \ln k^2 , \tag{8.56}$$

Taking the thermodynamic limit and employing dimensional regularization, $\int dk \ln k^2 - \to 0$, and the last term vanishes. Therefore

$$\sum_k \ln \left(\frac{k^2}{\alpha} + \frac{A}{k^2} \right) = V \int \frac{d^d k}{(2\pi)^d} \ln \left(\frac{k^2}{\sqrt{\alpha}} + i\sqrt{A} \right) + V \int \frac{d^d k}{(2\pi)^d} \ln \left(\frac{k^2}{\sqrt{\alpha}} - i\sqrt{A} \right) \sim \alpha^{\frac{d}{4}} \tag{8.57}$$

In the limit $\alpha \to 0$, this term also vanishes. In the end,

$$\ln \det[Q_{\mu\nu}\delta^{ab}] = (N^2-1)(d-1) \int \frac{d^d k}{(2\pi)^d} \ln \left(\frac{\xi^2 g^2 N}{2(N^2-1)k^2} + \frac{4k^2}{\beta} \right), \tag{8.58}$$

which could be rewritten as

$$\ln \det[Q_{\mu\nu}\delta^{ab}] = (N^2-1)(d-1)\left[\int \frac{d^dk}{(2\pi)^d}\ln\left(\beta\xi^2 g^2 N + 4k^2\right) - \int \frac{d^dk}{(2\pi)^d}\ln\left(2\beta(N^2-1)k^2\right)\right]. \tag{8.59}$$

In dimensional regularization, not only the last term is zero, but also the first one, as we should still take the limit $\beta \to 0$. We conclude that

$$f(\xi) = \xi^2 , \tag{8.60}$$

as we work in the thermodynamic limit, $V \to \infty$. The gap equation, viz. the equation for the critical point for a saddle point evaluation, thus gives the trivial solution

$$\xi = 0 , \tag{8.61}$$

to $f'(\xi) = 0$. So, up to leading order, the no-pole condition does not change the partition function at all, see (8.45) in conjunction with (8.61).

8.4.3 Absence of radiative corrections in the presence of the Gribov-Zwanziger horizon

In order to extend the result (8.61) to all orders, we must prove first that the topological BS theory in (A)SDL gauges remains tree-level exact in the presence of the Gribov-Zwanziger horizon function. Along the lines of [56], we need the tree-level propagators in order to show that all n-point functions are tree-level exact. The non-vanishing tree-level propagators which are relevant for the present work are computed from the action S_{loc} (8.69), which is composed of the BS action action in (A)SDL gauges (5.7) added to the GZ horizon function $h(A)$ in its local form,

$$S_{loc} = S_0 + S_{gf} + h(A) \tag{8.62}$$

where

$$h(A) = -\int d^4x \left(\bar{\varphi}^{ac}_\mu \mathcal{M}^{ab}(A)\varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab}(A)\omega^{bc}_\mu + \gamma^2\; g f^{abc} A^a_\mu (\varphi + \bar{\varphi})^{bc}_\mu\right) \;, \tag{8.63}$$

see (7.80). The corresponding non-vanishing tree-level propagators of S_{loc} are

$$\begin{aligned}
\langle U^{ab}_\mu(-k) U^{cd}_\nu(k)\rangle &= -\frac{1}{k^2}\delta_{\mu\nu}\delta^{abcd} \;, \\
\langle V^{ab}_\mu(-k) V^{cd}_\nu(k)\rangle &= -\frac{1}{k^2}\delta_{\mu\nu}\delta^{abcd} \;, \\
\langle b^a(-k) b^b(k)\rangle &= -2Ng^2\gamma^4\frac{1}{k^4}\delta^{ab} \;, \\
\langle B^a_{\mu\nu}(-k) B^b_{\alpha\beta}(k)\rangle &= -Ng^2\gamma^4\frac{1}{k^4}\delta_{\mu\nu\alpha\beta}\delta^{ab} \;, \\
\langle A^a_\mu(-k) b^b(k)\rangle &= -i\frac{k_\mu}{k^2}\delta^{ab} \;, \\
\langle A^a_\mu(-k) B^b_{\alpha\beta}(k)\rangle &= i\frac{1}{k^2}\Sigma_{\mu\alpha\beta}\delta^{ab} \;, \\
\langle b^a(-k) U^{bc}_\mu(k)\rangle &= i\sqrt{2}\gamma^2\frac{k_\mu}{k^4}f^{abc} \;, \\
\langle B^a_{\mu\nu}(-k) U^{bc}_\alpha(k)\rangle &= i\sqrt{2}g\gamma^2\frac{1}{k^4}\Sigma_{\alpha\mu\nu}f^{abc} \;,
\end{aligned} \tag{8.64}$$

while the vanishing tree-level propagators are

$$\begin{aligned}
\langle A^a_\mu(-k)A^b_\nu(k)\rangle = \langle A^a_\mu(-k)U^{bc}_\nu(k)\rangle = \langle b^a(-k)B^b_{\mu\nu}(k)\rangle &= 0 \;, \\
\langle V^{ab}_\mu(-k)A^c_\nu(k)\rangle = \langle V^{ab}_\mu(-k)U^{cd}_\nu(k)\rangle = \langle V^{ab}_\mu(-k)B^c_{\alpha\beta}(k)\rangle = \langle V^{ab}_\mu(-k)b^c(k)\rangle &= 0 \;,
\end{aligned} \tag{8.65}$$

with

$$\begin{aligned}
\varphi^{ab}_\mu &= \frac{\sqrt{2}}{2}\left(U + iV\right)^{ab}_\mu \;, \\
\bar{\varphi}^{ab}_\mu &= \frac{\sqrt{2}}{2}\left(U - iV\right)^{ab}_\mu \;,
\end{aligned} \tag{8.66}$$

and U and V being real fields. Moreover,

$$\begin{aligned}
\Sigma_{\alpha\mu\nu} &= \frac{1}{2}\left(\delta_{\alpha\mu}k_\nu - \delta_{\alpha\nu}k_\mu\right) \ , \\
\delta^{abcd} &= \frac{1}{2}\left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right) \ , \\
\delta_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \ ,
\end{aligned} \tag{8.67}$$

according to the symmetries of Lorentz and color indices. The remaining propagators can be found in Section 5.3.2, cf. [54; 56]. Hence, if we compare the present situation with the scenario of [56], we have the extra non-vanishing propagators given by (8.64) together with four new vertices (see the local action (8.69)), namely: (i) $\bar{\varphi}A\varphi$, (ii) $\bar{\omega}A\omega$, (iii) $\bar{\omega}\varphi c$, and (iv) $\bar{\omega}A\varphi c$. Again, there is no vertex with b, so we cannot use $\langle bb\rangle$ to propagate b to a loop diagram. Using the propagator $\langle BB\rangle$, we can only propagate an external B to the vertex BAA, increasing the number of A fields. This is the same cascade effect that occurs with the $\langle AB\rangle$ propagator as in [56]. The new vertices (i), (ii) and (iii) have one A-leg. To not produce an internal A-leg we need to propagate it to an external field, but again A only propagates through $\langle AB\rangle$ and $\langle Ab\rangle$, producing only B and b as external legs, since the propagators with A and the new fields vanish: $\langle A\omega\rangle = \langle A\bar{\omega}\rangle = \langle A\varphi\rangle = \langle A\bar{\varphi}\rangle = 0$.

The only possible problematic vertex is (iii), which does not possess A-legs, but we cannot propagate a vertex (iii) to another vertex (iii) because $\bar{\omega}$ only propagates to the vertex (ii) through $\langle\bar{\omega}\omega\rangle$; c only propagates to the vertex $\bar{c}Ac$ through $\langle\bar{c}c\rangle$; and φ only to vertex (i) through $\langle\bar{\varphi}\varphi\rangle$, or to external legs B and b through $\langle\varphi B\rangle$ and $\langle\varphi b\rangle$, or to the vertex BAA through $\langle\varphi B\rangle$. In the end, we can only propagate the vertex (iii) to vertices with internal A-legs or to external legs B and b. We conclude that all loop diagrams vanish, because we fall back to the same situation in which we can only construct a loop diagram with B and b as external legs, in order to avoid internal A-legs, but $\langle B \cdots Bb \cdots b \rangle = \langle s(\text{something}) \rangle = 0$, due to BRST cohomology. Otherwise, it is impossible to close non-vanishing loops as we need gauge propagators to do it, and $\langle AA \rangle$ also vanishes in the presence of the local Gribov terms (8.76).

8.4.4 Extension to all orders

Let us now extend the result (8.61) and prove that is valid to all orders in perturbation theory. Therefore, we will rely on the local version of the horizon function. Following the steps of [160; 169], see Section 7.4, the restriction to the region Ω to all orders is given by considering the following partition function,

$$Z = \int \mathcal{D}\Phi \mathrm{e}^{-S+\gamma^4 h(A) - 4V\gamma^4(N^2-1)} \,, \tag{8.68}$$

where $S = S_0[A] + S_{gf}[\Phi]$ is defined in (5.7) and $h(A)$ is the Gribov-Zwanziger horizon function described in eq. (7.72). We must remember that $h(A)$ reduces to $\frac{g^2 N}{V} \int d^d x A \frac{1}{\partial^2} A$ at lowest order, in fact recovering $\sigma(0, A)$ of the no-pole condition at one-loop (7.47). In the all-order Gribov-Zwanziger formalism, the Θ-function is also replaced by a δ-function in the thermodynamic limit [160; 169], $V \to \infty$, as we have made explicit before.

The non-local horizon function $h(A)$ can be equivalently written in a local form through a pair of bosonic auxiliary fields $(\bar{\varphi}, \varphi)^{ab}_\mu$ and a pair of anticommuting fields $(\bar{\omega}, \omega)^{ab}_\mu$ [169], see Section 7.4.1. In the current case, it means replacing the exponent of (8.68) by the local action

$$S_{loc} = S - \int d^4 x \left(\bar{\varphi}^{ac}_\mu \mathcal{M}^{ab}(A) \varphi^{bc}_\mu - \bar{\omega}^{ac}_\mu \mathcal{M}^{ab}(A) \omega^{bc}_\mu + \gamma^2 \, g f^{abc} A^a_\mu (\varphi + \bar{\varphi})^{bc}_\mu \right) \,. \tag{8.69}$$

In the local formulation, the gap equation reads, cf. eq. (7.84),

$$\frac{\partial \varepsilon}{\partial \gamma^2} = 0 \ . \tag{8.70}$$

This relation connects the semi-classical method characterized by the no-pole ghost condition with the Zwanziger horizon function. Indeed, for the reader's belief, let us analyze the leading order limit.

At one-loop order, the geometric interpretation of thermodynamic limit is very simple: the Gribov no-pole condition (7.46), replacing $\frac{A^a_\mu(k)}{\sqrt{k^2}}$ by $x^a_{\mu \vec{k}} \equiv x_{\vec{k}}$, could be written as

$$\frac{1}{V}\sum_{\vec{k}} x_{\vec{k}} x_{-\vec{k}} < r^2 \ , \tag{8.71}$$

where $r^2 = \frac{4(N^2-1)}{g^2}$. The expression above can be interpreted as an hypersphere in an infinite dimensional space. As it is well-known for hyperspheres, as the dimension grows, the volume of a hypersphere is getting more and more concentrated on the boundary, i.e., on the hypersurface defined, in our case, by the ellipsoid

$$\frac{1}{V}\sum_{\vec{k}} x_{\vec{k}} x_{-\vec{k}} = r^2 \ , \tag{8.72}$$

which means that the Θ-function that represents (8.71) could effectively be replaced by a δ-function in the thermodynamic limit. The collapse of the Θ-function into the δ-function is then expressed by

$$\int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{d\xi^2}{2\pi i \xi^2} e^{\xi^2(1-\sigma(0,A))} \to \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{d\xi^2}{2\pi i} e^{\xi^2(1-\sigma(0,A))} = \int_{-\infty}^{+\infty} \frac{d\xi^2}{2\pi} e^{i\xi^2(1-\sigma(0,A))} \ , \tag{8.73}$$

after a Wick rotation. In practice we just canceled the ξ^2 in the denominator, responsible for the second term in (8.53). The behavior of ξ, in turn, is only determined by the gap equation (8.72). The vacuum energy can be computed from the $\ln\det$ originating from the action (8.69), leading to exactly the same result as in the previous subsection, upon identifying ξ^2 and γ^4.

We conclude, without inconsistency between the both methods, that the Gribov copies (still at one-loop so far) are inoffensive to the $SU(2)$ topological Yang-Mills theory in the (A)SDL gauges, since the gap equation forbids the introduction of a Gribov massive parameter in the thermodynamic limit,

$$\xi\sigma(0,A) \sim \gamma^4 \int d^4k A \frac{1}{k^2} A \to 0 \; . \tag{8.74}$$

Finally, let us look to what happens beyond the $\ln\det$-level. Then, the vertices of the theory will start to play role. Based on the vertex structure of S_{loc}, it is easy to see that any vacuum diagram beyond one-loop will contain at least one AA-propagator. However, by inverting the quadratic form in (8.50), this propagator is given by[1]

$$D^{ab}_{\mu\nu} = \delta^{ab}\left[\frac{\beta}{4}\frac{p^2}{(p^4+\beta N g^2\gamma^4/2)}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\alpha\frac{p^2}{(p^4+2\alpha N g^2\gamma^4)}\frac{p_\mu p_\nu}{p^2}\right] , \tag{8.75}$$

i.e.,

$$\langle AA\rangle = 0 \tag{8.76}$$

if we take α, $\beta \to 0$, irrespective of γ^2. We immediately get that all higher order terms to the vacuum energy vanish, just as the $\ln\det$. As such, by employing the gap equation (8.70) which is valid to all orders, we can infer that the massive Gribov parameter vanishes to all orders in the thermodynamic limit. In this way, the global (topological and cohomological) properties of the original action

[1] We have listed all propagators in previous section, where we proved that the absence of radiative corrections [56] remains valid for the inclusion of the GZ horizon function.

are not violated and we come to the main result of this paper: quantization of the topological theory remains valid as it is, the resolution of the infinitesimal ("small") Gribov copy problem is trivial as the intrinsic topological features of the theory self-consistently forbid the introduction of the Gribov mass, the crux of the Gribov-Zwanziger restriction [42; 160; 169] when it comes to changing the structure of the theory.

One might wonder if it actually makes sense to have computed the above effective action by expanding around the trivial $A = 0$ sector, thinking about the importance of the instanton configurations for topological field theories.

Exactly the topological nature of the theory saves the day here. Let us first remark that it is possible to write down a BRST invariant version of the Gribov restriction, that is, if γ were to be nonzero, whilst preserving equivalence with the above formalism[1], see [176; 177] for details. As already reminded before, the topological partition function does not depend on the coupling g. This can also be shown using a BRST cohomological argument, as we reiterate in the next subsection. This means all observables can be computed in the $g \to 0$ limit. Expanding around a nontrivial instanton background rather than around $A = 0$ would lead to corrections of the type e^{-1/g^2} into the effective action, but the latter vanish exponentially fast once $g \to 0$ is considered. As such, we can a priori work around $A = 0$.

This is good news, as explicit instanton computations are usually performed in a background gauge setting, being virtually impossible in other gauges such as Landau gauge. The above reasoning prevents us that we should resort to another gauge, such as the background Landau gauge, for which the Gribov problem and resolution is a bit different and actually far more complicated, in particular when BRST invariance is to be preserved [178; 179]. In [179], such computation was

[1] In the sense that all correlation functions will be identical.

presented for a constant temporal background, already complicated enough. For an x-dependent instanton background, the methodology of [179] simply looks technically impossible.

8.4.5 Further evidence

Before ending, we find it instructive to present yet another argument why a null Gribov parameter is also in full accordance with the possibility of a vanishing β-function discussed in [55]. Indeed, the variation of the full action with respect to the coupling constant gives a BRST-exact term (up to boundary terms),

$$\delta_g S = s\left(\Delta^{(-1)}\right) \ , \tag{8.77}$$

where $\Delta^{(-1)}$ is a polynomial in the fields and parameters, with ghost number equal to minus one. (We point out the strength of (8.77) in the BS theory, as such a condition is off-shell BRST exact.) This result is independent of the gauge choice. Since the expectation values of BRST exact terms vanish, (8.77) implies that

$$\delta_g Z = \left\langle s\left(\Delta^{(-1)}\right)\right\rangle = 0 \ , \tag{8.78}$$

without requiring the equations of motion, with Z the generating functional. It means that the BS theory is insensitive to changes of the coupling constant, in other words, that the theory has no scale. This can be re-expressed by the theory not having a β-function, which makes impossible the feature of dimensional transmutation. Indeed, the Gribov gap equation is nothing but a tool giving $\gamma^2 \propto \Lambda^2$, $\Lambda \sim \mu e^{-\frac{1}{\beta_0 g^2(\mu)}}$ being the fundamental scale of the theory if μ is the renormalization group scale; a quantity directly related to the β-function [180]. However, in the absence of the latter, it holds that $\Lambda \equiv 0$ and a classically massless (or better said scale invariant) theory will remain so at the quantum level. A rather similar situation showed up in the super $N = 4$ Yang-Mills theory which possesses a vanishing β-function. The absence of a renormalization group invariant scale makes it impossible to attach a dynamical meaning to the Gribov parameter, in such a way that the restriction to the first Gribov region is not required [181].

Chapter 9

Conclusions and perspectives

As we known, the Witten's TQFT is obtained via the *twist* transformation of the $N = 2$ SYM [46], whereas the Baulieu-Singer one, via the BRST gauge fixing of an action composed of topological invariants [48]. By analyzing the symmetries of the BS model, we first prove, as a consequence of the rich set of Ward identities in the (anti-)self-dual Landau gauges, that all two-point functions are tree-level exact. In particular, as a consequence of the vector supersymmetry present in Landau gauges [53], we show that the gauge field propagator vanishes to all orders in perturbation theory, cf. eq. (5.109), which makes it impossible the construction of non-vanishing loop diagrams, according to the vertex structure of the BS action in the (A)SDL gauges [54]. Thus we prove the absence of radiative corrections in the BS model in this particular gauge choice, *i.e.*, that not only the two-point functions but all n-point Green function are tree-level exact in (A)SDL gauges.

Hence we prove that the twisted $N = 2$ SYM and Baulieu-Singer topological quantum field theory do not possess the same quantum properties. Such a conclusion can be inferred from the relation given by eq. (4.57), where we can see that the difference between the Witten and BS actions does not belong to the trivial part of cohomology. While the $N = 2$ SYM β-function receives one-loop contributions in the Landau gauge [49], the Baulieu-Singer one vanishes in the (anti-)self-dual Landau gauges (5.4)-(5.6), since it does not receive any radiative correction at the quantum level. Such a result is protected by the topological BRST cohomology [56], see Sec. 6.3. The quantum correspondence occurs in the weak coupling limit of the twisted $N = 2$ as in this limit both β-functions vanishes. This correspondence is in complete agreement with the equivalence between Witten's TQFT (constructed in the limit $g^2 \to 0$ of the twsited $N = 2$) and the BS theory, which share the same observables [50; 51] given by the Donaldson polynomials [15; 16; 17].

We also demonstrate the existence of a new non-linear bosonic symmetry that relates the Faddeev-Popov ghost with the topological one based on the transformation $\psi^a_\mu \to D^{ab}_\mu c^b$, see the Ward identity described in eq.'s (5.37) and (5.38), which allows us to reduce the independent renormalization parameters from four to one, as expressed by the general counterterm (5.66). After applying the quantum stability condition, the resulting Z-factor system (5.70), taking into account this new symmetry, showed up a kind of renormalization ambiguity [55], that can be explained in terms of the absence of a gauge field kinetic term out from the trivial BRST cohomology, and due to the absence of certain discrete symmetries that commonly appear in ordinary Yang-Mills theories, see Sec. 6.2. We investigated this renormalization ambiguity for generalized classes of renormalizable gauges, and the result was the same. We verify that the ambiguity in the renormalization of the gauge field is automatically transferred to the renormalization of the coupling constant, see eq. (6.12), elucidating the non-physical character of the β-function in the topological gauge theory of BS type, as the coupling constant, together with the metric, only appears in the trivial cohomology sector, thus behaving as a non-physical gauge parameter.

By using these results, we study the Gribov problem [42] in *off-shell* topological Yang-Mills theories of BS type. Such a theory has three gauge ambiguities to be fixed, see (4.2)-(4.4). First we prove the equivalence between the Fadeev-Popov and topological BRST gauge-fixing procedures, see Sec. 8.1, then, analyzing the corresponding copy equations in (A)SDL gauges, we conclude that, to preserve the instantons degrees of freedom that characterizes the dimension of the moduli space, the usual Gribov restriction is sufficient to eliminate the infinitesimal gauge copies, see Sec. 8.3. After computing the no-pole condition for the Faddeev-Popov and topological ghost sector at one loop order, we show that the Gribov horizon in the topological BS Yang-Mills theory is identical to the one obtained in ordinary Yang-Mills theory. Therefore, restricting the Feynman path integral domain to the Gribov region, *i.e.*, inside the Gribov horizon, we prove the triviality of the gap equation, in other words, that the gap equation forbids the introduction of an infrared massive parameter of Gribov type in the gauge field propagator, as described by eq. (8.61). This result was generalized to all orders. Such a generalization was achieved by proving the absence of radiative corrections in the presence of the Gribov-Zwanziger horizon.

In few words, the topological Yang-Mills symmetry structure, together with the conformal property of the BS theory in (A)SDL gauges, hides a mechanism that turns out the Gribov restriction inoffensive in this case case, making it impossible the introduction of a massive parameter that would affect the infrared dynamics, preserving the original topological properties of the theory, once the Gribov horizon would have introduced a metric dependent term out from the trivial BRST cohomology. Such a behavior could eventually shed some light on the asymptotic behavior of Yang-Mills theories dominated by vacuum topological configurations in extreme energy scales, indicating the possible existence of a topological phase.

Perspectives. We can consider the possibility of introducing a pure topological phase in Yang-Mills theories by employing the BS approach. For that we must study the spontaneous symmetry breaking in *off-shell* topological gauge theories, and provide a physical interpretation concerning the liberation of the local degrees of freedom. The broken phase would produce new interactions involving the bosonic and topological ghosts, that could affect the quantum level of the theory, by giving nontrivial contributions to the loop diagrams. This approach can be used to study a topological phase of the Lovelock-Cartan gravity [121], starting from the Pontryagin and Gauss-Bonet topological invariants, following [182], (in [183], for instance, the authors proposed an ultraviolet topological phase of gravity, in which the Einstein-Hilbert action is recovered from a spontaneous symmetry breaking via Higgs mechanism. We emphasize that, in the BS approach, the Higgs mechanism would be only possible through the introduction of new degrees of freedom, *i.e.*, new fields given by "topological BRST partners" that was never observed in nature, in order to not explicitly break the topological BRST symmetry. For this reason, the study of other methods for symmetry breaking in topological BS models seems to be necessary to introduce a topological phase in Yang-Mills theory.

The geometric interpretation in the extended space $M \times \mathcal{A}/\mathcal{G}$ of the symmetry between the FP and topological ghosts was not analyzed as well. It could reveal new aspects of the global observables in the BS theory, as the operator corresponding to the Ward identity of this symmetry ($\mathcal{T}$), see (5.38), also defines a cohomology problem for the most general counterterm Σ^c, *i.e.*, $\mathcal{T}_\Sigma \Sigma^c = 0$ with $\mathcal{T}_\Sigma \mathcal{T}_\Sigma = 0$, which possesses a bosonic nature.

Besides that, considering the indirect evidence of topological configurations in strong interactions [26], whose effect helped to explain the QCD spectra, the connection between topological quantum field theory and the AdS/CFT correspondence [103; 104] could be explored to investigate new aspects of topological phases of matter. As we know, the AdS/CFT correspondence in strong interactions, known as AdS/QCD correspondence, has been used to study the quark-gluon plasma, such as glueball sates [184; 185]. The determination of instanton background effects by using the Gribov copies, see [179], provides a second method to investigate topological quantum effects to glueballs. (It is a well-known result in literature that instantons could provoke quantum forces in glueballs [66; 67], that can be used to study the strong CP violation.) The comparison between holographic models and the Gribov copies could uncover correspondences between the AdS/CFT duality and topological gauge theories.

Appendix A

Conventions for Green functions generators

In this section we employ the conventions of Euclidean QFT as in [98]. Let us write the most relevant relations that we will employ. The Green functional is defined as

$$Z[J] = N \int D\Phi e^{-\Sigma - \int d^4z J^A \Phi^A} \ , \tag{A.1}$$

where $N = 1/Z[0]$ is the usual normalization, Φ^A stands for all fields, J^A are Schwinger sources introduced for each field and A is a multiple index ranging all fields. The functional measure is then $D\Phi = \prod_A d\Phi^A$. The connected Green functional $W[J]$ is defined as

$$e^{-W[J]} = Z[J] \ . \tag{A.2}$$

Hence, the quantum action (vertex functional) is given by

$$\Gamma[\Phi] = W[J] - \int d^4z J^A \Phi^A \Big|_{\Phi^A = \frac{\delta W}{\delta J^A}} \ , \tag{A.3}$$

whose inverse reads

$$W[J] = \Gamma[\Phi] + \int d^4z J^A \Phi^A \Big|_{J^A = (-1)^{(g_A+1)} \frac{\delta \Gamma}{\delta \Phi^A}} \ , \tag{A.4}$$

where g_A stands for the statistics of the field Φ^A (+1 for fermions and 0 for bosons). And, as usual,

$$\left.\frac{\delta W}{\delta J^A}\right|_{J^A=0} = \left.\frac{\delta \Gamma}{\delta \Phi^A}\right|_{\Phi^A=0} = 0 \ . \tag{A.5}$$

The connected two-point functions will be denoted by

$$\langle \Phi^A(x)\Phi^B(y)\rangle = -\left.\frac{\delta^2 W}{\delta J^B(y)\delta J^A(x)}\right|_{J=0} \ . \tag{A.6}$$

In momentum space, we have,

$$\langle \Phi^A(x)\Phi^B(y)\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \langle \Phi^A \Phi^B\rangle(p) \ . \tag{A.7}$$

For the amputated two-point functions we define

$$\Gamma^{AB}_{\Phi\Phi}(x,y) = \left.\frac{\delta^2 \Gamma}{\delta \Phi^B(y)\delta \Phi^A(x)}\right|_{\Phi=0} \ , \tag{A.8}$$

and the corresponding Fourier transform reads

$$\Gamma^{AB}_{\Phi\Phi}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \Gamma^{AB}_{\Phi\Phi}(p) \ . \tag{A.9}$$

Appendix B

Proof of $\Gamma^{ab}_{(AA)\mu\nu}(p) = 0$

To proof the exact result (5.95), we consider the Slavnov-Taylor identity (C.1) for the vertex functional Γ,

$$\mathcal{S}(\Gamma) = \int d^4z \left[\left(\psi^c_\alpha(z) - \frac{\delta\Gamma}{\delta\Omega^c_\alpha(z)} \right) \frac{\delta\Gamma}{\delta A^c_\alpha(z)} + \ldots \right] . \tag{B.1}$$

Varying (B.1) *w.r.t.* $\psi^a_\mu(x)$ and $A^b_\nu(y)$ we get

$$\int d^4z \left[\left(\delta^{ca}\delta_{\alpha\mu}\delta(z-x) - \frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} \right) \frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta A^c_\alpha(z)} + \ldots \right] = 0 , \tag{B.2}$$

which simplifies to

$$\frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta A^a_\mu(x)} - \int d^4z \left[\frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} \frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta A^c_\alpha(z)} + \ldots \right] = 0 . \tag{B.3}$$

At vanishing sources and fields (B.3) yields

$$\Gamma^{ab}_{(AA)\mu\nu}(x,y) - \int d^4z \left[\frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} \frac{\delta^2\Gamma}{\delta A^b_\nu(y)\delta A^c_\alpha(z)} \right]^{J^A=0}_{\Phi^A=0} = 0 . \tag{B.4}$$

Now, to show that the second term in (B.4) vanishes we develop

$$\begin{aligned} \frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} &= \sum_A \int d^4w \frac{\delta^2 W}{\delta J^A_{(\Phi)}(w)\delta\Omega^c_\alpha(z)} \frac{\delta J^A_{(\Phi)}(w)}{\delta\psi^a_\mu(x)} \\ &= \sum_A (-1)^{g_A+1} \int d^4w \frac{\delta^2 W}{\delta J^A_{(\Phi)}(w)\delta\Omega^c_\alpha(z)} \frac{\delta^2\Gamma}{\delta\psi^a_\mu(x)\delta\Phi^A(w)} . \end{aligned} \tag{B.5}$$

Evoking the dFPs, the only fields Φ^A that may generate non-vanishing two-point functions are the fields with ghost number -1. Hence

$$\frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} = \int d^4w \left[\frac{\delta^2 W}{\delta J^d_{(\bar{c})}(w)\delta\Omega^c_\alpha(z)}\frac{\delta^2\Gamma}{\delta\psi^a_\mu(x)\delta\bar{c}^d(w)} + \frac{\delta^2 W}{\delta J^d_{(\bar{\eta})}(w)\delta\Omega^c_\alpha(z)}\frac{\delta^2\Gamma}{\delta\psi^a_\mu(x)\delta\bar{\eta}^d(w)} + \frac{\delta^2 W}{\delta J^d_{(\bar{\chi})\sigma\gamma}(w)\delta\Omega^c_\alpha(z)}\frac{\delta^2\Gamma}{\delta\psi^a_\mu(x)\delta\bar{\chi}^d_{\sigma\gamma}(w)}\right] . \tag{B.6}$$

At vanishing sources and fields, this last expression reads

$$\frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} = \int d^4w \left[\langle D^{ce}_\alpha c^e(z)\bar{c}^d(w)\rangle\Gamma^{da}_{(\bar{c}\psi)\mu}(w,x) + \langle D^{ce}_\alpha c^e(z)\bar{\eta}^d(w)\rangle\Gamma^{da}_{(\bar{\eta}\psi)\mu}(w,x) + \langle D^{ce}_\alpha c^e(z)\bar{\chi}^d_{\sigma\gamma}(w)\rangle\Gamma^{da}_{(\bar{\chi}\psi)\sigma\gamma\mu}(w,x)\right] . \tag{B.7}$$

It is easy to see, from the BRST transformations (4.5) and (4.28), that the above composite propagators can be written as (omitting the spacetime dependence and indices)

$$\begin{aligned}
\langle Dc\bar{c}\rangle &= -\langle s(A\bar{c})\rangle + \langle\psi\bar{c}\rangle + \langle Ab\rangle \;=\; \langle\psi\bar{c}\rangle + \langle Ab\rangle \, , \\
\langle Dc\bar{\eta}\rangle &= -\langle s(A\bar{\eta})\rangle + \langle\psi\bar{\eta}\rangle \;=\; \langle\psi\bar{\eta}\rangle \, , \\
\langle Dc\bar{\chi}\rangle &= -\langle s(A\bar{\chi})\rangle + \langle\psi\bar{\chi}\rangle + \langle AB\rangle \;=\; \langle\psi\bar{\chi}\rangle + \langle AB\rangle \, ,
\end{aligned} \tag{B.8}$$

where the known fact that the expectation value of BRST exact quantities are zero was used. Moreover, due to (5.105) and (5.115), we get

$$\begin{aligned}
\langle Dc\bar{c}\rangle &= \langle Ab\rangle \, , \\
\langle Dc\bar{\eta}\rangle &= -\langle s(A\bar{\eta})\rangle + \langle\psi\bar{\eta}\rangle \;=\; \langle\psi\bar{\eta}\rangle \, , \\
\langle Dc\bar{\chi}\rangle &= 0 \, ,
\end{aligned} \tag{B.9}$$

Hence,

$$
\begin{aligned}
\frac{\delta^2\Gamma}{\psi^a_\mu(x)\delta\Omega^c_\alpha(z)} &= \int d^4w \left[\langle A^c_\alpha(z)b^d(w)\rangle\Gamma^{da}_{(\bar{c}\psi)\mu}(w,x) + \langle\psi^c_\alpha(z)\bar{\eta}^d(w)\rangle\Gamma^{da}_{(\bar{\eta}\psi)\mu}(w,x)\right] \\
&= \int d^4w \left[\langle A^c_\alpha(z)b^d(w)\rangle - \langle\psi^c_\alpha(z)\bar{\eta}^d(w)\rangle\right]\Gamma^{da}_{(\bar{c}\psi)\mu}(w,x) \\
&= 0 \,, \qquad \text{(B.10)}
\end{aligned}
$$

where, in the second line, we used the fact that $\Gamma^{da}_{(\bar{c}\psi)\mu}(w,x) = -\Gamma^{da}_{(\bar{\eta}\psi)\mu}(w,x)$ (see (5.83) and (5.94)). In the third line, the relations (5.101) and (5.103) were employed. Therefore, we finally achieve

$$
\Gamma^{ab}_{(AA)\mu\nu}(x,y) = 0 \,, \qquad \text{(B.11)}
$$

as we wanted to show.

Appendix C

Renormalizability proof of the α-gauges

The aim of this first appendix is to prove the renormalizability of the action (6.6), *i.e.*, the renormalizability of the topological Yang-Mills theories at the α-gauges. The action (6.6) displays a few Ward identities:

(i) Slavnov-Taylor identity due the BRST invariance:

$$\mathcal{S}(\Sigma_\alpha) = 0 \; , \tag{C.1}$$

where

$$\begin{aligned}\mathcal{S}(\Sigma_\alpha) &= \int d^4z \left[\left(\psi^a_\mu - \frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu}\right)\frac{\delta\Sigma_\alpha}{\delta A^a_\mu} + \frac{\delta\Sigma_\alpha}{\delta\tau^a_\mu}\frac{\delta\Sigma_\alpha}{\delta\psi^a_\mu} + \left(\phi^a + \frac{\delta\Sigma_\alpha}{\delta L^a}\right)\frac{\delta\Sigma_\alpha}{\delta c^a} + \frac{\delta\Sigma_\alpha}{\delta E^a}\frac{\delta\Sigma_\alpha}{\delta\phi^a}\right.\\ &+ \left. b^a\frac{\delta\Sigma_\alpha}{\delta\bar{c}^a} + \bar{\eta}^a\frac{\delta\Sigma_\alpha}{\delta\bar{\phi}^a} + B^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta\bar{\chi}^a_{\mu\nu}} + \Omega^a_\mu\frac{\delta\Sigma_\alpha}{\delta\tau^a_\mu} + L^a\frac{\delta\Sigma_\alpha}{\delta E^a} + K^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta\Lambda^a_{\mu\nu}}\right] \; . \end{aligned} \tag{C.2}$$

(ii) Gauge-fixing and anti-ghost equations:

$$\frac{\delta\Sigma_\alpha}{\delta b^a} = \partial_\mu A^a_\mu + \alpha b^a \; ; \qquad \frac{\delta\Sigma_\alpha}{\delta\bar{c}^a} - \partial_\mu\frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu} = -\partial_\mu\psi^a_\mu \; . \tag{C.3}$$

(iii) Second gauge-fixing and anti-ghost equations:

$$\frac{\delta\Sigma_\alpha}{\delta\bar{\eta}^a} = \partial_\mu\psi^a_\mu \; ; \qquad \frac{\delta\Sigma_\alpha}{\delta\bar{\phi}^a} - \partial_\mu\frac{\delta\Sigma_\alpha}{\delta\tau^a_\mu} = 0 \; . \tag{C.4}$$

(iv) First non-linear bosonic symmetry:

$$T^{(1)}(\Sigma_\alpha) = 0 \,, \tag{C.5}$$

where

$$\begin{aligned}
T^{(1)}(\Sigma_\alpha) &= \int d^4z \left[\frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu}\frac{\delta\Sigma_\alpha}{\delta\psi^a_\mu} + \left(\phi^a - \frac{\delta\Sigma_\alpha}{\delta L^a}\right)\frac{\delta\Sigma_\alpha}{\delta\phi^a} + c^a\frac{\delta\Sigma_\alpha}{\delta c^a} - \bar{\phi}^a\frac{\delta\Sigma_\alpha}{\delta\bar{\phi}^a} - \bar{\eta}^a\left(\frac{\delta\Sigma_\alpha}{\delta\bar{\eta}^a} + \frac{\delta\Sigma_\alpha}{\delta\bar{c}^a}\right)\right. \\
&- \left. \Omega^a_\mu\frac{\delta\Sigma_\alpha}{\Omega^a_\mu} - \tau^a_\mu\frac{\delta\Sigma_\alpha}{\tau^a_\mu} - 2L^a\frac{\delta\Sigma_\alpha}{\delta L^a} - 2E^a\frac{\delta\Sigma_\alpha}{\delta E^a} - K^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta K^a_{\mu\nu}} - \Lambda^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta\Lambda^a_{\mu\nu}}\right] \,.
\end{aligned} \tag{C.6}$$

(v) Second non-linear bosonic symmetry:

$$T^{(2)}\left(\Sigma_\alpha\right) = 0 \,, \tag{C.7}$$

where

$$\begin{aligned}
T^{(2)}(\Sigma_\alpha) &= \int d^4z \left[\frac{\delta\Sigma_\alpha}{\delta K^a_{\mu\nu}}\frac{\delta\Sigma_\alpha}{\delta B^a_{\mu\nu}} + c^a\frac{\delta\Sigma_\alpha}{\delta c^a} - \bar{c}^a\left(\frac{\delta\Sigma_\alpha}{\delta\bar{c}^a} + \frac{\delta\Sigma_\alpha}{\delta\bar{\eta}^a}\right) + \phi^a\frac{\delta\Sigma_\alpha}{\delta\phi^a} - \bar{\phi}^a\frac{\delta\Sigma_\alpha}{\delta\bar{\phi}^a} - \Omega^a_\mu\frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu}\right. \\
&- \left. \tau^a_\mu\frac{\delta\Sigma_\alpha}{\delta\tau^a_\mu} - 2L^a\frac{\delta\Sigma_\alpha}{\delta L^a} - 2E^a\frac{\delta\Sigma_\alpha}{\delta E^a} - \Lambda^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta\Lambda^a_{\mu\nu}} - K^a_{\mu\nu}\frac{\delta\Sigma_\alpha}{\delta K^a_{\mu\nu}}\right] \,.
\end{aligned} \tag{C.8}$$

(vi) Global ghost supersymmetry:

$$\mathcal{G}_3\Sigma_\alpha = 0 \,, \tag{C.9}$$

where

$$\mathcal{G}_3 = \int d^4z \left[\bar{\phi}^a\left(\frac{\delta}{\delta\bar{\eta}^a} + \frac{\delta}{\delta\bar{c}^a}\right) - c^a\frac{\delta}{\delta\phi^a} + \tau^a_\mu\frac{\delta}{\delta\Omega^a_\mu} + 2E^a\frac{\delta}{\delta L^a} + \Lambda^a_{\mu\nu}\frac{\delta}{\delta K^a_{\mu\nu}}\right] . \tag{C.10}$$

We notice that, just like the (A)SDL gauges, the symmetries $T^{(1)}$ and $T^{(2)}$, in (C.5) and (C.7), respectively, can be combined to compose a more suitable symmetry operator,

$$T(\Sigma_\alpha) = T^{(1)}(\Sigma_\alpha) - T^{(2)}(\Sigma_\alpha) = 0 \,, \tag{C.11}$$

such that

$$T(\Sigma_\alpha) = \int d^4z \left[\frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu} \frac{\delta\Sigma_\alpha}{\delta\psi^a_\mu} - \frac{\delta\Sigma_\alpha}{\delta L^a} \frac{\delta\Sigma_\alpha}{\delta\phi^a} - \frac{\delta\Sigma_\alpha}{\delta K^a_{\mu\nu}} \frac{\delta\Sigma_\alpha}{\delta B^a_{\mu\nu}} + (\bar{c}^a - \bar{\eta}^a) \left(\frac{\delta\Sigma_\alpha}{\delta\bar{c}^a} + \frac{\delta\Sigma_\alpha}{\delta\bar{\eta}^a} \right) \right] . \tag{C.12}$$

To study the perturbative quantum stability of action (6.6) one adds to the classical action (6.6) the most general counterterm Σ^c_α by means of

$$\Gamma^{(1)} = \Sigma_\alpha + \epsilon\Sigma^c_\alpha . \tag{C.13}$$

Following (5.26), the Ward identities of the model implies that the counterterm Σ^c_α should satisfy the constraints

$$\begin{aligned}
\mathcal{S}_{\Sigma_\alpha}\Sigma^c_\alpha &= 0 , & \text{(C.14)}\\
\frac{\delta\Sigma^c_\alpha}{\delta b^a} &= 0 , & \text{(C.15)}\\
\frac{\delta\Sigma^c_\alpha}{\delta\bar{c}^a} - \partial_\mu \frac{\delta\Sigma^c_\alpha}{\delta\Omega^a_\mu} &= 0 , & \text{(C.16)}\\
\frac{\delta\Sigma^c_\alpha}{\delta\bar{\eta}^a} &= 0 , & \text{(C.17)}\\
\frac{\delta\Sigma^c_\alpha}{\delta\bar{\phi}^a} - \partial_\mu \frac{\delta\Sigma^c_\alpha}{\delta\tau^a_\mu} &= 0 , & \text{(C.18)}\\
T_{\Sigma_\alpha}\Sigma^c_\alpha &= 0 , & \text{(C.19)}\\
\mathcal{G}_3\Sigma^c_\alpha &= 0 , & \text{(C.20)}
\end{aligned}$$

where the linearized Slavnov-Taylor operator is given by

$$\begin{aligned}
\mathcal{S}_{\Sigma_\alpha} &= \int d^4z \left[\left(\psi^a_\mu - \frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu} \right) \frac{\delta}{\delta A^a_\mu} - \frac{\delta\Sigma_\alpha}{\delta A^a_\mu} \frac{\delta}{\delta\Omega^a_\mu} + \frac{\delta\Sigma_\alpha}{\delta\tau^a_\mu} \frac{\delta}{\delta\psi^a_\mu} + \left(\Omega^a_\mu + \frac{\delta\Sigma_\alpha}{\delta\psi^a_\mu} \right) \frac{\delta}{\delta\tau^a_\mu} \right.\\
&+ \left(\phi^a + \frac{\delta\Sigma_\alpha}{\delta L^a} \right) \frac{\delta}{\delta c^a} + \frac{\delta\Sigma_\alpha}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta\Sigma_\alpha}{\delta E^a} \frac{\delta}{\delta\phi^a} + \left(L^a + \frac{\delta\Sigma_\alpha}{\delta\phi^a} \right) \frac{\delta}{\delta E^a} + b^a \frac{\delta}{\delta\bar{c}^a}\\
&+ \left. \bar{\eta}^a \frac{\delta}{\delta\bar{\phi}^a} + B^a_{\mu\nu} \frac{\delta}{\delta\bar{\chi}^a_{\mu\nu}} + K^a_{\mu\nu} \frac{\delta}{\delta\Lambda^a_{\mu\nu}} \right] ,
\end{aligned} \tag{C.21}$$

and the linearized bosonic symmetry operator is

$$
\begin{aligned}
T_{\Sigma_\alpha} &= \int d^4z \left[\frac{\delta\Sigma_\alpha}{\delta\Omega^a_\mu}\frac{\delta}{\delta\psi^a_\mu} + \frac{\delta\Sigma_\alpha}{\delta\psi^a_\mu}\frac{\delta}{\delta\Omega^a_\mu} - \frac{\delta\Sigma_\alpha}{\delta L^a}\frac{\delta}{\delta\phi^a} - \frac{\delta\Sigma_\alpha}{\delta\phi^a}\frac{\delta}{\delta L^a} - \frac{\delta\Sigma_\alpha}{\delta K^a_{\mu\nu}}\frac{\delta}{\delta B^a_{\mu\nu}} - \frac{\delta\Sigma_\alpha}{\delta B^a_{\mu\nu}}\frac{\delta}{\delta K^a_{\mu\nu}} \right. \\
&+ \left. (\bar{c}^a - \bar{\eta}^a)\left(\frac{\delta}{\delta\bar{c}^a} + \frac{\delta}{\delta\bar{\eta}^a}\right)\right] . \qquad \text{(C.22)}
\end{aligned}
$$

Since the operator S_{Σ_α} is nilpotent, it defines a cohomology problem for Σ^c_α. The cohomology is trivial and the Slavnov-Taylor identity is free of anomalies [53; 54]. Hence, the general solution of (C.14) is

$$
\Sigma^c = \mathcal{S}_{\Sigma_\alpha}\Delta^{(-1)} , \qquad \text{(C.23)}
$$

where $\Delta^{(-1)}$ is an integrated local polynomial in the fields, sources and their derivatives, and parameters bounded by dimension 4 and ghost number -1. From (C.23) and the constraints (C.15) — (C.20), it is straightforward to conclude that the most general counterterm allowed is given by (5.66) — the same counterterm in the (A)SDL gauges case. To check if the the α-gauges are stable is to check if the counterterm (5.66) can be reabsorbed by the classical action Σ_α by means of the redefinition of the fields, sources and parameters as in (5.67) and (5.68). It is easy to check that the solution is given by (5.70) and (6.7), which completes the proof of renormalizability of topological Yang-Mills quantized at the α-gauges.

Appendix D

Renormalizability proof of the β-gauges

The renormalizability proof of topological Yang-Mills theories at the β-gauges follows the same procedures of the α-gauges discussed in the previous appendix. The starting action is now (6.8) and its symmetries are described by the following Ward identities:

(i) Slavnov-Taylor identity:

$$\mathcal{S}(\Sigma_\beta) = 0 \,, \tag{D.1}$$

where

$$\mathcal{S}(\Sigma_\beta) = \mathcal{S}(\Sigma_\alpha)\big|_{\Sigma_\alpha \to \Sigma_\beta} \,, \tag{D.2}$$

where $\mathcal{S}(\Sigma_\alpha)$ was defined in (C.1).

(ii) Gauge-fixing and anti-ghost equations:

$$\frac{\delta \Sigma_\beta}{\delta b^a} = \partial_\mu A^a_\mu \,; \qquad \frac{\delta \Sigma_\beta}{\delta \bar{c}^a} - \partial_\mu \frac{\delta \Sigma_\beta}{\delta \Omega^a_\mu} = -\partial_\mu \psi^a_\mu \,. \tag{D.3}$$

(iii) Second gauge-fixing and anti-ghost equations:

$$\frac{\delta \Sigma_\beta}{\delta \bar{\eta}^a} = \partial_\mu \psi^a_\mu \,; \qquad \frac{\delta \Sigma_\beta}{\delta \bar{\phi}^a} - \partial_\mu \frac{\delta \Sigma_\beta}{\delta \tau^a_\mu} = 0 \,. \tag{D.4}$$

(iv) First non-linear bosonic symmetry:

$$T^{(1)}(\Sigma_\beta) = 0 \, , \tag{D.5}$$

where

$$T^{(1)}(\Sigma_\alpha) = T^{(1)}(\Sigma_\alpha)\big|_{\Sigma_\alpha \to \Sigma_\beta} \, , \tag{D.6}$$

where $T^{(1)}(\Sigma_\alpha)$ was defined in (C.6).

(v) Bosonic ghost equation:

$$\mathcal{G}^a_\phi \Sigma_\beta = \Delta^a_\phi \, , \tag{D.7}$$

where

$$\begin{aligned}
\mathcal{G}^a_\phi &= \int d^4z \left(\frac{\delta}{\delta \phi^a} - g f^{abc} \bar{\phi}^b \frac{\delta}{\delta b^c} \right) , \\
\Delta^a_\phi &= g f^{abc} \int d^4z \left(\tau^b_\mu A^c_\mu + E^b c^c + \Lambda^b_{\mu\nu} \bar{\chi}^c_{\mu\nu} \right) \; .
\end{aligned} \tag{D.8}$$

(vi) Second Faddeev-Popov ghost equation:

$$\mathcal{G}^a_2 \Sigma_\beta = \Delta^a \, , \tag{D.9}$$

where

$$\begin{aligned}
\mathcal{G}^a_2 &= \int d^4z \left[\frac{\delta}{\delta c^a} - g f^{abc} \left(\bar{\phi}^b \frac{\delta}{\delta \bar{c}^c} + A^b_\mu \frac{\delta}{\delta \psi^c_\mu} + c^b \frac{\delta}{\delta \phi^c} - \bar{\eta}^b \frac{\delta}{\delta b^c} + E^b \frac{\delta}{\delta L^c} + \tau^b_\mu \frac{\delta}{\delta \Omega^c_\mu} \right) \right] \, , \\
\Delta^a &= g f^{abc} \int d^4z \left(E^b \phi^c - \Omega^b_\mu A^c_\mu - \tau^b_\mu \psi^c_\mu - L^b c^c + \Lambda^b_{\mu\nu} B^c_{\mu\nu} - K^b_{\mu\nu} \bar{\chi}^c_{\mu\nu} \right) \; .
\end{aligned} \tag{D.10}$$

(vi) Global ghost supersymmetry:

$$\mathcal{G}_3 \Sigma_\beta = 0 \, , \tag{D.11}$$

where

$$\mathcal{G}_3 = \int d^4z \left[\bar{\phi}^a \left(\frac{\delta}{\delta \bar{\eta}^a} + \frac{\delta}{\delta \bar{c}^a} \right) - c^a \frac{\delta}{\delta \phi^a} + \tau^a_\mu \frac{\delta}{\delta \Omega^a_\mu} + 2E^a \frac{\delta}{\delta L^a} + \Lambda^a_{\mu\nu} \frac{\delta}{\delta K^a_{\mu\nu}} \right] . \tag{D.12}$$

The perturbative quantum stability of action (6.8) is studied just like the previous case. We start by adding to the classical action (6.8) the most general counterterm Σ^c_β by means of

$$\Gamma^{(1)} = \Sigma_\beta + \epsilon \Sigma^c_\beta \ . \tag{D.13}$$

Then we impose the validity of all Ward identities valid for the classical action (6.8) to the quantum action, so that the counterterm Σ^c_β must satisfy the following constraints

$$\mathcal{S}_{\Sigma_\beta} \Sigma^c_\beta = 0 \ , \tag{D.14}$$

$$\frac{\delta \Sigma^c_\beta}{\delta b^a} = 0 \ , \tag{D.15}$$

$$\frac{\delta \Sigma^c_\beta}{\delta \bar{c}^a} - \partial_\mu \frac{\delta \Sigma^c_\beta}{\delta \Omega^a_\mu} = 0 \ , \tag{D.16}$$

$$\frac{\delta \Sigma^c_\beta}{\delta \bar{\eta}^a} = 0 \ , \tag{D.17}$$

$$\frac{\delta \Sigma^c_\beta}{\delta \bar{\phi}^a} - \partial_\mu \frac{\delta \Sigma^c_\beta}{\delta \tau^a_\mu} = 0 \ , \tag{D.18}$$

$$T^{(1)}_{\Sigma_\beta} \Sigma^c_\beta = 0 \ , \tag{D.19}$$

$$\mathcal{G}^a_\phi \Sigma^c_\beta = 0 \ , \tag{D.20}$$

$$\mathcal{G}^a_2 \Sigma^c_\beta = 0 \ , \tag{D.21}$$

$$\mathcal{G}_3 \Sigma^c_\beta = 0 \ , \tag{D.22}$$

where the linearized Slavnov-Taylor operator is given by

$$\mathcal{S}_{\Sigma_\beta} = \mathcal{S}_{\Sigma_\alpha} \Big|_{\Sigma_\alpha \to \Sigma_\beta} \ , \tag{D.23}$$

where $\mathcal{S}_{\Sigma_\alpha}$ was defined in (C.21), and $T^{(1)}_{\Sigma_\beta}$ is given by

$$
\begin{aligned}
T^{(1)}_{\Sigma_\beta} &= \int d^4x \left[\frac{\delta\Sigma_\beta}{\delta\Omega^a_\mu}\frac{\delta}{\delta\psi^a_\mu} + \left(\phi^a - \frac{\delta\Sigma_\beta}{\delta L^a}\right)\frac{\delta}{\delta\phi^a} + c^a\frac{\delta}{\delta c^a} - \bar{\phi}^a\frac{\delta}{\delta\bar{\phi}^a} - \bar{\eta}^a\left(\frac{\delta}{\delta\bar{c}^a} + \frac{\delta}{\delta\bar{\eta}^a}\right)\right. \\
&+ \left(\frac{\delta\Sigma_\beta}{\delta\psi^a_\mu} - \Omega^a_\mu\right)\frac{\delta}{\delta\Omega^a_\mu} - \tau^a_\mu\frac{\delta}{\tau^a_\mu} - \left(\frac{\delta\Sigma_\beta}{\delta\phi^a} + 2L^a\right)\frac{\delta}{\delta L^a} - 2E^a\frac{\delta}{\delta E^a} - K^a_{\mu\nu}\frac{\delta}{\delta K^a_{\mu\nu}} \\
&- \left.\Lambda^a_{\mu\nu}\frac{\delta}{\delta\Lambda^a_{\mu\nu}}\right] . \qquad \text{(D.24)}
\end{aligned}
$$

The operator S_{Σ_β} is also nilpotent. Henceforth, it defines a cohomology problem for Σ^c_β. Once again the trivial BRST cohomology implies that the general solution of (D.14) is

$$
\Sigma^c_\beta = \mathcal{S}_{\Sigma_\beta}\Delta^{(-1)} , \qquad \text{(D.25)}
$$

where $\Delta^{(-1)}$ is an integrated local polynomial in the fields, sources and their derivatives, and parameters bounded by dimension 4 and ghost number -1. From (D.25) and the constraints (D.15) – (D.22), it is straightforward to show that the most general counterterm allowed is actually given by

$$
\begin{aligned}
\Sigma^c_\beta &= S_\Sigma \int d^4x \left(a_1\,\bar{\chi}^a_{\mu\nu}\partial_\mu A^a_\nu + a_2\,gf^{abc}\bar{\chi}^a_{\mu\nu}A^b_\mu A^c_\nu + a_4\beta\bar{\chi}^a_{\mu\nu}B^a_{\mu\nu}\right) \\
&= \int d^4x \left\{a_1\left[B^a_{\mu\nu}\partial_\mu A^a_\nu - \bar{\chi}^a_{\mu\nu}\partial_\mu\left(\psi^a_\nu - \frac{\delta\Sigma}{\delta\Omega^a_\nu}\right)\right] + \right. \\
&+ \left. a_2\left[gf^{abc}B^a_{\mu\nu}A^b_\mu A^c_\nu - 2gf^{abc}\bar{\chi}^a_{\mu\nu}\left(\psi^b_\mu - \frac{\delta\Sigma}{\delta\Omega^b_\mu}\right)A^c_\nu\right] + a_4\beta B^a_{\mu\nu}B^a_{\mu\nu}\right\} \qquad \text{(D.26)}
\end{aligned}
$$

As discussed in Section 6.2.2, the analysis of the quantum stability of the β-gauges via (5.67) and (5.68) leads to the relation $a_2 = a_1/2$, showing that the theory possesses two independent renormalization parameters. This simplification reduces (D.26) to (6.9). The solution for the Z factors are given by (5.70) and (6.10) and the proof of renormalizability of topological Yang-Mills quantized at the β-gauges is complete.

For the case in which α, $\beta \neq 0$, we have to collect the common symmetries between both cases, *i.e.*, between (C.14)-(C.20) and (D.14)-(D.22). Aftermath we conclude that the term (6.5) survives at the quantum level, which cannot be reabsorbed by the classical action $\Sigma(\alpha, \beta)$ (6.3), proving that the theory is not renormalizable for α, $\beta \neq 0$.

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